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Proof of some conjectural hypergeometric supercongruences via curious identities

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ABSTRACT

In this paper, we prove several supercongruences conjectured by Z.-W. Sun ten years ago via certain strange hypergeometric identities. For example, for any prime $p > 3$, we show that

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k}\equiv 0 \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\begin{cases}\binom{(2p-2)/3}{(p-1)/3} \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ p/\binom{(2p+2)/3}{(p+1)/3} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

We also obtain some other results of such types.

1. Introduction

For $n, r \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $\alpha_0, \ldots, \alpha_r, \beta_1, \ldots, \beta_r, z \in \mathbb{C}$ the truncated hypergeometric series ${}_{r+1}F_r$ are defined by

$${}_{r+1}F_r\left[\begin{matrix}\alpha_0 & \alpha_1 & \cdots & \alpha_r\\ & \beta_1 & \cdots & \beta_r\end{matrix}\,\middle|\, z\right]_n := \sum_{k=0}^{n}\frac{(\alpha_0)_k\cdots(\alpha_r)_k}{(\beta_1)_k\cdots(\beta_r)_k}\cdot\frac{z^k}{k!},$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ is the Pochhammer symbol (rising factorial). Since $(-\alpha)_k/(1)_k = (-1)^k\binom{\alpha}{k}$, sometimes we may write the truncated hypergeometric series as sums involving products of

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binomial coefficients. In the past decades, supercongruences involving truncated hypergeometric series have been widely studied (cf. for example, [6,7,9–12,15,16,24–26,30,31]).

Via the p-adic Gamma function and the Gross-Koblitz formula, E. Mortenson [15,16] proved that for any prime $p > 3$ we have

$$\begin{aligned} {}_2F_1\left[\begin{matrix}\frac12 & \frac12\\ & 1\end{matrix}\middle|\, 1\right]_{p-1} &\equiv \left(\frac{-1}{p}\right) \pmod{p^2}, & {}_2F_1\left[\begin{matrix}\frac13 & \frac23\\ & 1\end{matrix}\middle|\, 1\right]_{p-1} &\equiv \left(\frac{-3}{p}\right) \pmod{p^2},\\ {}_2F_1\left[\begin{matrix}\frac14 & \frac34\\ & 1\end{matrix}\middle|\, 1\right]_{p-1} &\equiv \left(\frac{-2}{p}\right) \pmod{p^2}, & {}_2F_1\left[\begin{matrix}\frac16 & \frac56\\ & 1\end{matrix}\middle|\, 1\right]_{p-1} &\equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \end{aligned} \tag{1.1}$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol. Actually, these congruences were first conjectured in [19] motivated by hypergeometric families of Calabi-Yau manifolds. For any prime $p > 3$, Z.-W. Sun [25] showed further that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3}, \tag{1.2}$$

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}^2}{16^k} \equiv -2p^2 E_{p-3} \pmod{p^3}, \tag{1.3}$$

where E_{p-3} is the $(p-3)$th Euler number. In fact, (1.2) and (1.3) are refinements of the first congruence in (1.1). To see this, we note that $(\frac12)_k/(1)_k = \binom{2k}{k}/4^k$ for any $k \in \mathbb{N}$. Z.-W. Sun [26] also gave some extensions of (1.1). In 2014, Z.-H. Sun [24] found that the congruences in (1.1) can be extended to a unified form. For any odd prime p let $\mathbb{Z}_p$ denote the ring of all p-adic integers. For any $\alpha \in \mathbb{Z}_p$, we use $\langle \alpha \rangle_p$ to denote the least nonnegative residue of α modulo p, i.e., the unique integer lying in $\{0, 1, \ldots, p-1\}$ such that $\langle \alpha \rangle_p \equiv \alpha \pmod{p}$. Z.-H. Sun [24] proved that for any $\alpha \in \mathbb{Q} \cap \mathbb{Z}_p$ we have

$${}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\middle|\, 1\right]_{p-1} \equiv (-1)^{\langle -\alpha \rangle_p} \pmod{p^2}. \tag{1.4}$$

It is easy to see that the congruences in (1.1) are special cases of (1.4).

It is worth noting that Mortenson's congruences in (1.1) all concern the hypergeometric series with variable $z = 1$. In this paper, we shall confirm several congruences involving hypergeometric series with variable $z \neq 1$, as conjectured by Z.-W. Sun.

It is rare that a sum of the form $\sum_{k=0}^{p-1} a_k$ is always congruent to 0 modulo p^2 for any prime $p > 3$. The only known example we can recall is Wolstenholme's congruence $H_{p-1} \equiv 0 \pmod{p^2}$ (cf. [32]) for any prime $p > 3$, where H_n denotes the harmonic number $\sum_{0<k\le n} 1/k$. Nevertheless, we establish the following curious result which was first conjectured by Sun [25, Conjecture 5.14(i)].

Theorem 1.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv 0 \pmod{p^2}. \tag{1.5}$$

Remark 1.1. Sun [29, Conjecture 17] conjectured further that for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv \frac{5}{12} p^2 B_{p-2}\left(\frac13\right) \pmod{p^3} \tag{1.6}$$

and

$$p^2\sum_{k=1}^{p-1}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}\equiv 4\left(\frac{p}{3}\right)+4p \pmod{p^2}, \tag{1.7}$$

where $B_{p-2}(x)$ is the Bernoulli polynomial of degree $p-2$. It is worth mentioning that (1.7) is related to Sun's conjectural identity

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}=\frac{15}{2}\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2} \tag{1.8}$$

(cf. [28, (1.23)]) and Sun would like to offer \$480 as the prize for the first proof of (1.8).

Our second theorem concerns a variant of the second congruence in (1.1) and confirms a conjecture of Z.-W. Sun in [25, Conjecture 5.13] and [29, Conjecture 16(i)].

Theorem 1.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\begin{cases}\binom{(2p-2)/3}{(p-1)/3} \pmod{p^2} & \text{if } p\equiv 1 \pmod 3,\\ p/\binom{(2p+2)/3}{(p+1)/3} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3.\end{cases} \tag{1.9}$$

Remark 1.2. By Sun [26, (1.20)], for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k} \pmod{p^2}.$$

Theorem 3.2 of Sun [27] with $x=y=-z$ and $a=1$ gives the following p-adic analogue of the Clausen identity (cf. [1, p. 116]):

$$\left({}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\,\middle|\, z\right]_{p-1}\right)^2\equiv {}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \frac{1}{2}\\ & 1 & 1\end{matrix}\,\middle|\, 4z(1-z)\right]_{p-1} \pmod{p^2} \tag{1.10}$$

for any odd prime p and $\alpha,z\in\mathbb{Z}_p$; this was given by Z.-H. Sun in the cases $\alpha=1/3,1/4,1/6$ (cf. [21–23]). Applying (1.10) with $\alpha=1/3$ and $z=9/8$ and noting that $(1/3)_k(2/3)_k/(1)_k^2=\binom{2k}{k}\binom{3k}{k}/27^k$ we obtain that

$$\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\right)^2\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k} \pmod{p^2}. \tag{1.11}$$

It is known (cf. [2,3]) that for any prime $p\equiv 1 \pmod 3$ with $4p=x^2+27y^2$ $(x,y\in\mathbb{Z})$ we have

$$\binom{(2p-2)/3}{(p-1)/3}\equiv\left(\frac{x}{3}\right)\left(\frac{p}{x}-x\right) \pmod{p^2}. \tag{1.12}$$

Combining Theorem 1.2, (1.11) and (1.12) we immediately obtain the following result which was conjectured by Z.-W. Sun in [25, Conjecture 5.6] and [29, Conjecture 24(i)] and partially proved by Z.-H. Sun [21, Theorem 4.2].

Corollary 1.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \ \&\ 4p = x^2 + 27y^2\ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

The next result gives a companion of Theorem 1.2.

Theorem 1.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{(k+1)\binom{3k}{k}\binom{2k}{k}}{24^k} \equiv \begin{cases} p/\binom{(2p-2)/3}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ -(p+1)\binom{(2p+2)/3}{(p+1)/3} \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases} \tag{1.13}$$

Combining Theorems 1.2 and 1.3 we confirm the following two conjectures of Sun [25, Conjecture 5.13].

Corollary 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{(k+1)24^k} \equiv \frac{1}{2}\binom{2(p - (\frac{p}{3}))/3}{(p - (\frac{p}{3}))/3} \pmod{p}. \tag{1.14}$$

When $p \equiv 1 \pmod 3$ and $4p = x^2 + 27y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 2 \pmod 3$, we have

$$\sum_{k=0}^{p-1} \frac{k+2}{24^k}\binom{2k}{k}\binom{3k}{k} \equiv x \pmod{p^2}. \tag{1.15}$$

Remark 1.3. To obtain (1.14), we note the following congruence relation obtained by Sun [26]:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{(k+1)m^k} \equiv p + \frac{m-27}{6}\sum_{k=0}^{p-1} \frac{k\binom{2k}{k}\binom{3k}{k}}{m^k} \pmod{p^2},$$

where $p > 3$ is a prime and m is an integer with $p \nmid m$. To get (1.15) we only need to substitute (1.12) into (1.9) and (1.13).

The proofs of the above three theorems depend on some new hypergeometric identities motivated by the strange identities obtained by S.B. Ekhad [5] and the Pfaff transformation (cf. [1, p. 68]); they will be given in Sections 2–4. We can also prove some other results similar to Theorems 1.1–1.3 in the same way, however, we will not give the detailed proofs of them; we shall list them and sketch their proofs in the last section.

2. Proof of Theorem 1.1

Let p be an odd prime. Let us recall the concept of the p-adic Gamma function introduced by Y. Morita [14] as a p-adic analogue of the classical Gamma function. For each integer $n \geq 1$, define the p-adic Gamma function

$$\Gamma_p(n) := (-1)^n \prod_{\substack{1 \leq k < n \\ p \nmid k}} k.$$

Moreover, set $\Gamma_p(0) = 1$. It is clear that the definition of Γ_p can be extended to $\mathbb{Z}_p$ since $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ with respect to p-adic norm $|\cdot|_p$. Thus, for all $x \in \mathbb{Z}_p$ we can define

$$\Gamma_p(x) := \lim_{\substack{n\in\mathbb{N}\\ |x-n|_p\to 0}} \Gamma_p(n),$$

so that the function $\Gamma_p(x)$ is p-adically continuous. The reader is referred to [14,18] for some properties of the p-adic Gamma function. It is known (cf. [18, p. 369]) that for any $x \in \mathbb{Z}_p$ we have

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{p-\langle -x\rangle_p} \tag{2.1}$$

and

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } p \nmid x, \\ -1 & \text{if } p \mid x. \end{cases} \tag{2.2}$$

The following lemma gives a p-adic expansion of Γ_p.

Lemma 2.1. *For any prime $p > 3$ and $\alpha, t \in \mathbb{Z}_p$ we have*

$$\Gamma_p(\alpha + tp) \equiv \Gamma_p(\alpha)\left(1 + tp(\Gamma_p'(0) + H_{p-1-\langle -\alpha\rangle_p})\right) \pmod{p^2}, \tag{2.3}$$

where $\Gamma_p'(x)$ denotes the derivative of $\Gamma_p(x)$ for any $x \in \mathbb{Z}_p$.

Proof. It is known (cf. [11, Theorem 14]) that

$$\Gamma_p(\alpha + tp) \equiv \Gamma_p(\alpha)(1 + tp\Gamma_p'(\alpha)) \pmod{p^2}.$$

By [31, Lemma 2.4] we have

$$\frac{\Gamma_p'(\alpha)}{\Gamma_p(\alpha)} \equiv \Gamma_p'(0) + H_{p-1-\langle -\alpha\rangle_p} \pmod{p}.$$

Combining the above we immediately get the desired result. □

The following hypergeometric identity plays a key role in the proof of Theorem 1.1.

Lemma 2.2. *For any nonnegative integer n and $\delta \in \{0, 1\}$, we have*

$$\sum_{k=0}^{n} \frac{(4n+2\delta-k+1)(-n)_k(\frac{1}{2}-\delta-n)_k}{(2n+\delta+k+1)(1)_k(-4n-2\delta)_k}\left(\frac{4}{3}\right)^k = \left(\frac{3}{4}\right)^{2n+\delta} \frac{\Gamma(\frac{1}{2})\Gamma(2n+1+\delta)}{\Gamma(2n+\delta+\frac{1}{2})}. \tag{2.4}$$

Proof. Denote the left-hand side of (2.4) by $S(n)$. Via Zeilberger's algorithm (cf. [17]) in `Mathematica`, we find that

$$9(n+1)(2n+2\delta+1)S(n) - 2(4n+2\delta+1)(4n+2\delta+3)S(n+1) = 0$$

for any $n \in \mathbb{N}$ and $\delta \in \{0, 1\}$. Then the identities can be verified by induction on n. □

For a prime p and an integer $a \not\equiv 0 \pmod{p}$, we use $q_p(a)$ to denote the Fermat quotient $(a^{p-1} - 1)/p$.

Lemma 2.3 *(Lehmer [8]). For any prime $p > 3$, modulo p, we have the following congruences:*

$$H_{\lfloor p/2\rfloor} \equiv -2q_p(2),\ H_{\lfloor p/3\rfloor} \equiv -\frac{3}{2}q_p(3),\ H_{\lfloor p/4\rfloor} \equiv -3q_p(2),\ H_{\lfloor p/6\rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3).$$

Proof of Theorem 1.1. Note that $(1/4)_k(3/4)_k/(1)_k^2 = \binom{4k}{2k}\binom{2k}{k}/64^k$. Thus we can rewrite (1.5) as follows:

$$\sum_{k=0}^{p-1} \frac{2k(\frac{1}{4})_k(\frac{3}{4})_k}{(2k+1)(1)_k^2}\left(\frac{4}{3}\right)^k \equiv 0 \pmod{p^2}.$$

Let

$$f_k(x) := \frac{(x-k)(\frac{1-x}{4})_k(\frac{3-x}{4})_k}{(\frac{x}{2}+k+\frac{1}{2})(1-x)_k(1)_k}\left(\frac{4}{3}\right)^k$$

and

$$f(x) := \sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1} f_k(x).$$

Clearly, for any $t \in \mathbb{Z}_p$ we have

$$f(tp) \equiv f(0) + tpf'(0) \pmod{p^2},$$

where $f'(x)$ stands for the derivative of $f(x)$. Thus we can easily obtain that

$$f(0) \equiv \frac{1}{2}(3f(p) - f(3p)) \pmod{p^2}. \tag{2.5}$$

On the other hand, it is easy to see that

$$f(0) = -\sum_{k=0}^{p-1} \frac{2k(\frac{1}{4})_k(\frac{3}{4})_k}{(2k+1)(1)_k^2}\left(\frac{4}{3}\right)^k + \varepsilon, \tag{2.6}$$

where

$$\varepsilon := \frac{(p-1)(\frac{1}{4})_{(p-1)/2}(\frac{3}{4})_{(p-1)/2}}{p(1)_{(p-1)/2}^2}\left(\frac{4}{3}\right)^{(p-1)/2}.$$

Combining (2.5) and (2.6) we arrive at

$$\sum_{k=0}^{p-1} \frac{2k(\frac{1}{4})_k(\frac{3}{4})_k}{(2k+1)(1)_k^2}\left(\frac{4}{3}\right)^k \equiv \varepsilon - \frac{3}{2}f(p) + \frac{1}{2}f(3p) \pmod{p^2}. \tag{2.7}$$

We first consider ε modulo p^2. Clearly,

$$\frac{\Gamma(-\frac{1}{4}+\frac{p}{2})\Gamma(\frac{1}{4}+\frac{p}{2})}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})} = \frac{p\Gamma_p(-\frac{1}{4}+\frac{p}{2})\Gamma_p(\frac{1}{4}+\frac{p}{2})}{4\Gamma_p(\frac{1}{4})\Gamma_p(\frac{3}{4})}.$$

Thus, by Lemma 2.1 we have

$$\begin{aligned}\varepsilon &= \frac{(p-1)\Gamma(-\frac{1}{4}+\frac{p}{2})\Gamma(\frac{1}{4}+\frac{p}{2})\Gamma(1)^2}{p\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})\Gamma(\frac{1}{2}+\frac{p}{2})^2}\left(\frac{4}{3}\right)^{(p-1)/2}\\ &= \frac{(p-1)\frac{p}{4}\Gamma_p(-\frac{1}{4}+\frac{p}{2})\Gamma_p(\frac{1}{4}+\frac{p}{2})\Gamma_p(1)^2}{p\Gamma_p(\frac{1}{4})\Gamma_p(\frac{3}{4})\Gamma_p(\frac{1}{2}+\frac{p}{2})^2}\left(\frac{4}{3}\right)^{(p-1)/2}\\ &\equiv \frac{(-p-1)}{\Gamma_p(\frac{1}{2})^2}\left(1+pH_{\lfloor p/4\rfloor}-pH_{(p-1)/2}\right)\left(\frac{4}{3}\right)^{(p-1)/2} \pmod{p^2}.\end{aligned}$$

Note that

$$\left(\frac{4}{3}\right)^{(p-1)/2} = 3^{(p-1)/2}\left(\frac{2}{3}\right)^{p-1} \equiv 3^{(p-1)/2}(1+p\,q_p(2)-p\,q_p(3)) \pmod{p^2}.$$

Then, by (2.1) and Lemma 2.3 we immediately obtain

$$\varepsilon \equiv (-3)^{(p-1)/2}(1+p-p\,q_p(3)) \pmod{p^2}. \tag{2.8}$$

Now we evaluate $f(p)$ modulo p^2. Obviously, $f_{(p-1)/2}(p)=0$ since

$$\left(-\left\lfloor\frac{p}{4}\right\rfloor\right)_{(p-1)/2}=0.$$

Taking $n=\lfloor p/4\rfloor$ and $\delta=(1-(\frac{-1}{p}))/2$ in Lemma 2.2 and using (2.1), Lemmas 2.1 and 2.3 we obtain

$$\begin{aligned}f(p) &= f(p)+f_{(p-1)/2}(p) = \left(\frac{3}{4}\right)^{(p-1)/2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} = -\frac{3^{(p-1)/2}}{2^{p-1}}\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{p+1}{2})}{\Gamma_p(\frac{p}{2})}\\ &\equiv -3^{(p-1)/2}\Gamma_p\left(\frac{1}{2}\right)^2\left(1+\frac{p}{2}H_{(p-1)/2}\right)(1-p\,q_p(2))\\ &\equiv (-3)^{(p-1)/2}(1-2p\,q_p(2)) \pmod{p^2}.\end{aligned} \tag{2.9}$$

Finally, we consider $f(3p)$ modulo p^2. Taking $n=\lfloor 3p/4\rfloor$ and $\delta=(1+(\frac{-1}{p}))/2$ in Lemma 2.2 we have

$$f(3p)+f_{(p-1)/2}(3p) = \left(\frac{3}{4}\right)^{(3p-1)/2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3p+1}{2})}{\Gamma(\frac{3p}{2})}. \tag{2.10}$$

Similarly, as in (2.9), we deduce that

$$\begin{aligned}\left(\frac{3}{4}\right)^{(3p-1)/2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3p+1}{2})}{\Gamma(\frac{3p}{2})} &= \left(\frac{3}{4}\right)^{(3p-1)/2}\frac{p\Gamma_p(\frac{1}{2})\Gamma_p(\frac{3p+1}{2})}{\frac{p}{2}\Gamma_p(\frac{3p}{2})\Gamma_p(1)}\\ &\equiv \frac{3}{2}(-3)^{(p-1)/2}(1+p\,q_p(3)-6p\,q_p(2)) \pmod{p^2}.\end{aligned} \tag{2.11}$$

Observe that

$$\begin{aligned}f_{(p-1)/2}(3p) &= \frac{(5p+1)\Gamma(\frac{1}{4}-\frac{p}{4})\Gamma(-\frac{1}{4}-\frac{p}{4})\Gamma(1)\Gamma(1-3p)}{4p\Gamma(\frac{3}{4}-\frac{3p}{4})\Gamma(\frac{1}{4}-\frac{3p}{4})\Gamma(\frac{1}{2}+\frac{p}{2})\Gamma(\frac{1}{2}-\frac{5p}{2})}\left(\frac{4}{3}\right)^{(p-1)/2}\\ &= \frac{-\frac{p}{2}(5p+1)\Gamma_p(\frac{1}{4}-\frac{p}{4})\Gamma_p(-\frac{1}{4}-\frac{p}{4})\Gamma_p(1)\Gamma_p(1-3p)}{4p\Gamma_p(\frac{3}{4}-\frac{3p}{4})\Gamma_p(\frac{1}{4}-\frac{3p}{4})\Gamma_p(\frac{1}{2}+\frac{p}{2})\Gamma_p(\frac{1}{2}-\frac{5p}{2})}\left(\frac{4}{3}\right)^{(p-1)/2},\end{aligned}$$

where we note that

$$\frac{\Gamma(\frac{1}{4}-\frac{p}{4})\Gamma(-\frac{1}{4}-\frac{p}{4})}{\Gamma(\frac{3}{4}-\frac{3p}{4})\Gamma(\frac{1}{4}-\frac{3p}{4})}=-\frac{p\Gamma_p(\frac{1}{4}-\frac{p}{4})\Gamma_p(-\frac{1}{4}-\frac{p}{4})}{2\Gamma_p(\frac{3}{4}-\frac{3p}{4})\Gamma_p(\frac{1}{4}-\frac{3p}{4})}.$$

Furthermore, in view of (2.1), Lemmas 2.1 and 2.3, we arrive at

$$f_{(p-1)/2}(3p)\equiv\frac{(-3)^{(p-1)/2}}{2}(1+4p-6p\,q_p(2)-p\,q_p(3)) \pmod{p^2}. \tag{2.12}$$

Now combining (2.7)–(2.12) we finally obtain

$$\sum_{k=0}^{p-1}\frac{2k(\frac{1}{4})_k(\frac{3}{4})_k}{(2k+1)(1)_k^2}\left(\frac{4}{3}\right)^k\equiv 0 \pmod{p^2}.$$

This completes the proof. □

3. Proof of Theorem 1.2

Lemma 3.1. *For any nonnegative integer n we have*

$${}_2F_1\left[\begin{matrix}-n & \frac{1}{3}-n\\ & -3n\end{matrix}\,\middle|\,\frac{9}{8}\right]_n=\frac{(\frac{1}{2})_n}{2^n(\frac{1}{3})_n} \tag{3.1}$$

and

$${}_2F_1\left[\begin{matrix}-n & -\frac{1}{3}-n\\ & -3n-1\end{matrix}\,\middle|\,\frac{9}{8}\right]_n=\frac{(\frac{5}{6})_n}{2^n(\frac{2}{3})_n}. \tag{3.2}$$

Proof. Denote the left-hand sides of (3.1) and (3.2) by $F(n)$ and $G(n)$ respectively. By applying the Zeilberger algorithm in `Mathematica`, we find that

$$-3(2n+1)F(n)+4(3n+1)F(n+1)=0$$

and

$$-(6n+5)G(n)+4(3n+2)G(n+1)=0.$$

Then Lemma 3.1 follows by induction on n. □

The next lemma is a p-adic analogue of the classical Gauss multiplication formula.

Lemma 3.2 *(Robert [18, p. 371]). Let p be an odd prime. Then, for any $x\in\mathbb{Z}_p$ and $m\in\mathbb{Z}^+$ we have*

$$\prod_{0\le j<m}\Gamma_p\left(x+\frac{j}{m}\right)=\Gamma_p(mx)\prod_{0\le j<m}\Gamma_p\left(\frac{j}{m}\right)m^{((1-p)mx+\langle -mx\rangle_p)/p}. \tag{3.3}$$

Proof of Theorem 1.2. Assume that $p\equiv 1 \pmod 3$. Clearly, for any $\alpha,t\in\mathbb{Z}_p$ we have

$$(\alpha+tp)_k\equiv(\alpha)_k\left(1+tp\sum_{j=0}^{k-1}\frac{1}{\alpha+j}\right) \pmod{p^2}. \tag{3.4}$$

Therefore, it is easy to verify that

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}\frac{1-p}{3} & \frac{2-p}{3}\\ & 1-p\end{matrix}\middle| \frac{9}{8}\right]_{p-1}\\
&\quad\equiv \sum_{k=0}^{p-1}\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k\left(1-p\sum_{j=0}^{k-1}\frac{1}{3j+1}-p\sum_{j=0}^{k-1}\frac{1}{3j+2}+pH_k\right)\pmod{p^2}
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}\frac{2-2p}{3} & \frac{1-2p}{3}\\ & 1-2p\end{matrix}\middle| \frac{9}{8}\right]_{p-1}\\
&\quad\equiv \sum_{k=0}^{p-1}\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k\left(1-2p\sum_{j=0}^{k-1}\frac{1}{3j+1}-2p\sum_{j=0}^{k-1}\frac{1}{3j+2}+2pH_k\right)\pmod{p^2}.
\end{aligned}
$$

Combining the above two congruences we arrive at

$$
\sum_{k=0}^{p-1}\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k\equiv 2\,{}_2F_1\left[\begin{matrix}\frac{1-p}{3} & \frac{2-p}{3}\\ & 1-p\end{matrix}\middle| \frac{9}{8}\right]_{p-1}-{}_2F_1\left[\begin{matrix}\frac{2-2p}{3} & \frac{1-2p}{3}\\ & 1-2p\end{matrix}\middle| \frac{9}{8}\right]_{p-1}\pmod{p^2}.
$$

Moreover, letting $n=(p-1)/3$ in (3.1) and $n=(2p-2)/3$ in (3.2) we obtain

$$
\sum_{k=0}^{p-1}\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k\equiv\sigma_1-\sigma_2\pmod{p^2},\tag{3.5}
$$

where

$$
\sigma_1:=2^{(4-p)/3}\frac{(\frac12)_{(p-1)/3}}{(\frac13)_{(p-1)/3}}\quad\text{and}\quad\sigma_2:=2^{(2-2p)/3}\frac{(\frac56)_{(2p-2)/3}}{(\frac23)_{(2p-2)/3}}.
$$

We first compute σ_1 modulo p^2. Note that

$$
\frac{(\frac12)_{(p-1)/3}}{(1)_{(p-1)/3}}=\frac{\binom{(2p-2)/3}{(p-1)/3}}{4^{(p-1)/3}}=4^{(1-p)/3}\frac{\Gamma(\frac13+\frac{2p}{3})}{\Gamma(\frac23+\frac p3)^2}.\tag{3.6}
$$

Therefore, by Lemmas 2.1 and 2.3 we have

$$
\begin{aligned}
\sigma_1&=2^{(4-p)/3}\frac{(\frac12)_{(p-1)/3}(1)_{(p-1)/3}}{(1)_{(p-1)/3}(\frac13)_{(p-1)/3}}\\
&\equiv\frac{\Gamma_p(\frac13+\frac{2p}{3})\Gamma_p(\frac13)}{\Gamma_p(\frac23+\frac p3)\Gamma_p(\frac p3)}(2-2p\,q_p(2))\\
&\equiv\frac{\Gamma_p(\frac13)^2}{\Gamma_p(\frac23)}(2-2p\,q_p(2))\left(1+\frac{2p}{3}H_{(2p-2)/3}-\frac p3H_{(p-1)/3}-\frac p3H_{p-1}\right)\\
&\equiv\Gamma_p\left(\frac13\right)^3(p\,q_p(3)+2p\,q_p(2)-2)\pmod{p^2},
\end{aligned}\tag{3.7}
$$

where in the last step we note that $\Gamma_p(1/3)\Gamma_p(2/3)=(-1)^{1+(2p-2)/3}=-1$ by (2.1) and $H_{p-1-k}\equiv H_k\pmod p$ for any $k\in\{0,1,\dots,p-1\}$.

We now evaluate σ_2 modulo p^2 in a similar way. By (3.6), Lemmas 2.1 and 2.3 we have

$$\begin{aligned}\sigma_2 &= 2^{(2-2p)/3}\frac{\Gamma(\frac{1}{6}+\frac{2p}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{6})\Gamma(\frac{2p}{3})} = \frac{\Gamma(\frac{2}{3}+\frac{p}{3})^2\Gamma(\frac{1}{6}+\frac{2p}{3})\Gamma(\frac{2}{3})\Gamma(\frac{1}{6}+\frac{p}{3})\Gamma(1)}{\Gamma(\frac{1}{3}+\frac{2p}{3})\Gamma(\frac{2p}{3})\Gamma(\frac{5}{6})\Gamma(\frac{1}{2})\Gamma(\frac{2}{3}+\frac{p}{3})}\\
&\equiv \frac{\Gamma_p(\frac{2}{3}+\frac{p}{3})\Gamma_p(\frac{1}{6}+\frac{2p}{3})\Gamma_p(\frac{2}{3})\Gamma_p(\frac{1}{6}+\frac{p}{3})}{\Gamma_p(\frac{1}{3}+\frac{2p}{3})\Gamma_p(\frac{2p}{3})\Gamma_p(\frac{5}{6})\Gamma_p(\frac{1}{2})}\\
&\equiv \frac{\Gamma_p(\frac{2}{3})^2\Gamma_p(\frac{1}{6})^2}{\Gamma_p(\frac{1}{3})\Gamma_p(\frac{5}{6})\Gamma_p(\frac{1}{2})}\left(1-\frac{p}{3}H_{(p-1)/3}+pH_{(p-1)/6}\right)\\
&\equiv \frac{\Gamma_p(\frac{2}{3})^2\Gamma_p(\frac{1}{6})^2}{\Gamma_p(\frac{1}{3})\Gamma_p(\frac{5}{6})\Gamma_p(\frac{1}{2})}(1-2p\,q_p(2)-p\,q_p(3)) \pmod{p^2}.\end{aligned}$$

In view of Lemma 3.2, we arrive at

$$\frac{\Gamma_p(\frac{2}{3})^2\Gamma_p(\frac{1}{6})^2}{\Gamma_p(\frac{1}{3})\Gamma_p(\frac{5}{6})\Gamma_p(\frac{1}{2})} = \frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{3})^2}{\Gamma_p(\frac{1}{3})\Gamma_p(\frac{5}{6})} = \frac{\Gamma_p(\frac{1}{3})^2}{\Gamma_p(\frac{2}{3})} = -\Gamma_p\left(\frac{1}{3}\right)^3.$$

So we have

$$\sigma_2 \equiv -\Gamma_p\left(\frac{1}{3}\right)^3(1-2p\,q_p(2)-p\,q_p(3)) \pmod{p^2}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.5) we obtain

$$\sum_{k=0}^{p-1}\frac{(\frac{1}{3})_k(\frac{2}{3})_k}{(1)_k^2}\left(\frac{9}{8}\right)^k \equiv -\Gamma_p\left(\frac{1}{3}\right)^3 \pmod{p^2}.$$

Thus it suffices to show that

$$\binom{(2p-2)/3}{(p-1)/3} \equiv -\Gamma_p\left(\frac{1}{3}\right)^3 \pmod{p^2}.$$

In fact, it is routine to verify that

$$\begin{aligned}\binom{(2p-2)/3}{(p-1)/3} = -\frac{\Gamma_p(\frac{1}{3}+\frac{2p}{3})}{\Gamma_p(\frac{2}{3}+\frac{p}{3})^2} &\equiv -\Gamma_p\left(\frac{1}{3}\right)^3\left(1+\frac{2p}{3}\left(H_{(2p-2)/3}-H_{(p-1)/3}\right)\right)\\
&\equiv -\Gamma_p\left(\frac{1}{3}\right)^3 \pmod{p^2}.\end{aligned}$$

This proves Theorem 1.2 in the case $p\equiv 1 \pmod 3$.

Below we handle the case $p\equiv 2 \pmod 3$ in a similar way. It is easy to see that

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\frac{1-2p}{3} & \frac{2-2p}{3}\\ & 1-2p\end{matrix}\middle|\ \frac{9}{8}\right]_{p-1}\\
&\equiv \sum_{k=0}^{p-1}\frac{(\frac{1}{3})_k(\frac{2}{3})_k}{(1)_k^2}\left(\frac{9}{8}\right)^k\left(1-2p\sum_{j=0}^{k-1}\frac{1}{3j+1}-2p\sum_{j=0}^{k-1}\frac{1}{3j+2}+2pH_k\right) \pmod{p^2}\end{aligned}$$

and

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}\frac{2-p}{3} & \frac{1-p}{3} \\ & 1-p\end{matrix}\middle|\ \frac{9}{8}\right]_{p-1} \\
&\equiv \sum_{k=0}^{p-1}\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k\left(1-p\sum_{j=0}^{k-1}\frac{1}{3j+1}-p\sum_{j=0}^{k-1}\frac{1}{3j+2}+pH_k\right) \pmod{p^2}.
\end{aligned}
$$

Combining the above two congruences and putting $n=(2p-1)/3$ in (3.1) and $n=(p-2)/3$ in (3.2) we deduce that

$$
\sum_{k=0}^{p-1}\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k \equiv \sigma_3-\sigma_4 \pmod{p^2}, \tag{3.9}
$$

where

$$
\sigma_3 := 2^{(5-p)/3}\frac{(\frac56)_{(p-2)/3}}{(\frac23)_{(p-2)/3}} \quad \text{and} \quad \sigma_4 := 2^{(1-2p)/3}\frac{(\frac12)_{(2p-1)/3}}{(\frac13)_{(2p-1)/3}}.
$$

We first consider σ_3 modulo p^2. Note that

$$
\frac{(\frac12)_{(2p-1)/3}}{(1)_{(2p-1)/3}} = \frac{\binom{(4p-2)/3}{(2p-1)/3}}{4^{(2p-1)/3}} = 4^{(1-2p)/3}\frac{\Gamma(\frac{4p+1}{3})}{\Gamma(\frac{2p+2}{3})^2}.
$$

Therefore, in view of (2.2) we have

$$
\begin{aligned}
\sigma_3 &= 4\times 2^{p-1}\times 4^{(1-2p)/3}\frac{\Gamma(\frac16+\frac p3)\Gamma(\frac23)}{\Gamma(\frac56)\Gamma(\frac p3)} = 4\times 2^{p-1}\frac{\Gamma(\frac16+\frac{2p}{3})\Gamma(\frac23+\frac{2p}{3})\Gamma(\frac16+\frac p3)\Gamma(\frac23)}{\Gamma(\frac12)\Gamma(\frac13+\frac{4p}{3})\Gamma(\frac56)\Gamma(\frac p3)} \\
&\equiv \frac{p\Gamma_p(\frac16)^2\Gamma_p(\frac23)^2}{3\Gamma_p(\frac12)\Gamma_p(\frac13)\Gamma_p(\frac56)} \pmod{p^2},
\end{aligned}
$$

where we note that $\Gamma(1/6+2p/3)/\Gamma(1/2), \Gamma(2/3+2p/3)/\Gamma(1/3+4p/3)$ and $\Gamma(1/6+p/3)/\Gamma(5/6)$ contain factors $p/2, 1/p$ and $p/6$, respectively. With the help of Lemma 3.2 and noting that $\Gamma_p(1/3)\Gamma_p(2/3) = (-1)^{2(p+1)/3}=1$, we obtain

$$
\frac{\Gamma_p(\frac16)^2\Gamma_p(\frac23)^2}{\Gamma_p(\frac12)\Gamma_p(\frac13)\Gamma_p(\frac56)} = \frac{\Gamma_p(\frac13)^2}{\Gamma_p(\frac23)} = \Gamma_p\left(\frac13\right)^3.
$$

Thus we get

$$
\sigma_3 \equiv \frac p3\Gamma_p\left(\frac13\right)^3 \pmod{p^2}. \tag{3.10}
$$

We now consider σ_4. Clearly,

$$
\begin{aligned}
\sigma_4 &= 2^{1-2p}\binom{(4p-2)/3}{(2p-1)/3}\frac{(1)_{(2p-1)/3}}{(\frac13)_{(2p-1)/3}} = 2^{1-2p}\frac{\Gamma(\frac13+\frac{4p}{3})\Gamma(\frac13)}{\Gamma(\frac23+\frac{2p}{3})\Gamma(\frac{2p}{3})} \\
&= 2^{1-2p}\frac{p\Gamma_p(\frac13+\frac{4p}{3})\Gamma_p(\frac13)}{3\Gamma_p(\frac23+\frac{2p}{3})\Gamma_p(\frac{2p}{3})} \equiv \frac{p\Gamma_p(\frac13)^2}{6\Gamma_p(\frac23)} = \frac p6\Gamma_p\left(\frac13\right)^3 \pmod{p^2}.
\end{aligned} \tag{3.11}
$$

Substituting (3.10) and (3.11) into (3.9) we have

$$\sum_{k=0}^{p-1}\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k\equiv\frac{p}{6}\Gamma_p\left(\frac13\right)^3 \pmod{p^2}. \tag{3.12}$$

On the other hand,

$$\frac{p}{\binom{(2p+2)/3}{(p+1)/3}}=-\frac{p\Gamma_p(\frac43+\frac{p}{3})^2}{\Gamma_p(\frac53+\frac{2p}{3})}\equiv\frac{p}{6}\Gamma_p\left(\frac13\right)^3 \pmod{p^2}.$$

This, together with (3.12), proves Theorem 1.2 in the case $p\equiv 2 \pmod 3$.

The proof of Theorem 1.2 is now complete. □

4. Proof of Theorem 1.3

Lemma 4.1. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n}(3n+k+2)\frac{(-n)_k(\frac13-n)_k}{(1)_k(-3n)_k}\left(\frac98\right)^k=3\times 2^{4/3-n}\frac{\Gamma(\frac23)\Gamma(\frac76+n)}{\Gamma(\frac12)\Gamma(\frac13+n)} \tag{4.1}$$

and

$$\sum_{k=0}^{n}(3n+k+3)\frac{(-n)_k(-\frac13-n)_k}{(1)_k(-3n-1)_k}\left(\frac98\right)^k=3\times 2^{1-n}\frac{\Gamma(\frac23)\Gamma(\frac32+n)}{\Gamma(\frac12)\Gamma(\frac23+n)}. \tag{4.2}$$

Proof. Denote the left-hand sides of (4.1) and (4.2) by $J(n)$ and $K(n)$ respectively. Via the Zeilberger algorithm, we find that

$$(6n+7)J(n)-4(3n+1)J(n+1)=0$$

and

$$3(2n+3)K(n)-4(3n+2)K(n+1)=0.$$

Then the identities can be verified by induction on n. □

The following lemma is the well-known Gauss multiplication formula whose p-adic analogue has been stated in Lemma 3.2.

Lemma 4.2 *(Robert [18, p. 371]). For any $z\in\mathbb{C}$ and $m\in\{2,3,\ldots\}$, we have*

$$\prod_{0\le j<m}\Gamma\left(z+\frac{j}{m}\right)=(2\pi)^{(m-1)/2}m^{(1-2mz)/2}\Gamma(mz). \tag{4.3}$$

Proof of Theorem 1.3. Suppose that $p\equiv 1 \pmod 3$. As in the proof of Theorem 1.2, by Lemma 4.1 we can easily prove

$$\sum_{k=0}^{p-1}(k+1)\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k\equiv\sigma_5-\sigma_6 \pmod{p^2}, \tag{4.4}$$

where

$$\sigma_5 := 6 \times 2^{(5-p)/3}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{2p+5}{6})}{\Gamma(\frac{1}{2})\Gamma(\frac{p}{3})} \quad \text{and} \quad \sigma_6 := 3 \times 2^{(5-2p)/3}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{4p+5}{6})}{\Gamma(\frac{1}{2})\Gamma(\frac{2p}{3})}.$$

Note that

$$-\frac{\Gamma_p(\frac{1}{6}+\frac{p}{3})}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{2}{3}+\frac{p}{3})} = \frac{(\frac{1}{2})_{(p-1)/3}}{(1)_{(p-1)/3}} = \frac{\binom{(2p-2)/3}{(p-1)/3}}{4^{(p-1)/3}} = -2^{(2-2p)/3}\frac{\Gamma_p(\frac{1}{3}+\frac{2p}{3})}{\Gamma_p(\frac{2}{3}+\frac{p}{3})^2}.$$

Therefore, by Lemmas 2.1, 3.2 and 4.2 we obtain

$$\begin{aligned}\sigma_5 &= -2^{(11-2p)/3}\frac{\Gamma(-\frac{1}{3})\Gamma(\frac{2p+5}{6})}{\Gamma(\frac{p}{6})\Gamma(\frac{p+3}{6})} = -\frac{2^{(8-2p)/3}p}{3}\frac{\Gamma_p(-\frac{1}{3})\Gamma_p(\frac{5}{6}+\frac{p}{3})}{\Gamma_p(\frac{p}{6})\Gamma_p(\frac{1}{2}+\frac{p}{6})} \\ &\equiv -\frac{4p\Gamma_p(\frac{1}{6})\Gamma_p(\frac{2}{3})^2\Gamma_p(\frac{5}{6})}{\Gamma_p(\frac{1}{2})^2\Gamma_p(\frac{1}{3})} = -\frac{4p}{\Gamma_p\left(\frac{1}{3}\right)^3} \pmod{p^2}.\end{aligned} \tag{4.5}$$

Also,

$$\begin{aligned}\sigma_6 &= -\,2^{(5-2p)/3}\frac{\Gamma(\frac{-1}{3})\Gamma(\frac{5}{6}+\frac{2p}{3})}{\Gamma(\frac{1}{2})\Gamma(\frac{2p}{3})} = -\frac{2^{(2-2p)/3}p\Gamma_p(\frac{-1}{3})\Gamma_p(\frac{5}{6}+\frac{2p}{3})}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{2p}{3})} \\ &\equiv -\,\frac{3p\Gamma_p(\frac{1}{6})\Gamma_p(\frac{2}{3})^2\Gamma_p(\frac{5}{6})}{\Gamma_p(\frac{1}{2})^2\Gamma_{(}\frac{1}{3})} = -\frac{3p}{\Gamma_p(\frac{1}{3})^3} \pmod{p^2}.\end{aligned} \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) we get

$$\sum_{k=0}^{p-1}(k+1)\frac{(\frac{1}{3})_k(\frac{2}{3})_k}{(1)_k^2}\left(\frac{9}{8}\right)^k \equiv -\frac{p}{\Gamma_p(\frac{1}{3})^3}.$$

On the other hand, from the proof of Theorem 1.2 we know

$$\binom{(2p-2)/3}{(p-1)/3} \equiv -\Gamma_p\left(\frac{1}{3}\right)^3 \pmod{p^2}.$$

This proves Theorem 1.3 in the case $p \equiv 1 \pmod 3$.

Now we assume $p \equiv 2 \pmod 3$. By Lemma 4.1 it is easy to verify that

$$\sum_{k=0}^{p-1}(k+1)\frac{(\frac{1}{3})_k(\frac{2}{3})_k}{(1)_k^2}\left(\frac{9}{8}\right)^k \equiv \sigma_7 - \sigma_8 \pmod{p^2}, \tag{4.7}$$

where

$$\sigma_7 := 12 \times 2^{(2-p)/3}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{2p+5}{6})}{\Gamma(\frac{1}{2})\Gamma(\frac{p}{3})} \quad \text{and} \quad \sigma_8 := 3 \times 2^{(5-2p)/3}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{4p+5}{6})}{\Gamma(\frac{1}{2})\Gamma(\frac{2p}{3})}.$$

Note that

$$-\frac{\Gamma_p(-\frac{1}{6}+\frac{p}{3})}{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{3}+\frac{p}{3})} = \frac{(\frac{1}{2})_{(p-2)/3}}{(1)_{(p-2)/3}} = \frac{\binom{(2p-4)/3}{(p-2)/3}}{4^{(p-2)/3}} = -2^{(4-2p)/3}\frac{\Gamma_p(-\frac{1}{3}+\frac{2p}{3})}{\Gamma_p(\frac{1}{3}+\frac{p}{3})^2}.$$

By Lemmas 2.1 and 2.3 we have

$$
\begin{aligned}
\sigma_7 &= -4\times 2^{(2-p)/3}\frac{\Gamma_p(-\frac13)\Gamma_p(\frac56+\frac p3)}{\Gamma_p(\frac p3)\Gamma_p(\frac12)}\\
&\equiv -8\times 2^{1-p}\frac{\Gamma_p(-\frac13+\frac{2p}{3})\Gamma_p(-\frac13)\Gamma_p(\frac56+\frac p3)}{\Gamma_p(-\frac16+\frac p3)\Gamma_p(\frac13+\frac p3)\Gamma_p(\frac p3)}\\
&\equiv -12\times 2^{1-p}\frac{\Gamma_p(\frac23)^2\Gamma_p(\frac56)}{\Gamma_p(\frac56)\Gamma_p(\frac13)}\left(1+\frac p3 H_{(p-2)/3}\right)\\
&\equiv -\frac{12}{\Gamma_p(\frac13)^3}\left(1-p\,q_p(2)-\frac p2 q_p(3)\right)\pmod{p^2}.
\end{aligned}
\tag{4.8}
$$

Also,

$$
\begin{aligned}
\sigma_8 &= -2^{(5-4p)/3}\frac{\Gamma(-\frac13)\Gamma(\frac56+\frac{2p}{3})}{\Gamma(\frac p3)\Gamma(\frac12+\frac p3)} = -3\times 2^{(5-4p)/3}\frac{\Gamma_p(\frac23)\Gamma_p(\frac56+\frac{2p}{3})}{\Gamma_p(\frac p3)\Gamma_p(\frac12+\frac p3)}\\
&\equiv -\frac{3\Gamma_p(-\frac16+\frac p3)^2\Gamma_p(\frac13+\frac p3)^2\Gamma_p(\frac23)\Gamma_p(\frac56+\frac{2p}{3})}{2\Gamma_p(\frac12)^2\Gamma_p(-\frac13+\frac{2p}{3})^2\Gamma_p(\frac p3)\Gamma_p(\frac12+\frac p3)}\\
&\equiv -\frac{6}{\Gamma_p(\frac13)^3}\left(1-2p\,q_p(2)-p\,q_p(3)\right)\pmod{p^2}.
\end{aligned}
\tag{4.9}
$$

Combining (4.7)–(4.9) we have

$$
\sum_{k=0}^{p-1}(k+1)\frac{(\frac13)_k(\frac23)_k}{(1)_k^2}\left(\frac98\right)^k \equiv -\frac{6}{\Gamma_p(\frac13)^3}\pmod{p^2}.
$$

On the other hand, it is routine to verify that

$$
\binom{(2p+2)/3}{(p+1)/3}\equiv 6(1-p)\frac{\Gamma_p(\frac23+\frac{2p}{3})}{\Gamma_p(\frac13+\frac p3)^2}\equiv \frac{6(1-p)}{\Gamma_p(\frac13)^3}\pmod{p^2}.
$$

Comparing the above two congruences we immediately obtain the desired result.

The proof of Theorem 1.3 is now complete. □

5. More results similar to Theorems 1.1–1.3

In this section, we list some congruences that can also be proved by some strange identities. However, we only give the outlines of their proofs since the proofs are quite similar to the ones of Theorems 1.1–1.3.

Sun [25, Conjecture 5.14(i)] posed the following conjecture as a variant of the third congruence in (1.1).

Conjecture 5.1. *Let $p>3$ be a prime. If $p\equiv 1 \pmod 3$ and $p=x^2+3y^2$ $(x,y\in\mathbb{Z})$ with $x\equiv 1\pmod 3$, then we have*

$$
\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{48^k}\equiv 2x-\frac{p}{2x}\pmod{p^2}
\tag{5.1}
$$

and

$$
\sum_{k=0}^{p-1}\frac{k+1}{48^k}\binom{2k}{k}\binom{4k}{2k}\equiv x\pmod{p^2}.
\tag{5.2}
$$

If $p \equiv 2 \pmod 3$, *then we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{48^k} \equiv \frac{3p}{2\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}. \tag{5.3}$$

(5.1) has been proved by G.-S. Mao and H. Pan [12] and its proof depends on the results concerning Legendre polynomials obtained by M.J. Coster and L. Van Hamme [4]. In fact, we can confirm Conjecture 5.1 completely by using some strange hypergeometric identities.

Theorem 5.1. *For any prime* $p > 3$ *we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{48^k} \equiv \begin{cases} \binom{(p-1)/2}{(p-1)/6}\left(1 + \frac{2p}{3}q_p(2) - \frac{3p}{4}q_p(3)\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ 3p/\left(2\binom{(p+1)/2}{(p+1)/6}\right) \pmod{p^2} & \text{if } p \equiv 2 \pmod 3, \end{cases} \tag{5.4}$$

and

$$\begin{aligned} &\sum_{k=0}^{p-1} \frac{(2k+1)\binom{2k}{k}\binom{4k}{2k}}{48^k} \\ &\equiv \begin{cases} p/\binom{(p-1)/2}{(p-1)/6} \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ \binom{(p+1)/2}{(p+1)/6}\left(-\frac{2}{3} - \frac{2p}{3} - \frac{4p}{9}q_p(2) + \frac{p}{2}q_p(3)\right) \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases} \end{aligned} \tag{5.5}$$

Remark 5.1. It is known (cf. [2, p. 283]) that for any prime $p = x^2 + 3y^2 \equiv 1 \pmod 6$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$ we have

$$\binom{(p-1)/2}{(p-1)/6} \equiv \left(2x - \frac{p}{2x}\right)\left(1 - \frac{2p}{3}q_p(2) + \frac{3p}{4}q_p(3)\right) \pmod{p^2}.$$

This, together with (5.4), gives (5.1) and (5.2).

Applying (1.10) with $\alpha = 1/4$ and $z = 4/3$ and noting that $(1/4)_k(3/4)_k/(1)_k^2 = \binom{4k}{2k}\binom{2k}{k}/64^k$ we arrive at

$$\left(\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{48^k}\right)^2 \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{k}}{(-144)^k} \pmod{p^2}. \tag{5.6}$$

Therefore, combining (5.6) with Theorem 5.1 we can easily obtain the following result which was conjectured and partially proved by Z.-H. Sun (cf. [20, Conjecture 2.2] and [22, Theorem 5.1]).

Corollary 5.1. *Let* $p > 3$ *be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-144)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \ \&\ p = x^2 + 3y^2\ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

To show Theorem 5.1, we need the following identities which can be easily verified by Zeilberger's algorithm.

Lemma 5.1. *Let n be a nonnegative integer. Then we have the following identities:*

$$\sum_{k=0}^{n} \frac{(-n)_k(\frac{1}{2}-n)_k}{(1)_k(-4n)_k}\left(\frac{4}{3}\right)^k = \left(\frac{9}{16}\right)^n \frac{(\frac{2}{3})_{2n}}{(\frac{1}{2})_{2n}}, \tag{5.7}$$

$$\sum_{k=0}^{n} \frac{(-n)_k(-\frac{1}{2}-n)_k}{(1)_k(-4n-2)_k}\left(\frac{4}{3}\right)^k = \left(\frac{3}{4}\right)^{2n+1} \frac{(\frac{2}{3})_{2n+1}}{(\frac{1}{2})_{2n+1}}, \tag{5.8}$$

$$\sum_{k=0}^{n} \frac{(2n+k+1)(-n)_k(\frac{1}{2}-n)_k}{(1)_k(-4n)_k}\left(\frac{4}{3}\right)^k = \frac{3^{2n+1}(\frac{1}{3})_{2n}}{16^n(\frac{1}{2})_{2n}}, \tag{5.9}$$

$$\sum_{k=0}^{n} \frac{(2n+k+2)(-n)_k(-\frac{1}{2}-n)_k}{(1)_k(-4n-2)_k}\left(\frac{4}{3}\right)^k = \frac{9^{n+1}(\frac{1}{3})_{2n+2}}{4^{2n+1}(\frac{1}{2})_{2n+1}}. \tag{5.10}$$

Proof of Theorem 5.1. We should divide the proof into four cases that $p \equiv 1, 5, 7, 11 \pmod{12}$. We only prove (5.4) for $p \equiv 1 \pmod{12}$ briefly since (5.4) in the other cases can be handled similarly.

By Lemma 5.1, it is easy to verify that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{48^k} \equiv \frac{3}{2}\left(\frac{9}{16}\right)^{(p-1)/4} \frac{(\frac{2}{3})_{(p-1)/2}}{(\frac{1}{2})_{(p-1)/2}} - \frac{1}{2}\left(\frac{3}{4}\right)^{(3p-1)/2} \frac{(\frac{2}{3})_{(3p-1)/2}}{(\frac{1}{2})_{(3p-1)/2}} \pmod{p^2}.$$

From [13, Lemma 4.1] we know that for any positive integer n and integer a not divisible by p we have

$$a^{(p-1)/2} \equiv \left(\frac{a}{p}\right)\sum_{k=0}^{n-1}\binom{\frac{1}{2}}{k}(p\,q_p(a))^k \pmod{p^n}. \tag{5.11}$$

Thus we have

$$\begin{aligned}\frac{3}{2}\left(\frac{3}{4}\right)^{(p-1)/2} \frac{(\frac{2}{3})_{(p-1)/2}}{(\frac{1}{2})_{(p-1)/2}} &\equiv \frac{3}{2}\left(1+\frac{1}{2}p\,q_p(3) - p\,q_p(2)\right)\frac{\Gamma_p(\frac{1}{6}+\frac{p}{2})\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{2}{3})\Gamma_p(\frac{p}{2})}\\ &\equiv -\frac{3}{2}\left(1-2p\,q_p(2)-\frac{p}{4}q_p(3)\right)\Gamma_p\left(\frac{1}{3}\right)^3 \pmod{p^2}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\left(\frac{3}{4}\right)^{(3p-1)/2} \frac{(\frac{2}{3})_{(3p-1)/2}}{(\frac{1}{2})_{(3p-1)/2}} &\equiv \frac{3}{8}\left(1+\frac{3p}{2}q_p(3) - 3p\,q_p(2)\right)\frac{\frac{2p}{3}\Gamma_p(\frac{1}{6}+\frac{3p}{2})\Gamma_p(\frac{1}{2})}{\frac{p}{2}\Gamma_p(\frac{2}{3})\Gamma_p(\frac{3p}{2})}\\ &\equiv -\frac{1}{2}\left(1-6p\,q_p(2)-\frac{3p}{4}q_p(3)\right)\Gamma_p\left(\frac{1}{3}\right)^3 \pmod{p^2}.\end{aligned}$$

Then we obtain

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{48^k} \equiv -\Gamma_p\left(\frac{1}{3}\right)^3 \pmod{p^2}.$$

On the other hand, it is routine to verify that

$$\binom{(p-1)/2}{(p-1)/6}\left(1+\frac{2p}{3}q_p(2)-\frac{3p}{4}q_p(3)\right) \equiv -\Gamma_p\left(\frac{1}{3}\right)^3 \pmod{p^2}.$$

This completes the proof. $\square$

Sun [25, Conjecture 5.14(iii)] made the following conjecture.

Conjecture 5.2. *For any prime $p>3$, if $p\equiv 1 \pmod 4$ and $p=x^2+4y^2$ $(x,y\in\mathbb{Z})$ with $x\equiv 1 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{72^k}\equiv\left(\frac{6}{p}\right)\left(2x-\frac{p}{2x}\right) \pmod{p^2} \tag{5.12}$$

and

$$\sum_{k=0}^{p-1}\frac{(1-k)\binom{2k}{k}\binom{4k}{2k}}{72^k}\equiv\left(\frac{6}{p}\right)x \pmod{p^2}; \tag{5.13}$$

if $p\equiv 3 \pmod 4$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{72^k}\equiv\left(\frac{6}{p}\right)\frac{2p}{3\binom{(p+1)/2}{(p+1)/4}} \pmod{p^2}. \tag{5.14}$$

We shall prove these congruences by establish the following result.

Theorem 5.2. *For any prime $p>3$ we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{72^k}\equiv\begin{cases}(\frac{6}{p})\binom{(p-1)/2}{(p-1)/4}\left(1-\frac{p}{2}q_p(2)\right) \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ 2p(\frac{6}{p})/\left(3\binom{(p+1)/2}{(p+1)/4}\right) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases} \tag{5.15}$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{(2k-1)\binom{2k}{k}\binom{4k}{2k}}{72^k}\\ &\quad\equiv\begin{cases}-p(\frac{6}{p})/\binom{(p-1)/2}{(p-1)/4} \pmod{p^2} & \text{if } p\equiv 1 \pmod 4,\\ (\frac{6}{p})\binom{(p+1)/2}{(p+1)/4}\left(\frac{3}{2}+\frac{3p}{2}-\frac{3p}{4}q_p(2)\right) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{5.16}$$

Remark 5.2. From [2, p. 281] we know for any prime $p>3$, if $p\equiv 1 \pmod 4$ and $p=x^2+4y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 1 \pmod 4$ we have

$$\binom{(p-1)/2}{(p-1)/4}\equiv\left(2x-\frac{1}{2x}\right)\left(1+\frac{p}{2}q_p(2)\right) \pmod{p^2}.$$

Combining this with Theorem 5.2 we obtain (5.12) and (5.13).

Applying (1.10) with $\alpha=1/4$ and $z=8/9$ we arrive at

$$\left(\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{72^k}\right)^2\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{k}}{648^k} \pmod{p^2}. \tag{5.17}$$

Therefore, combining (5.17) with Theorem 5.2 we can easily obtain the following result which was conjectured and partially proved by Z.-H. Sun (cf. [20, Conjecture 2.1] and [22, Theorem 5.1]).

Corollary 5.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{648^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 4 \text{ \& } p = x^2 + 4y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

To prove Theorem 5.2 we need the following preliminary result which can also be showed by Zeilberger's algorithm.

Lemma 5.2. *Let n be a nonnegative integer. Then*

$$\sum_{k=0}^{n} \frac{(-n)_k(\frac{1}{2}-n)_k}{(1)_k(-4n)_k} \left(\frac{8}{9}\right)^k = \frac{\Gamma(\frac{1}{2})\Gamma(3n)}{3^{2n-1}\Gamma(2n+\frac{1}{2})\Gamma(n)}, \tag{5.18}$$

$$\sum_{k=0}^{n} \frac{(-n)_k(-\frac{1}{2}-n)_k}{(1)_k(-4n-2)_k} \left(\frac{8}{9}\right)^k = \frac{3^{n+1}\Gamma(n+\frac{5}{6})\Gamma(n+\frac{7}{6})}{2\Gamma(\frac{1}{2})\Gamma(2n+\frac{3}{2})}, \tag{5.19}$$

$$\sum_{k=0}^{n} \frac{(10n-k+3)(-n)_k(\frac{1}{2}-n)_k}{(1)_k(-4n)_k} \left(\frac{8}{9}\right)^k = \frac{3^{n+2}\Gamma(n+\frac{5}{6})\Gamma(n+\frac{7}{6})}{\Gamma(\frac{1}{2})\Gamma(2n+\frac{1}{2})}, \tag{5.20}$$

$$\sum_{k=0}^{n} \frac{(10n-k+8)(-n)_k(-\frac{1}{2}-n)_k}{(1)_k(-4n-2)_k} \left(\frac{8}{9}\right)^k = \frac{2\Gamma(\frac{1}{2})\Gamma(3n+3)}{9^n\Gamma(2n+\frac{3}{2})\Gamma(n+1)}. \tag{5.21}$$

Proof of Theorem 5.2. We only prove (5.15) for $p \equiv 1 \pmod 4$ since (5.15) in the case $p \equiv 3 \pmod 4$ can be deduced in a similar way.

By Lemma 5.2 we can easily verify that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{72^k} \equiv \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3p-3}{4})}{2 \times 3^{(p-1)/2}\Gamma(\frac{p}{2})\Gamma(\frac{p-1}{4})} - \frac{3^{(3p+1)/4}\Gamma(\frac{9p+1}{12})\Gamma(\frac{9p+5}{12})}{4\Gamma(\frac{1}{2})\Gamma(\frac{3p}{2})} \pmod{p^2}.$$

Clearly, by Lemma 2.3 and (5.11) we have

$$\begin{aligned} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3p-3}{4})}{2 \times 3^{(p-1)/2}\Gamma(\frac{p}{2})\Gamma(\frac{p-1}{4})} &= \frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{3p-3}{4})}{2 \times 3^{(p-1)/2}\Gamma_p(\frac{p}{2})\Gamma_p(\frac{p-1}{4})} \\ &\equiv \left(\frac{3}{p}\right)\left(\frac{1}{2} - \frac{3p}{4}q_p(3)\right)\left(1 + \frac{3p}{4}H_{(p+3)/4} - \frac{p}{4}H_{(p-5)/4}\right)\frac{\Gamma_p(\frac{1}{2})\Gamma_p(-\frac{3}{4})}{\Gamma_p(-\frac{1}{4})} \\ &\equiv (-1)^{(p+3)/4}\left(\frac{3}{p}\right)\left(\frac{1}{2} - \frac{p}{4}q_p(3) - \frac{3p}{4}q_p(2)\right)\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2 \pmod{p^2}. \end{aligned}$$

Moreover, by Lemma 4.2 we have

$$\begin{aligned} \frac{3^{(3p+1)/4}\Gamma(\frac{9p+1}{12})\Gamma(\frac{9p+5}{12})\Gamma(\frac{3p+3}{4})}{4\Gamma(\frac{1}{2})\Gamma(\frac{3p}{2})\Gamma(\frac{3p+3}{4})} &= \frac{3^{(2-6p)/4}\Gamma(\frac{1}{2})\Gamma(\frac{1}{4}+\frac{9p}{4})}{2\Gamma(\frac{3p}{2})\Gamma(\frac{3}{4}+\frac{3p}{4})} = \frac{3^{(2-6p)/4} \times \frac{3p}{2}\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4}+\frac{9p}{4})}{2 \times \frac{p}{2}\Gamma_p(\frac{3p}{2})\Gamma_p(\frac{3}{4}+\frac{3p}{4})} \\ &\equiv (-1)^{(p+3)/4}\left(\frac{3}{p}\right)\left(\frac{1}{2} - \frac{3p}{4}q_p(3)\right)\left(1 + \frac{3p}{2}H_{(p-1)/4}\right)\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2 \end{aligned}$$

$$\equiv (-1)^{(p+3)/4}\left(\frac{3}{p}\right)\left(\frac{3}{2}-\frac{3p}{4}q_p(3)-\frac{9p}{4}q_p(2)\right)\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2 \pmod{p^2}.$$

Thus we obtain

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{72^k}\equiv\left(\frac{6}{p}\right)\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2 \pmod{p^2},$$

since

$$\left(\frac{2}{p}\right)\left(\frac{3}{p}\right)=\left(\frac{6}{p}\right) \text{ and } \left(\frac{2}{p}\right)=(-1)^{(p-1)^2/8-(p^2-1)/8}=(-1)^{(p-1)/4}.$$

On the other hand, one may easily verify that

$$\binom{(p-1)/2}{(p-1)/4}\left(1-\frac{p}{2}q_p(2)\right)\equiv\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)^2 \pmod{p^2}.$$

This completes the proof. □

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