Journal of Mathematical Analysis and Applications **259,** 582–595 (2001)
doi:10.1006/jmaa.2001.7432, available online at http://www.idealibrary.com on IDEAL®

Extension of a Theorem of Ferenc Lukács from Single to Double Conjugate Series[1]

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Submitted by Daniel Waterman

Received November 9, 1999

A theorem of Ferenc Lukács states that if a periodic function f is integrable in the Lebesgue sense and has a discontinuity of the first kind at some point x, then the mth partial sum of the conjugate series of its Fourier series diverges at x at the rate of $\log m$. The aim of the present paper is to extend this theorem to the rectangular partial sum of the conjugate series of a double Fourier series when conjugation is taken with respect to both variables. We also consider functions of two variables which are of bounded variation over a rectangle in the sense of Hardy and Krause. As a corollary, we obtain that the terms of the Fourier series of a periodic function f of bounded variation over the square $[-\pi, \pi] \times [-\pi, \pi]$ determine the atoms of the finite Borel measure induced by f.

Key Words: Fourier series; conjugate series; rectangular partial sum; rate of divergence; function of bounded variation over a rectangle in the sense of Hardy and Krause; sector limits of a function in two variables; induced Borel measure; criterion of nonatomic measure.

1. CONJUGATE SERIES ON T

Given a periodic function $f \in L^1(\mathsf{T})$, $\mathsf{T} = [-\pi, \pi)$, with Fourier series

$$\sum_{j \in \mathsf{Z}} \hat{f}(j) e^{ijx}, \qquad \hat{f}(j) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-iju}\, du,$$

[1] This research was completed while the author visited the Department of Mathematics at the University of Tennessee, Knoxville, in the academic year 1999–2000; it was also partially supported by the Hungarian National Foundation for Scientific Research under Grant T 029094.

0022-247X/01 $35.00

its conjugate series is defined by

$$\sum_{j \in \mathbb{Z}} (-i \operatorname{sign} j) \hat{f}(j) e^{ijx}, \tag{1.1}$$

where

$$\operatorname{sign} j := \begin{cases} 1 & \text{if } j > 0, \\ 0 & \text{if } j = 0, \\ -1 & \text{if } j < 0. \end{cases}$$

It is well known that the symmetric partial sum of series (1.1) can be represented in the form

$$\tilde{s}_m(f; u) := \sum_{|j| \le m} (-i \operatorname{sign} j) \hat{f}(j) e^{ijx} = \frac{1}{\pi} \int_0^\pi \psi_x(f; u) \tilde{D}_m(u)\, du,$$

where

$$\psi_x(f; u) := f(x-u) - f(x+u)$$

and

$$\tilde{D}_m(u) := \sum_{j=1}^{m} \sin ju = \frac{\cos(u/2) - \cos(m + 1/2)u}{2\sin(u/2)}, \qquad m = 1, 2, \ldots, \tag{1.2}$$

is the conjugate Dirichlet kernel.

The theorem of Ferenc Lukács[2] reads as follows (see [3; 5, Vol. 1, p. 60]).

THEOREM 1.1. *Let $f \in L^1(\mathbb{T})$ and $x \in \mathbb{T}$. If there exists a number $d_x(f)$ such that*

$$\lim_{h \to 0+} \frac{1}{h} \int_0^h |\psi_x(f; u) - d_x(f)|\, du = 0,$$

then

$$\lim_{m \to \infty} \frac{\tilde{s}_m(f; x)}{\log m} = \frac{d_x(f)}{\pi}. \tag{1.3}$$

By log we mean the natural logarithm.

[2] Ferenc Lukács (1891–1918), a very capable Hungarian mathematician, was an Associate Professor at the Technical University of Budapest.

The following corollary was also deduced by Ferenc Lukács.

COROLLARY 1.2. *If $f \in L^1(\mathsf{T})$ and the finite limit*

$$\lim_{u \to 0+} \psi_x(f; u) =: d_x(f)$$

exists at some point $x \in \mathsf{T}$, then we have (1.3).

In particular, Corollary 1.2 applies at any point $x \in \mathsf{T}$ if f is of bounded variation on the interval $[-\pi, \pi]$ (by periodicity, $f(\pi) = f(-\pi)$) with

$$d_x(f) := f(x - 0) - f(x + 0).$$

The aim of the present paper is to extend these results from single to double conjugate series.

2. DOUBLE CONJUGATE SERIES ON T^2

Given a periodic function $f \in L^1(\mathsf{T}^2)$ (in each variable) with Fourier series

$$\sum_{(j,k) \in \mathsf{Z}^2} \hat{f}(j, k) e^{i(jx+ky)}, \tag{2.1}$$

where

$$\hat{f}(j, k) := \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(u, v) e^{-i(ju+kv)}\, du\, dv,$$

its double conjugate series with respect to both variables is defined by

$$\sum_{(j,k) \in \mathsf{Z}^2} (-i \operatorname{sign} j)(-i \operatorname{sign} k) \hat{f}(j, k) e^{i(jx+ky)}. \tag{2.2}$$

The following representation of the symmetric rectangular partial sum of series (2.2) is also well known,

$$\begin{aligned} \tilde{s}_{mn}(f; x, y) &:= \sum_{|j| \le m} \sum_{|k| \le n} (-i \operatorname{sign} j)(-i \operatorname{sign} k) \hat{f}(j, k) e^{i(jx+ky)} \\ &= \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \psi_{xy}(f; u, v) \tilde{D}_m(u) \tilde{D}_n(v)\, du\, dv, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \psi_{xy}(f; u, v) := f(x - u, y - v) - f(x + u, y - v) - f(x - u, y + v) \\ + f(x + u, y + v). \end{aligned}$$

Our main result reads as follows.

THEOREM 2.1. *Let $f \in L^1(\mathsf{T}^2)$ and $(x, y) \in \mathsf{T}^2$. If there exists a number $d_{xy}(f)$ such that for*

(2.4)
$$\Psi(h,k) = \Psi_{xy}(f;h,k) := \int_0^h \int_0^k |\psi_{xy}(f;u,v) - d_{xy}(f)|\, du\, dv$$

we have

$$\lim_{h,k \to 0+} \Psi(h,k)/hk = 0 \tag{2.5}$$

and

$$\Psi(h,k) \le C \min\{h,k\}, \qquad 0 < h,k \le \pi, \tag{2.6}$$

where C is a constant, then

$$\lim_{m,n \to \infty} \frac{\tilde{s}_{mn}(f;x,y)}{(\log m)(\log n)} = \frac{d_{xy}(f)}{\pi^2}. \tag{2.7}$$

A few remarks about conditions (2.5) and (2.6) are appropriate here. It is well known (see, e.g., [5, Vol. 2, p. 306]) that if $f \in L^1 \log^+ L(\mathsf{T}^2)$, then its double integral (in the Lebesgue sense) is strongly differentiable at almost every point; and that the integrability condition imposed on f is best possible. Hence it follows in a routine way that almost every point is a Lebesgue point of f in the strong sense. Consequently condition (2.5) is satisfied with $d_{xy}(f) := 0$ at almost every point $(x, y) \in \mathsf{T}^2$. On the other hand, if the finite limit

$$\lim_{u,v \to 0+} \psi_{x,y}(f;u,v) =: d_{xy}(f) \tag{2.8}$$

exists at some point $(x, y) \in \mathsf{T}^2$, then condition (2.5) is satisfied trivially at that point (cf. Definition (2.4)).

As to condition (2.6), consider, for example, the case when $0 < h \le k \le \pi$. It is easy to see that

$$\frac{\Psi(h,k)}{h} \le \frac{1}{h} \int_0^h \int_{-\pi}^{\pi} |f(x-u,v) - f(x+u,v)|\, du\, dv + \pi |d_{xy}(f)|.$$

So, condition (2.6) is surely satisfied in this case if

$$\int_{-\pi}^{\pi} |f(x-u,v) - f(x+u,v)|\, dv \le C, \qquad 0 < u \le \pi. \tag{2.6$'$}$$

Condition (2.6′) is clearly satisfied for all $(x, y) \in \mathsf{T}^2$ if

$$\int_{-\pi}^{\pi} |f(u,v)|\,dv \le C, \qquad u \in \mathsf{T}. \tag{2.6″}$$

This is always the case if the function f is bounded.

The case when $0 < k \le h \le \pi$ can be treated in an analogous way. In particular, the symmetric counterpart of (2.6″) is the requirement that

$$\int_{-\pi}^{\pi} |f(u,v)|\,du \le C, \qquad v \in \mathsf{T}. \tag{2.6‴}$$

We formulate explicitly only the following corollary of Theorem 2.1.

COROLLARY 2.2. *Let $f \in L^1(\mathsf{T}^2)$ be such that conditions* (2.6″) *and* (2.6‴) *are satisfied. If the finite limit* (2.8) *exists at some point $(x, y) \in \mathsf{T}^2$, then we have* (2.7).

3. PROOF OF THEOREM 2.1

The following known limit relation plays an important role in our proof:

$$\lim_{m\to\infty} \frac{l_m}{\log m} = 1, \quad \text{where } l_m := \int_0^{\pi} \tilde{D}_m(u)\,du. \tag{3.1}$$

Note that Definition (1.2) can be rewritten in the form

$$\tilde{D}_m(u) = \frac{1 - \cos mu}{u} + (1 - \cos mu)\left(\frac{1}{2\tan(u/2)} - \frac{1}{u}\right) + \frac{1}{2}\sin mu.$$

The second and third terms on the right are bounded for $0 < u \le \pi$, while

$$\int_0^{\pi} \frac{1 - \cos mu}{u}\,du = \int_0^{m\pi} \frac{1 - \cos u}{u}\,du$$
$$= C + \int_1^{m\pi} \frac{du}{u} - \int_1^{m\pi} \frac{\cos u}{u}\,du,$$

where C is a constant. Hence it follows that

$$\lim_{m\to\infty} \frac{1}{\log m} \int_0^{\pi} \frac{1 - \cos mu}{u}\,du$$
$$= \lim_{m\to\infty} \frac{1}{\log m}\left(\log m\pi - \sum_{j=1}^{m-1} \int_{j\pi}^{(j+1)\pi} \frac{\cos u}{u}\,du\right) = 1.$$

Relying on (2.3) and (3.1), in order to prove (2.7) it is enough to prove that

$$\lim_{m,n\to\infty} \frac{1}{l_m l_n} \int_0^\pi \int_0^\pi \left[\psi_{xy}(f;u,v) - d_{xy}(f) \right] \tilde{D}_m(u) \tilde{D}_n(v)\, du\, dv = 0. \tag{3.2}$$

We shall actually prove (3.2) in its stronger form, where the integrand is replaced by its absolute value. In the interest of brevity, we introduce the notation

$$P(u,v) = P_{xy}(f;u,v) := |\psi_{xy}(f;u,v) - d_{xy}(f)|.$$

To begin with, by (2.5), given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Psi(h,k) \le \varepsilon hk, \qquad 0 < h,k \le \delta. \tag{3.3}$$

We also recall the following elementary estimate of the conjugate Dirichlet kernel:

$$|\tilde{D}_m(u)| \le \min\{m, \pi/u\}, \qquad m \ge 1 \quad \text{and} \quad 0 < u \le \pi. \tag{3.4}$$

Now, we consider integers $m,n > 1/\delta$, where δ is from (3.3), and decompose the double integral in (3.2) (while putting the integrand between absolute value bars) as

$$\begin{aligned} &\left\{ \int_0^{1/m} + \int_{1/m}^{\delta} + \int_{\delta}^{\pi} \right\} \\ &\qquad \times \left\{ \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right\} P(u,v) |\tilde{D}_m(u) \tilde{D}_n(v)|\, du\, dv \\ &=: I_{11} + I_{12} + I_{13} + I_{21} + \cdots + I_{33}, \quad \text{say.} \end{aligned} \tag{3.5}$$

By (2.4), (3.3), and (3.4), we have

$$I_{11} \le mn\Psi(1/m, 1/n) \le \varepsilon. \tag{3.6}$$

Again by (2.4), (3.3), and (3.4), Fubini's theorem and integration by parts yield

$$\begin{aligned} I_{21} &\le \pi n \int_{1/m}^{\delta} \left(\int_0^{1/n} P(u,v)\, dv \right) \frac{du}{u} \\ &= \pi n \left[\frac{1}{u} \int_0^u \left(\int_0^{1/n} P(u_1,v)\, dv \right) du_1 \right]_{u=1/m}^{\delta} \\ &\quad + \pi n \int_{1/m}^{\delta} \left(\int_0^u \left(\int_0^{1/n} P(u_1,v)\, dv \right) du_1 \right) \frac{du}{u^2} \\ &\le \frac{\pi n}{\delta} \Psi\left(\delta, \frac{1}{n} \right) + \pi n \int_{1/m}^{\delta} \frac{\Psi(u, 1/n)}{u^2}\, du \\ &\le \pi\varepsilon(1 + \log m\delta); \end{aligned} \tag{3.7}$$

and analogously,

$$I_{12} \le \pi\varepsilon(1 + \log n\delta). \tag{3.8}$$

By (3.4), we have

$$I_{22} \le \pi^2 \int_{1/m}^{\delta} \left(\int_{1/n}^{\delta} P(u,v) \frac{dv}{v} \right) \frac{du}{u}. \tag{3.9}$$

In order to estimate the right-hand side, we need two preliminary estimates: integration by parts with respect to v gives

$$\int_{1/n}^{\delta} P(u,v_1) \frac{dv}{v} \le \frac{1}{\delta} \int_0^{\delta} P(u,v_1)\,dv_1 + \int_{1/n}^{\delta} \left(\int_0^{v} P(u,v_1)\,dv_1 \right) \frac{dv}{v^2}, \tag{3.10}$$

while integration by parts with respect to u gives

$$\begin{aligned} \int_{1/m}^{\delta} P(u,v_1) \frac{du}{u} \le{}& \frac{1}{\delta} \int_0^{\delta} P(u_1,v_1)\,du_1 \\ &+ \int_{1/m}^{\delta} \left(\int_0^{u} P(u_1,v_1)\,du_1 \right) \frac{du}{u^2}. \end{aligned} \tag{3.11}$$

By Fubini's theorem, (3.9), and (3.10), we obtain

$$\begin{aligned} \pi^{-2} I_{22} &\le \int_{1/m}^{\delta} \left(\frac{1}{\delta} \int_0^{\delta} P(u,v_1)\,dv_1 \right) \frac{du}{u} \\ &\quad + \int_{1/m}^{\delta} \left(\int_{1/n}^{\delta} \left(\int_0^{v} P(u,v_1)\,dv_1 \right) \frac{dv}{v^2} \right) \frac{du}{u} \\ &= \frac{1}{\delta} \int_0^{\delta} \left(\int_{1/m}^{\delta} P(u,v_1) \frac{du}{u} \right) dv_1 \\ &\quad + \int_{1/n}^{\delta} \left(\int_0^{v} \left(\int_{1/m}^{\delta} P(u,v_1) \frac{du}{u} \right) dv_1 \right) \frac{dv}{v^2}, \end{aligned}$$

whence, making use of (3.11), we obtain

$$\begin{aligned} \pi^{-2} I_{22} \le{}& \frac{1}{\delta} \int_0^{\delta} \left(\frac{1}{\delta} \int_0^{\delta} P(u_1,v_1)\,du_1 \right) dv_1 \\ &+ \frac{1}{\delta} \int_0^{\delta} \left(\int_{1/m}^{\delta} \left(\int_0^{u} P(u_1,v_1)\,du_1 \right) \frac{du}{u^2} \right) dv_1 \\ &+ \int_{1/n}^{\delta} \left(\int_0^{v} \left(\frac{1}{\delta} \int_0^{\delta} P(u_1,v_1)\,du_1 \right) dv_1 \right) \frac{dv}{v^2} \\ &+ \int_{1/n}^{\delta} \left(\int_0^{v} \left(\int_{1/m}^{\delta} \left(\int_0^{u} P(u_1,v_1)\,du_1 \right) \frac{du}{u^2} \right) dv_1 \right) \frac{dv}{v^2}, \end{aligned}$$

whence, again by Fubini's theorem, (2.4), and (3.3), we obtain

$$\begin{aligned}(3.12)\quad \pi^{-2} I_{22} &\le \frac{1}{\delta^2}\Psi(\delta,\delta) + \frac{1}{\delta}\int_{1/m}^{\delta} \frac{\Psi(u,\delta)}{u^2}\,du \\ &\quad + \frac{1}{\delta}\int_{1/n}^{\delta}\frac{\Psi(\delta,v)}{v^2}\,dv + \int_{1/m}^{\delta}\int_{1/n}^{\delta}\frac{\Psi(u,v)}{u^2 v^2}\,du\,dv \\ &\le \varepsilon(1 + \log m\delta + \log n\delta + (\log m\delta)(\log n\delta)).\end{aligned}$$

By (2.4), (2.6), and (3.4), we have

$$(3.13)\quad I_{31} \le \frac{\pi n}{\delta}\int_{\delta}^{\pi}\int_0^{1/n} P(u,v)\,du\,dv \le \frac{\pi n}{\delta}\Psi\left(\pi,\frac{1}{n}\right) \le \frac{\pi C}{\delta};$$

and analogously,

$$(3.14)\qquad I_{13} \le \frac{\pi C}{\delta}.$$

By Fubini's theorem and integration by parts, while making use of (2.4), (2.6), and (3.4), we find that

$$\begin{aligned}(3.15)\qquad I_{32} &\le \frac{\pi^2}{\delta}\int_{1/n}^{\delta}\left(\int_{\delta}^{\pi} P(u,v)\,du\right)\frac{dv}{v} \\ &= \frac{\pi^2}{\delta}\left[\frac{1}{v}\int_0^{v}\left(\int_{\delta}^{\pi} P(u,v_1)\,du\right)dv_1\right]_{v=1/n}^{\delta} \\ &\quad + \frac{\pi^2}{\delta}\int_{1/n}^{\delta}\left(\int_0^{v}\left(\int_{\delta}^{\pi} P(u,v_1)\,du\right)dv_1\right)\frac{dv}{v^2} \\ &\le \frac{\pi^2}{\delta^2}\Psi(\pi,\delta) + \frac{\pi^2}{\delta}\int_{1/n}^{\delta}\frac{\Psi(\pi,v)}{v^2}\,dv \\ &\le \frac{\pi^2 C}{\delta}(1 + \log n\delta);\end{aligned}$$

and analogously,

$$(3.16)\qquad I_{23} \le \frac{\pi^2 C}{\delta}(1 + \log m\delta).$$

Finally, by (2.4), (2.6), and (3.4), we have

$$(3.17)\quad I_{33} \le \pi^2\int_{\delta}^{\pi}\int_{\delta}^{\pi} P(u,v)\frac{du}{u}\frac{dv}{v} \le \pi^2\Psi(\pi,\pi)/\delta^2 \le \pi^3 C/\delta^2.$$

Combining (3.2), (3.5)–(3.8), (3.12)–(3.17), we find that

$$\begin{aligned}
&\frac{1}{l_m l_n}\int_0^\pi\int_0^\pi P(u,v)|\tilde{D}_m(u)\tilde{D}_n(v)|\,du\,dv\\
&\quad\le \frac{1}{l_m l_n}\Bigg\{(1+\pi)^2\varepsilon+\left(2+2\pi+\frac{\pi^2}{\delta}\right)\frac{\pi C}{\delta}\\
&\qquad\qquad +\pi\left(\varepsilon+\pi\varepsilon+\frac{\pi C}{\delta}\right)\\
&\qquad\qquad \times(\log m\delta+\log n\delta)+\pi^2\varepsilon(\log m\delta)(\log n\delta)\Bigg\}\\
&\quad\le(\pi^2+1)\varepsilon,
\end{aligned}$$

provided m and n are large enough. Taking into account (3.1) this proves (2.7).

4. FUNCTIONS OF BOUNDED VARIATION IN TWO VARIABLES

We begin by recalling the definition. A function f defined on a bounded rectangle $\mathsf{R}=[a,b]\times[c,d]$ is said to be of bounded variation over R in the sense of Hardy and Krause (see [1] and see also the discussion in [2, Sect. 254]) if the following three quantities are finite,

$$\begin{aligned}
V_{11} &:= \sup_{\mathscr{P}_1\times\mathscr{P}_2}\sum_{j=1}^m\sum_{k=1}^n|f(x_j,y_k)-f(x_{j-1},y_k)\\
&\qquad -f(x_j,y_{k-1})+f(x_{j-1},y_{k-1})|,\\
V_{10} &:= \sup_{\mathscr{P}_1}\sum_{j=1}^m|f(x_j,c)-f(x_{j-1},c)|,
\end{aligned}$$

and

$$V_{01} := \sup_{\mathscr{P}_2}\sum_{k=1}^n|f(a,y_k)-f(a,y_{k-1})|,$$

where

$$\mathscr{P}_1 : a=x_0<x_1<x_2<\cdots<x_m=b$$

and

$$\mathscr{P}_2 : c=y_0<y_1<y_2<\cdots<y_n=d$$

are arbitrary finite partitions. The sum

$$V(f) = V(f; \mathsf{R}) := V_{11} + V_{10} + V_{01}$$

is called the total variation of f over R.

It is easy to see that if f is of bounded variation over R, then not only are the marginal functions $f(\cdot, c)$ and $f(a, \cdot)$ of bounded variation in the usual sense over the intervals $[a, b]$ and $[c, d]$, respectively, but so are the functions $f(\cdot, y)$ where $y \in [c, d]$ is fixed and $f(x, \cdot)$ where $x \in [a, b]$ is fixed. It is easy to check that the total variation of each of these functions $f(\cdot, y)$ over $[a, b]$ or $f(x, \cdot)$ over $[c, d]$ does not exceed $V(f; \mathsf{R})$.

The following characterization of bounded variation is due to Hardy [1].

THEOREM 4.1. *A necessary and sufficient condition that a function $f(x, y)$ be of bounded variation in the sense of Hardy and Krause over a bounded rectangle $\mathsf{R} := [a, b] \times [c, d]$ is that it can be represented in the form $f = f_1 - f_2$, where both f_1 and f_2 are bounded and nondecreasing functions in each variable on R and such that*

$$f_i(x, y) - f_i(\bar{x}, y) - f_i(x, \bar{y}) + f_i(\bar{x}, \bar{y}) \geq 0, \qquad i = 1, 2,$$

for all $a \leq x \leq \bar{x} \leq b$ and $c \leq y \leq \bar{y} \leq d$.

The functions f_1 and f_2 occurring in Theorem 4.1 are called "monotonely monotone" by W. H. Young and G. C. Young [4]. They belong to the class of the "quasi-monotone" functions as defined by Hobson [2, Sect. 255].

The next theorem (see, for example, [2, Sect. 307]) guarantees the existence of the so-called "sector" (or "quadrant") limits of a quasi-monotone function.

THEOREM 4.2. *If f is a quasi-monotone function on a bounded rectangle $\mathsf{R} := [a, b] \times [c, d]$, then the sector limits*

$$f(x_0 - 0, y_0 - 0) := \lim_{x \to x_0 -,\, y \to y_0 -} f(x, y),$$

$$f(x_0 + 0, y_0 - 0) := \lim_{x \to x_0 +,\, y \to y_0 -} f(x, y),$$

$$f(x_0 - 0, y_0 + 0) := \lim_{x \to x_0 -,\, y \to y_0 +} f(x, y),$$

$$f(x_0 + 0, y_0 + 0) := \lim_{x \to x_0 +,\, y \to y_0 +} f(x, y)$$

exist at each point $(x_0, y_0) \in \mathsf{R}$ at which the left and/or right approaches are available inside R.

Combining Theorems 4.1 and 4.2 yields that if f is of bounded variation in the sense of Hardy and Krause over R, then each of the four sector limits exists at the points of R where the left and/or right approaches are available inside R.

After these preliminaries, Corollary 2.2 and Theorems 4.1 and 4.2 give the following.

COROLLARY 4.3. *If a function $f(x, y)$ is periodic in each variable and of bounded variation in the sense of Hardy and Krause over the square $[-\pi, \pi] \times [-\pi, \pi]$, then we have* (2.7) *at each point $(x, y) \in \mathsf{T}^2$ with*

$$d_{xy}(f) := f(x-0, y-0) - f(x+0, y-0) - f(x-0, y+0) + f(x+0, y+0).$$

The values of f at the points of the upper horizontal and the right vertical edges of the square $[-\pi, \pi] \times [-\pi, \pi]$ are, by periodicity,

$$f(x, \pi) = f(x, -\pi), \qquad f(\pi, y) = f(-\pi, y) \quad \text{for } -\pi \le x, y < \pi,$$

and

$$f(\pi, \pi) = f(-\pi, -\pi)(= f(-\pi, \pi) = f(\pi, -\pi)).$$

Given an arbitrary function f defined on a bounded rectangle $\mathsf{R} := [a, b] \times [c, d]$, we can associate with it a so-called rectangle function F as follows. If $\mathsf{R}_1 := (a_1, b_1) \times (c_1, d_1)$ is an open subrectangle of R, then set

$$F(\mathsf{R}_1) := f(b_1 - 0, d_1 - 0) - f(a_1 + 0, d_1 + 0) - f(b_1 - 0, c_1 + 0) + f(a_1 + 0, c_1 + 0).$$

It is clear that the Borel measure induced by F on R is finite if and only if $V_{11} < \infty$.

We may also introduce marginal interval functions as follows: If $I := (a_1, b_1)$ is an open subinterval of $[a, b]$ and $J := (c_1, d_1)$ is an open subinterval of $[c, d]$, then set

$$F(I; c) := f(b_1 - 0, c) - f(a_1 + 0, c) \quad \text{and}$$
$$F(a; J) := f(a, d_1 - 0) - f(a, c_1 + 0).$$

It is clear again that the Borel measures induced by $F(\cdot; c)$ on $[a, b]$ and by $F(a; \cdot)$ on $[c, d]$ are finite if and only if $V_{10} < \infty$ and $V_{01} < \infty$, respectively. The interval functions $F(\cdot; y)$, where $y \in [c, d]$ is fixed and $F(x; \cdot)$, where $x \in [a, b]$ is fixed, can be defined analogously.

Obviously, the Borel measures induced by F on R and by $F(\cdot; y)$ and $F(x; \cdot)$ on $[a, b]$ and $[c, d]$, respectively, are all finite if and only if f is of bounded variation over R in the sense of Hardy and Krause.

Now, Corollary 4.3 says that the terms of the Fourier series of a periodic function f of bounded variation in the sense of Hardy and Krause determine the atoms of the Borel measure induced by f (via its rectangle function F). In particular, the Borel measure induced by f is nonatomic (or, equivalently, is continuous) if and only if the limit in (2.7) equals 0 at each point $(x, y) \in \mathsf{T}^2$. As to the nonatomic property of the Borel measures induced by $f(\cdot, y)$ (via its interval function $F(\cdot; y)$, where y is fixed) and by $f(x, \cdot)$ (via its interval function $F(x; \cdot)$ where x is fixed), respectively, we can draw conclusions by means of Corollary 1.2.

In our last corollary, we give simple sufficient conditions in terms of the Fourier coefficients of a periodic function f of bounded variation in order that the Borel measure induced by f be nonatomic.

COROLLARY 4.4. *If the Fourier coefficients $\hat{f}(j, k)$ of a periodic function f of bounded variation in the sense of Hardy and Krause over the square $[-\pi, \pi] \times [-\pi, \pi]$ satisfy the condition*

$$\lim_{|j|, |k| \to \infty} jk\hat{f}(j, k) = 0, \tag{4.1}$$

then the Borel measure induced by f is nonatomic.

In fact, given any $\varepsilon > 0$, by (4.1) there exists a positive integer j_0 such that

$$|jk\hat{f}(j, k)| \le \varepsilon \qquad \text{if } |j|, |k| > j_0.$$

By this and (2.3), for $m, n > j_0$ we have

$$\begin{aligned}
|\tilde{s}_{mn}(f; x, y)| &\le \sum_{|j|=1}^{m} \sum_{|k|=1}^{n} |\hat{f}(j, k)| \\
&\le \sum_{|j|=1}^{j_0} \sum_{|k|=1}^{j_0} |\hat{f}(j, k)| + \sum_{|j|=j_0+1}^{m} \sum_{|k|=1}^{j_0} \frac{V(f)}{\pi|j|} \\
&\quad + \sum_{|j|=1}^{j_0} \sum_{|k|=j_0+1}^{n} \frac{V(f)}{\pi|k|} + \sum_{|j|=j_0+1}^{m} \sum_{|k|=j_0+1}^{n} \frac{\varepsilon}{|jk|},
\end{aligned} \tag{4.2}$$

where we used the well-known estimate of the Fourier coefficients of a periodic function of bounded variation by its total variation over the interval $[-\pi, \pi]$ (see, e.g., [5, Vol. 1, p. 48]) in the following way:

$$\begin{aligned}
|\hat{f}(j,k)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(u,v) e^{-iju} \, du \right] e^{-ikv} \, dv \right| \\
&\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u,v) e^{-iju} \, du \right| dv \\
&\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{V(f(\cdot, v); [-\pi, \pi])}{\pi |j|} \, dv \\
&\le \frac{V(f)}{\pi |j|}, \quad \text{where } V(f) := V(f; [-\pi, \pi] \times [-\pi, \pi]),
\end{aligned}$$

provided $j \neq 0$. Analogously, the symmetric counterpart

$$|\hat{f}(j,k)| \le \frac{V(f)}{\pi |k|}$$

also holds, provided $k \neq 0$.

It follows from (4.2) that

$$\begin{aligned}
|\tilde{s}_{mn}(f; x, y)| &\le j_0^2 \|f\|_\infty + \frac{4 j_0 V(f)}{\pi} \log m \\
&\quad + \frac{4 j_0 V(f)}{\pi} \log n + 4\varepsilon (\log m)(\log n) \\
&\le 5\varepsilon (\log m)(\log n),
\end{aligned}$$

provided both m and n are large enough. This proves the limit relation

$$\lim_{m,n \to \infty} \frac{\tilde{s}_{mn}(f; x, y)}{(\log m)(\log n)} = 0.$$

Applying Corollary 4.3 gives $d_{xy}(f) = 0$, which is the statement of Corollary 4.4.

ACKNOWLEDGMENT

The author is grateful to Professor Daniel Waterman whose suggestions have led to improvements in the presentation. The present form of Corollary 4.4 is due to him.

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