



On the probability of fast exits and long stays of a planar Brownian motion in simply connected domains



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ABSTRACT

Let T^D denote the first exit time of a planar Brownian motion from a domain D . Given two simply connected planar domains $U, W \neq \mathbb{C}$ containing 0, we investigate the cases in which we are more likely to have fast exits (meaning $\mathbf{P}(T^U < t) > \mathbf{P}(T^W < t)$ for t small) from U than from W , or long stays (meaning $\mathbf{P}(T^U > t) > \mathbf{P}(T^W > t)$ for t large). We prove several results on these questions. In particular, we show that the primary factor in the probability of fast exits is the proximity of the boundary to the origin, while for long stays an important factor is the moments of the exit time. The complex analytic theory that motivated our inquiry is also discussed.

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1. Introduction

The distribution of the exit time of a planar Brownian motion from a domain measures in some sense the size of the domain. It can also be used for the study of analytic functions via the conformal invariance of Brownian motion.

Let $Z_t = X_t + iY_t$, $t \geq 0$ denote standard Brownian motion moving in the plane and starting from the origin (that is, $Z_0 = 0$ almost surely). We denote by \mathbf{P} and \mathbf{E} the corresponding probability measure and expectation, respectively. For a domain D containing 0 in the complex plane \mathbb{C} , we denote by T^D the first exit time of Z_t from D ; that is

$$T^D = \inf\{t > 0 : Z_t \notin D\}.$$

Suppose also that f is a univalent function in the well-known class \mathcal{S} . Thus f is a function univalent and holomorphic in the unit disk \mathbb{D} with $f(0) = 0$ and $f'(0) = 1$. By a classical result of P. Lévy, the image of

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Z_t under f is a Brownian motion with a time change. We describe now a precise version of this conformal invariance of Brownian motion (see [5, §2]). Let

$$\rho_f(s) = \int_0^s |f'(Z_t)|^2 dt, \quad 0 \leq s < T^{\mathbb{D}}.$$

Observe that ρ_f is almost surely strictly increasing and set

$$W_t = f(Z_{\rho_f^{-1}(t)}), \quad 0 \leq t < \rho_f(T^{\mathbb{D}}).$$

Define also $W_{\rho_f(T^{\mathbb{D}})} = \lim_{t \rightarrow T^{\mathbb{D}}} W_{\rho_f(t)}$ and

$$W_{\rho_f(T^{\mathbb{D}})+t} = W_{\rho_f(T^{\mathbb{D}})} + (Z_{T^{\mathbb{D}}+t} - Z_{T^{\mathbb{D}}}), \quad t > 0.$$

Note that f being univalent and holomorphic implies that $P(\rho_f(T^{\mathbb{D}}) < \infty) = 1$; this is a nontrivial statement, but is shown for instance in [3, p. 198]. Lévy's theorem now asserts that $\{W_t, t \geq 0\}$ is standard planar Brownian motion starting from the origin.

We set

$$\nu(f) = \rho_f(T^{\mathbb{D}}) = \int_0^{T^{\mathbb{D}}} |f'(Z_t)|^2 dt$$

and observe that $\nu(f)$ is the first exit time of W_t from $f(\mathbb{D})$; that is, $\nu(f) = T^{f(\mathbb{D})}$. “The distribution of $\nu(f)$ is an intuitively appealing measure of the size of $f(\mathbb{D})$ ” [5]. In several classical extremal problems for functions in the class \mathcal{S} , the identity function $I(z) = z$ is the “smallest” function while the Koebe function $k(z) = z/(1-z)^2$ is the “largest” function in \mathcal{S} . B. Davis [5] conjectured that

$$\mathbf{E}[\Phi(\nu(I))] \leq \mathbf{E}[\Phi(\nu(f))],$$

for all $f \in \mathcal{S}$ and all increasing convex functions $\Phi : [0, \infty) \rightarrow \mathbb{R}$, and suggested that perhaps

$$\mathbf{P}(\nu(I) > t) \leq \mathbf{P}(\nu(g) > t),$$

for all $t > 0$ and $g \in \mathcal{S}$. Davis also asked in what sense, with regard to $\nu(f)$, the Koebe function is the largest in \mathcal{S} .

Apropos the first conjecture T. McConnell [15] proved that

$$\mathbf{E}[\nu(I)^p] \leq \mathbf{E}[\nu(f)^p], \quad 0 < p < \infty,$$

but the full conjecture remains open, as far as we know. McConnell also disproved the second conjecture by finding a function $g \in \mathcal{S}$ such that

$$\mathbf{P}(\nu(I) > t) > \mathbf{P}(\nu(g) > t),$$

for all sufficiently small $t > 0$.

Motivated by these developments, we have considered the following questions. Given two simply connected planar domains $U, W \neq \mathbb{C}$ containing 0, what sufficient conditions can we place on the domains so that we are more likely to have fast exits (meaning $\mathbf{P}(T^U < t) > \mathbf{P}(T^W < t)$ for t small) from U than from

W , or long stays (meaning $\mathbf{P}(T^U > t) > \mathbf{P}(T^W > t)$ for t large). We have found that the primary factor influencing the probability of fast exits is the proximity of the boundary to the origin. In order to make this more precise, let us introduce the following notation. For any simply connected domain V containing 0, let

$$d(V) = \inf\{|z| : z \in \partial V\}.$$

We then have the following theorem, which is the main result of the paper.

Theorem 1. *Suppose that $d(U) < d(W)$. Then, for all sufficiently small $t > 0$,*

$$\mathbf{P}(T^U < t) > \mathbf{P}(T^W < t). \quad (1.1)$$

In fact,

$$\lim_{t \rightarrow 0^+} \frac{\mathbf{P}(T^U < t)}{\mathbf{P}(T^W < t)} = \infty.$$

Among other things, this shows that McConnell's choice of $g \in \mathcal{S}$ such that $\mathbf{P}(\nu(I) > t) > \mathbf{P}(\nu(g) > t)$ is not at all an isolated example, and that in fact this inequality holds for every $g \in \mathcal{S}$ for sufficiently small t . In this sense, the unit disk is indeed seen to be extremal in \mathcal{S} , except in the opposite manner to what Davis conjectured!

Naturally, it would be nice to have an analog for long stays. We believe that the important factor for long stays in domains is the moments of the exit time. To be precise, for domain V let

$$H(V) = \sup\{p > 0 : \mathbf{E}[(T^V)^p] < \infty\};$$

note that $H(V)$ is proved in [3] to be exactly equal to half of the Hardy number of V , a purely analytic quantity, as defined in [11], and is therefore calculable for a number of common domains. Furthermore $H(V) \geq \frac{1}{4}$ as long as $V \neq \mathbb{C}$, with equality only when V is a rotation of the Koebe domain $\mathbb{C} \setminus (-\infty, \frac{-1}{4}]$ ([3]). We have the following simple result.

Proposition 1. *Suppose that $H(U) > H(W)$. Then*

$$\limsup_{t \rightarrow \infty} \frac{\mathbf{P}(T^W > t)}{\mathbf{P}(T^U > t)} = \infty. \quad (1.2)$$

We conjecture that this proposition is true with the \limsup replaced by \lim , but have not been able to prove it (except when W is either a half-plane or quarter-plane). This is discussed in detail in the final section.

2. Proofs

In this section we prove Theorem 1 and Proposition 1.

2.1. Proof of Theorem 1

The proof of Theorem 1 is based on the strong Markov property and the explicit formula for the transition density of one-dimensional Brownian motion. Some of the estimates used are probably known to experts; we hope, however, that their elementary derivation and their use in the study of univalent functions are of some interest.

In what follows, X_t will denote one-dimensional Brownian motion. The corresponding probability measure with starting point $x \in \mathbb{R}$ will be denoted by \mathbf{P}^x . The first hitting time of a point $y \in \mathbb{R}$ will be denoted by τ_y . The following well known equality comes easily from the reflection principle (see [8, p. 23]):

$$\mathbf{P}^0(\tau_a \leq t) = 2\mathbf{P}^0(X_t \geq a), \quad a > 0.$$

We will use the standard notation for the transition density function of Brownian motion:

$$p_s(x) = \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s}, \quad x \in \mathbb{R}, \quad s > 0.$$

It follows from elementary calculus that, for any $\delta > 0$ there exists a constant $C > 1$ such that for every $y > \delta$,

$$C^{-1} \frac{e^{-y^2}}{y} \leq \int_y^\infty e^{-x^2} dx \leq C \frac{e^{-y^2}}{y}. \quad (2.1)$$

Note that this provides upper and lower bounds of the form $\frac{C}{\beta} e^{-\beta^2/2}$ on the quantity $\mathbf{P}^0(X_1 \geq \beta)$ (see [9, Sec. 7.1]).

We begin by proving a preliminary proposition, then show how it extends to prove Theorem 1.

Proposition 2. *Let $K_\alpha := \mathbb{C} \setminus (-\infty, -\alpha]$ for any α with $0 < \alpha < 1$. Then*

$$\lim_{t \rightarrow 0^+} \frac{\mathbf{P}(T^{K_\alpha} < t)}{\mathbf{P}(T^{\mathbb{D}} < t)} = \infty.$$

To prove this, we need several lemmas. In what follows, C will denote a generic constant that may change from line to line.

Lemma 1. *Given $r \in (0, 1)$, there exists a constant $C > 0$ such that, for any $t > 0$,*

$$\mathbf{P}^0(X_s = 0 \text{ for some } s \in [rt, t]) \geq C.$$

Proof. It is straightforward to verify by scaling time that $\mathbf{P}^0(X_s = 0 \text{ for some } s \in [rt, t])$ is independent of t , so we may assume $t = \frac{1}{r}$. The following string of inequalities is evident by symmetry, translation, the Markov property, and the intermediate value theorem.

$$\begin{aligned} \mathbf{P}^0(X_s = 0 \text{ for some } s \in [1, 1/r]) &\geq \mathbf{P}^0(X_1 > 0, X_{1/r} < 0) \\ &\geq \mathbf{P}^0(0 < X_1 < 1, X_{1/r} - X_1 < -1) \\ &= \mathbf{P}^0(0 < X_1 < 1) \mathbf{P}^0(X_{1/r-1} > 1) = C > 0. \quad \square \end{aligned}$$

Lemma 2. *Given $r \in (0, 1)$ and $\alpha > 0$, there exists a constant $C > 0$ such that for all t sufficiently small we have*

$$\mathbf{P}^0(X_s \leq -\alpha, \quad \forall s \in [rt, t]) \geq C \sqrt{t} e^{-\frac{\alpha^2}{2rt}} e^{-\frac{\alpha}{\sqrt{rt}}}.$$

Proof. By symmetry and time scaling, we have

$$\mathbf{P}^0(X_s \leq -\alpha, \quad \forall s \in [rt, t]) = \mathbf{P}^0\left(X_s \geq \frac{\alpha}{\sqrt{rt}}, \quad \forall s \in [1, 1/r]\right).$$

Set $\beta = \frac{\alpha}{\sqrt{rt}} + 1$. Then, by the Markov property, for β sufficiently large (and thus t sufficiently small)

$$\begin{aligned} \mathbf{P}^0(X_s \geq \beta - 1, \forall s \in [1, 1/r]) &\geq \mathbf{P}^0\left(X_1 \geq \beta, X_{1+s} - X_1 \geq -1 \forall s \in [0, \frac{1}{r} - 1]\right) \\ &= \mathbf{P}^0(X_1 \geq \beta) \mathbf{P}^0\left(X_s \geq -1 \forall s \in [0, \frac{1}{r} - 1]\right) \\ &= C \mathbf{P}^0(X_1 \geq \beta) \geq \frac{C}{\beta} e^{-\beta^2/2}. \end{aligned}$$

The last inequality follows from the discussion surrounding (2.1). Note that $\frac{1}{\beta} \geq C\sqrt{t}$ for t sufficiently small. Thus,

$$\begin{aligned} \mathbf{P}^0\left(X_s \geq \frac{\alpha}{\sqrt{rt}}, \forall s \in [1, 1/r]\right) &\geq C\sqrt{t} e^{-(\frac{\alpha}{\sqrt{rt}}+1)^2/2} \\ &= C\sqrt{t} e^{-\frac{\alpha^2}{2rt}} e^{-\frac{\alpha}{\sqrt{rt}}}, \end{aligned}$$

for t sufficiently small, as required. \square

We can now prove Proposition 2. Fix α with $0 < \alpha < 1$, and choose $r \in (\alpha^2, 1)$. Now choose $\varepsilon > 0$ so that $1 - \varepsilon > \frac{\alpha^2}{r}$. We may apply Lemma 1 and Lemma 2, using the independence of X_s and Y_s , to get

$$\begin{aligned} \mathbf{P}(T^{K_\alpha} \leq t) &= \mathbf{P}(Z_s \in (-\infty, -\alpha], \text{ for some } s \in (0, t)) \\ &\geq \mathbf{P}^0(X_s \leq -\alpha, \text{ for all } s \in [rt, t]) \\ &\quad \times \mathbf{P}^0(Y_s = 0, \text{ for some } s \in [rt, t]) \\ &\geq C\sqrt{t} e^{-\frac{\alpha^2}{2rt}} e^{-\frac{\alpha}{\sqrt{rt}}}, \end{aligned}$$

for t sufficiently small. On the other hand, it is proved in [15] that for all $t > 0$ and all positive integers $n \geq 3$,

$$\mathbf{P}(T^{\mathbb{D}} \leq t) \leq c(n) e^{-\frac{\cos^2(\pi/n)}{2t}}.$$

Fixing n large enough, we see that for all $t > 0$,

$$\mathbf{P}(T^{\mathbb{D}} \leq t) \leq C e^{-\frac{(1-\varepsilon)}{2t}}.$$

Thus,

$$\lim_{t \rightarrow 0^+} \frac{\mathbf{P}(T^{K_\alpha} < t)}{\mathbf{P}(T^{\mathbb{D}} < t)} \geq \lim_{t \rightarrow 0^+} C \frac{\sqrt{t} \exp\left(-\frac{\alpha^2/r + \alpha\sqrt{t/r}}{2t}\right)}{\exp\left(-\frac{(1-\varepsilon)}{2t}\right)} = \lim_{t \rightarrow 0^+} \sqrt{t} \exp\left(\frac{(1-\varepsilon) - (\alpha^2/r + \alpha\sqrt{t/r})}{2t}\right) = \infty,$$

since for t sufficiently small we will have $(1 - \varepsilon) - (\alpha^2/r + \alpha\sqrt{t/r}) > \delta > 0$ for some $\delta > 0$. \square

Now we are ready for the proof of Theorem 1. By the scale invariance of Brownian motion we can assume that $d(W) = 1$, and rotation invariance allows us to assume that $-\alpha \in \partial U$, where $\alpha = d(U) \in (0, 1)$. Then clearly $\mathbb{D} \subseteq W$, and although it is not necessarily true that $U \subseteq K_\alpha$, we may still use our estimates for K_α as a lower bound by the following lemma.

Lemma 3. *If U is a simply connected domain containing 0, and $\alpha = d(U) \in (0, 1)$, then*

$$\mathbf{P}(T^U < t) \geq \frac{1}{2}\mathbf{P}(T^{K_\alpha} < t),$$

where $K_\alpha := \mathbb{C} \setminus (-\infty, -\alpha]$ as above.

Proof. Note that the complex conjugate of Z_t, \bar{Z}_t , is also a Brownian motion. Let

$$\tilde{T}^U = \inf\{t > 0 : \bar{Z}_t \notin U\}.$$

We claim that $T^U \wedge \tilde{T}^U \leq T^{K_\alpha}$ a.s. If not, then the union of Brownian paths

$$\{Z_t : 0 \leq t \leq T^{K_\alpha}\} \cup \{\bar{Z}_t : 0 \leq t \leq T^{K_\alpha}\}$$

contains a closed curve separating $-\alpha$ from ∞ , and this contradicts simple connectivity. Thus,

$$\mathbf{P}(T^U \wedge \tilde{T}^U < t) \geq \mathbf{P}(T^{K_\alpha} < t).$$

But

$$\mathbf{P}(T^U \wedge \tilde{T}^U < t) \leq \mathbf{P}(T^U < t) + \mathbf{P}(\tilde{T}^U < t) = 2\mathbf{P}(T^U < t),$$

and the lemma follows. \square

As for Theorem 1,

$$\lim_{t \rightarrow 0^+} \frac{\mathbf{P}(T^U < t)}{\mathbf{P}(T^W < t)} \geq \lim_{t \rightarrow 0^+} \frac{\frac{1}{2}\mathbf{P}(T^{K_\alpha} < t)}{\mathbf{P}(T^{\mathbb{D}} < t)} = \infty,$$

completing the proof.

Remark. In fact, Lemma 3 holds with the constant 1 in place of $\frac{1}{2}$. However, the proof requires a number of results on symmetrization and polarization which are not related to the rest of this paper; for this reason we have given the simpler result and proof above, and postpone the proof of the stronger result until the end of Section 3.

2.2. Proof of Proposition 1

Let $p \in (\mathbf{H}(W), \mathbf{H}(U))$ and $\delta = \frac{\mathbf{H}(U)-p}{2}$. Since $\mathbf{E}[(T^U)^{p+\frac{3}{2}\delta}] < \infty$, the well-known Markov inequality (see e.g. [10, 6.17]) implies

$$\mathbf{P}(T^U > t) \leq \frac{\mathbf{E}[(T^U)^{p+\frac{3}{2}\delta}]}{t^{p+\frac{3}{2}\delta}}.$$

We now need a lower bound on $\mathbf{P}(T^W > t)$. For that purpose we will use the so-called “layer cake” representation for the p -th moment (see e.g. [10, 6.24]):

$$\mathbf{E}((T^W)^p) = p \int_0^{+\infty} t^{p-1} \mathbf{P}(T^W > t) dt. \quad (2.2)$$

We now claim that

$$\limsup_{t \rightarrow +\infty} \frac{\mathbf{P}(T^W > t)}{t^{-(p+\delta)}} = +\infty,$$

since otherwise $\frac{\mathbf{P}(T^W > t)}{t^{-(p+\delta)}}$ is bounded above by a constant, and then by (2.2) we get $\mathbf{E}[(T^W)^p] < +\infty$ which contradicts the definition of $H(W)$. We obtain

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \frac{\mathbf{P}(T^W > t)}{\mathbf{P}(T^U > t)} &\geq \limsup_{t \rightarrow +\infty} \frac{\mathbf{P}(T^W > t)}{t^{-(p+\delta)}} \frac{t^{-(p+\delta)} t^{p+\frac{3}{2}\delta}}{\mathbf{E}[(T^U)^{p+\frac{3}{2}\delta}]} \\ &= \limsup_{t \rightarrow +\infty} \frac{\mathbf{P}(T^W > t)}{t^{-(p+\delta)}} \frac{t^{\frac{\delta}{2}}}{\mathbf{E}[(T^U)^{p+\frac{3}{2}\delta}]} = +\infty, \end{aligned}$$

which ends the proof. \square

3. Concluding remarks

Standard theory (see [7]) shows that $\frac{1}{4} \leq d(U) \leq 1$ for any Schlicht domain U (recall that a Schlicht domain is the image of the unit disk \mathbb{D} under a function univalent and holomorphic on the disk with $f(0) = 0$ and $f'(0) = 1$), with equality only in the case of rotations of the Koebe domain $K_{1/4}$ and the unit disk \mathbb{D} , respectively. Theorem 1 therefore shows that the unit disk and Koebe domain are extremal among Schlicht domains for fast exits. We conjecture that the same is true, with the roles of the two domains switched, for large stays.

Conjecture 1. *Suppose U is a Schlicht domain. Then*

$$\mathbf{P}(T^{K_{1/4}} < t) \leq \mathbf{P}(T^U < t) \leq \mathbf{P}(T^{\mathbb{D}} < t)$$

for t sufficiently large, with equality holding only when U is \mathbb{D} or a rotation of K_α .

We remark that this conjecture is essentially the same as Davis' conjecture discussed in the introduction, however we venture it only for large t (recall that Davis' conjecture is false for small t , as was shown in [15] and as follows from the results in this paper). Proving this may not be so simple, since for large t these probabilities seem to be related to the size and shape of the domain on a large scale, and this is not easy to understand for arbitrary domains.

The moments of the exit time have been considered previously by several authors. In [3], it is shown that for the wedge $R_\theta = \{|\arg(z)| < \theta\}$, that is, the infinite wedge centered at the positive real axis of angular width 2θ , we have $\mathbf{E}[(T^{R_\theta})^p] < \infty$ if and only if $p < \frac{\pi}{4\theta}$, so

$$H(R_\theta) = \frac{\pi}{4\theta}. \quad (3.1)$$

A domain W is *spiral-like of order $\sigma \geq 0$ with center a* if, for any $z \in W$, the spiral $\{a + (z-a) \exp(te^{-i\sigma}) : t \leq 0\}$ also lies within W ; W is *star-like* if it is spiral-like of order $\sigma = 0$. The quantity $H(W)$ can be determined explicitly if W is star-like or spiral-like, as is shown in [14], with equivalent analytic results appearing in [11] and [12]. In particular, if we take $a = 0$ then, since W is spiral-like, the quantity

$$\mathcal{A}_{r,W} = \max\{m(E) : E \text{ is a subarc of } W \cap \{|z| = r\}\},$$

is non-increasing in r (here m denotes angular Lebesgue measure on the circle). We may therefore let $\mathcal{A}_W = \lim_{r \nearrow \infty} \mathcal{A}_{r,W}$, and then [14, Thm. 2] we have $H(W) = \frac{\pi}{2\mathcal{A}_W \cos^2 \sigma}$.

As mentioned before we suspect that in many cases the limsup in Proposition 1 is not necessary, and venture the following conjecture.

Conjecture 2. *Suppose that $H(U) > H(W)$ and W is spiral-like. Then*

$$\lim_{t \rightarrow \infty} \frac{\mathbf{P}(T^W > t)}{\mathbf{P}(T^U > t)} = \infty.$$

Note that this includes the case that W is star-like, as well as the wedge R_θ . This conjecture would make a natural complement to the main result of this paper, Theorem 1. It would follow from the following, if true.

Conjecture 3. *Suppose that W is spiral-like. Then for any $p > H(W)$ there is a constant $C > 0$ so that $\mathbf{P}(T^W > t) \geq \frac{C}{t^p}$.*

Our evidence for the truth of Conjecture 2 is as follows. First note that Markov's inequality, used as in Proposition 1, yields the following fact.

Proposition 3. *For any $p < H(U)$, there is a constant $C > 0$ so that $\mathbf{P}(T^U > t) \leq \frac{C}{t^p}$.*

Furthermore the bound required is true in the limsup sense, as is shown in the proof of Proposition 1. Next we remark that Conjecture 2 has already been proved when W is a wedge. This is because Conjecture 3 has been proved when W is a wedge, in [1]. However, the proof there is rather difficult and technical, so we include a simple proof for the easiest cases, which are when W is a half-plane or quarter-plane. Let $W = \{\operatorname{Re}(z) < 1\}$; recall from (3.1) that $H(W) = \frac{1}{2}$. Then, using the reflection principle, $\mathbf{P}(T^W > t)$ can be bounded below as follows.

$$\begin{aligned} & \mathbf{P}^0(X_s < 1, \quad \forall s \in [0, t]) \\ &= 1 - 2\mathbf{P}^0(X_t \geq 1) = 1 - 2 \frac{1}{\sqrt{2\pi t}} \int_1^\infty e^{-x^2/2t} dx \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2t}}} e^{-\xi^2} d\xi \geq \frac{C}{\sqrt{t}}. \end{aligned}$$

Now suppose $W = \{\operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0\}$; recall from (3.1) that $H(W) = 1$. Then, using the independence of the one dimensional components of a planar Brownian motion, and the calculation for the half-plane, we have

$$\mathbf{P}^{1+i}(T^W > t) = \mathbf{P}^1(X_s > 0, \quad \forall s \in [0, t])^2 \geq \frac{C}{t}$$

for t bounded away from 0.

Finally, we prove an improvement of Lemma 3, as we promised in Section 2. This result is based on the fact that if U is a simply connected domain with $d(U) = \alpha$, then its circular symmetrization (see [6] or [13]) is contained in K_α . Therefore, although in general U is not a subset of K_α , the slit domain K_α may be considered as “conformally larger” than U . Consequently, Brownian motion stays longer in K_α than in U .

Proposition 4. Let U be a simply connected domain with $0 \in U$ and let $\alpha = d(U) \in (0, \infty)$. If $K_\alpha = \mathbb{C} \setminus (-\infty, -\alpha]$, then for every $t > 0$,

$$\mathbf{P}(T^U > t) \leq \mathbf{P}(T^{K_\alpha} > t).$$

Proof. Let $p^U(t, 0, w)$ be the transition density function for Brownian motion killed upon hitting ∂U . Then (see e.g. [4, Theorem 2.4])

$$\mathbf{P}(T^U > t) = \int p^U(t, 0, w) A(dw), \quad 0 < t < +\infty,$$

where A denotes the area measure. A similar formula holds for $\mathbf{P}(T^{K_\alpha} > t)$. Therefore, it suffices to prove that

$$\int p^U(t, 0, w) A(dw) \leq \int p^{K_\alpha}(t, 0, w) A(dw), \quad 0 < t < +\infty. \quad (3.2)$$

We will prove (3.2) using the theory of polarization and symmetrization. We refer to [13], [6], [2] for the definitions and basic facts.

The function $p^U(t, 0, w)$ satisfies the heat equation on U . Let $P_H U$ denote the polarization of U with respect to a half-plane H with $0 \in \partial H$. This is defined as follows. Let R_H denote reflection in the boundary line of H . We divide U into three disjoint sets: the symmetric part of U , $U_s := U \cap R_H U$; the upper non-symmetric part of U , $U_u := (U \cap H) \setminus U_s$; and the lower non-symmetric part of U , $U_\ell := (U \cap R_H H) \setminus U_s$. Then $U = U_s \cup U_u \cup U_\ell$. The polarization of U with respect to the half-plane H is the set $P_H U := U_s \cup U_u \cup R_H U_\ell$. With this definition, it follows from [2, Theorem 9.4] that

$$p^U(t, 0, w) + p^U(t, 0, R_H w) \leq p^{P_H U}(t, 0, w) + p^{P_H U}(t, 0, R_H w), \quad 0 < t < \infty. \quad (3.3)$$

It follows from (3.3) that for every $r \in (0, +\infty)$,

$$\int_0^{2\pi} p^U(t, 0, re^{i\theta}) d\theta \leq \int_0^{2\pi} p^{P_H U}(t, 0, re^{i\theta}) d\theta, \quad 0 < t < \infty. \quad (3.4)$$

Applying a standard technique involving a sequence of polarizations (see [6, Section 4.1] or [2, Section 6]) to (3.4) leads to the inequality

$$\int_0^{2\pi} p^U(t, 0, re^{i\theta}) d\theta \leq \int_0^{2\pi} p^{U^*}(t, 0, re^{i\theta}) d\theta, \quad 0 < t < \infty, \quad 0 < r < \infty, \quad (3.5)$$

where U^* is the circular symmetrization of U with respect to the positive semi-axis (see [6, Section 4.1] or [2, Section 6]). Since $U^* \subset K_\alpha$, we have

$$\int_0^{2\pi} p^{U^*}(t, 0, re^{i\theta}) d\theta \leq \int_0^{2\pi} p^{K_\alpha}(t, 0, re^{i\theta}) d\theta, \quad 0 < t < \infty, \quad 0 < r < \infty. \quad (3.6)$$

By (3.5), (3.6), and integration over $r \in (0, \infty)$, we obtain (3.2). \square

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References

- [1] S. Brascosco, A note on planar Brownian motion, *Ann. Probab.* 20 (3) (1992) 1498–1503.
- [2] F. Brock, A. Solynin, An approach to symmetrization via polarization, *Trans. Am. Math. Soc.* 352 (4) (2000) 1759–1796.
- [3] D. Burkholder, Exit times of Brownian motion, harmonic majorization, and Hardy spaces, *Adv. Math.* 26 (2) (1977) 182–205.
- [4] K. Chung, Z. Zhao, *From Brownian Motion to Schrödinger's Equation*, vol. 312, Springer Science & Business Media, 2012.
- [5] B. Davis, Brownian motion and analytic functions, *Ann. Probab.* 7 (6) (1979) 913–932.
- [6] V. Dubinin, *Condenser Capacities and Symmetrization in Geometric Function Theory*, Springer, 2014.
- [7] P.L. Duren, *Univalent Functions*, Springer, 1983.
- [8] R. Durrett, *Brownian Motion and Martingales in Analysis*, Wadsworth Advanced Books & Software, 1984.
- [9] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 2, John Wiley & Sons, 2008.
- [10] G. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, 2013.
- [11] L. Hansen, Hardy classes and ranges of functions, *Mich. Math. J.* 17 (3) (1970) 235–248.
- [12] L. Hansen, The Hardy class of a spiral-like function, *Mich. Math. J.* 18 (3) (1971) 279–282.
- [13] W. Hayman, *Multivalent Functions*, vol. 110, Cambridge University Press, 1994.
- [14] G. Markowsky, The exit time of planar Brownian motion and the Phragmén–Lindelöf principle, *J. Math. Anal. Appl.* 422 (1) (2015) 638–645.
- [15] T. McConnell, The size of an analytic function as measured by Lévy's time change, *Ann. Probab.* 13 (3) (1985) 1003–1005.