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Some Existence Theorems for Elliptic Resonant Problems[1]

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Some existence theorems are obtained for the elliptic resonant problems at either the first eigenvalue or the higher eigenvalues with unbounded nonlinearity by using the minimax methods in critical point theory. © 2001 Elsevier Science

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1. INTRODUCTION AND MAIN RESULTS

Consider the Dirichlet boundary value problem

$$\begin{aligned} -\Delta u &= \lambda_k u + g(u) - h(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset R^N$ $(N \geq 1)$ is a bounded smooth domain, λ_k is the kth eigenvalue of the problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, $g \in C(R, R)$, and $h \in L^2(\Omega)$.

Under the condition that

$$\lim_{|t|\to\infty} \frac{g(t)}{t} = 0, \tag{2}$$

the problem (1) is called the elliptic resonant problem at the kth eigenvalue. The solvability of this problem has been studied by many authors.

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When $k = 1$, there are some well-known sufficient conditions, such as the Landesman–Lazer-type condition (see [1–4] and their references), the monotonicity condition (see [5]), the periodicity condition (see [6, 7]), the sign condition (see [8, 9] and their references), and the strong resonant condition (see [10, 11]). When $k > 1$, there are many well-known existence results (see [1, 5, 7, 10–16]). In this case, most of them are under the condition of boundedness for nonlinear terms, that is, $\sup\{|g(t)| \mid t \in R\} < +\infty$. The elliptic resonant problem with unbounded nonlinear terms has been considered in [12–16].

Recently a new Landesman–Lazer-type solvability condition was given for the two-point boundary value problem and the following theorem was obtained in [4].

THEOREM A ([4]). *Suppose that* $g \in C(R, R)$ *satisfies* (2). *Assume that* $h \in L^2(0, \pi)$, *satisfying*

$$\overline{F}(-\infty)\int_0^\pi \sin x\,dx < \int_0^\pi h(x)\sin x\,dx < \underline{F}(+\infty)\int_0^\pi \sin x\,dx,$$

where $\overline{F}(-\infty) = \limsup_{t\to-\infty} F(t)$, $\underline{F}(+\infty) = \liminf_{t\to+\infty} F(t)$ *and*

$$F(t) = \begin{cases} (2/t)\int_0^t g(s)\,ds - g(t) & t \neq 0, \\ g(0) & t = 0. \end{cases} \tag{3}$$

Then the two-point boundary value problem

$$-u'' = u + g(u) - h(x), \qquad u(0) = u(\pi) = 0,$$

has at least one solution.

This result was extended to the quasilinear elliptic resonant problem in [17].

In this paper we first replace the condition in (2) with the weaker one that $g \in C(R, R)$ such that

$$0 \le \liminf_{|t|\to\infty} \frac{g(t)}{t} \le \limsup_{|t|\to\infty} \frac{g(t)}{t} < \lambda_2 - \lambda_1, \tag{4}$$

and we obtain the same result for the semilinear elliptic problem. Then we extend the result in [4] to the semilinear elliptic resonant problem at higher eigenvalues. The main results are the following theorems, which are proved using the Saddle Point theorem.

THEOREM 1. *Suppose that* (4) *holds*. *Assume that* $h \in L^2(\Omega)$, *satisfying*

$$\overline{F}(-\infty)\int_\Omega \psi(x)\,dx < \int_\Omega h(x)\psi(x)\,dx < \underline{F}(+\infty)\int_\Omega \psi(x)\,dx, \tag{5}$$

where ψ *is the normal eigenfunction corresponding to* λ_1, $\psi(x) > 0$ *for all* $x \in \Omega$. *Then problem* (1), *where* $k = 1$, *has at least one solution in the Hilbert space* $H_0^1(\Omega)$.

Remark 1. Theorem 1 generalizes Theorem A in two directions: one is from one-dimension to a higher dimension and the other is with the growth of g; that is, g may grow linearly in our Theorem 1 and not in Theorem A. Besides this, the proof of Theorem A essentially relies on the condition in (2); one cannot prove Theorem 1 with the technique used in the proof of Theorem A.

Remark 2. There are functions g and h satisfying our Theorem 1 and not satisfying the corresponding results in [1–11]. For example (e.g. [4]), let

$$g(t) = \begin{cases} 1 - e^{-t^4|\sin t|}\ln(1+t^4), & t \geq 0, \\ 2e^t - 1, & t \leq 0, \end{cases} \tag{6}$$

and $h = 0$. In fact, on one hand $\overline{F}(-\infty) = -1$, and that $\underline{F}(+\infty) = 1$ follows from the fact that $\int_0^\infty e^{-t^4|\sin t|}\ln(1+t^2)\,dx < +\infty$, which can be checked without difficulty. On the other hand, g is not monotone, not periodic, and not unbounded from both above and below, does not satisfy the sign condition, and does not satisfy the Landesman–Lazer condition, where $\bar{g}(-\infty) = -1$ and $\underline{g}(+\infty) = -\infty$.

THEOREM 2. *Suppose that* $g \in C(R, R)$ *satisfies* (2). *Assume that* $h \in L^2(\Omega)$ *satisfying*

$$\int_\Omega hv\,dx < \underline{F}(+\infty)\int_\Omega v^+\,dx - \overline{F}(-\infty)\int_\Omega v^-\,dx \tag{7}$$

for all $v \in \mathrm{Ker}(\Delta + \lambda_k) \setminus \{0\}$, *where* $v^+(x) = \max\{v(x), 0\}$ *and* $v^- = (-v)^+$. *Then the problem* (1), *where* $k > 1$, *has at least one solution in* $H_0^1(\Omega)$.

Remark 3. There are functions g and h satisfying Theorem 2 and not satisfying the corresponding results in [1, 5, 7, 10–16]. For example, let g be given in (6) and $h = 0$. The reason is the same as that in Remark 2.

THEOREM 3. *Suppose that* $g \in C(R, R)$ *satisfies* (2). *Assume that* $h \in L^2(\Omega)$ *satisfying*

$$\int_\Omega hv\,dx < \underline{F}(-\infty)\int_\Omega v^+\,dx - \overline{F}(+\infty)\int_\Omega v^-\,dx$$

for all $v \in \mathrm{Ker}(\Delta + \lambda_k) \setminus \{0\}$. *Then the problem* (1), *where* $k > 1$, *has at least one solution in* $H_0^1(\Omega)$.

Remark 4. There are functions g and h satisfying Theorem 3 and not satisfying the corresponding results in [1, 5, 7, 10–16]. For example, let

$$g(t) = \begin{cases} e^{-t^4|\sin t|}\ln(1+t^2) - 1 & t \geq 0, \\ 1 - 2e^t & t \leq 0, \end{cases}$$

and $h = 0$. The reason in the same as that in Remark 2.

2. PROOF OF THEOREMS

Define φ on the Sobolev space $H_0^1(\Omega)$ by

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_k \|u\|^2_{L^2(\Omega)} - \int_\Omega G(u)\,dx + \int_\Omega hu\,dx,$$

where $G(t) = \int_0^t g(s)\,ds$ and $\|u\| = (\int_u |\nabla u|^2\,dx)^{1/2}$ is the norm in $H_0^1(\Omega)$. Then φ is continuously differentiable and

$$\langle \varphi'(u), v\rangle = \int_\Omega \nabla u \nabla v\,dx - \lambda_k \int_\Omega uv\,dx - \int_\Omega g(u)v\,dx - \int_\Omega hv\,dx$$

for u, $v \in H_0^1(\Omega)$. It is well-known that $u \in H_0^1(\Omega)$ is a solution of the problem (1) if and only if u is a critical point of φ. By Sobolev's inequality there exists a positive constant C such that

$$\|u\|_{L^1(\Omega)} \le C\|u\|, \qquad \|u\|_{L^2(\Omega)} \le C\|u\| \tag{8}$$

for all $u \in H_0^1(\Omega)$.

In order to prove our result we require the following lemmas.

LEMMA 1. *Assume that* (4) *and* (5) *hold. Then the functional* φ, *where* $k = 1$, *satisfies the* (*PS*) *condition.*

Proof. Suppose that (u_n) is a (PS) sequence of φ in $H_0^1(\Omega)$; that is, $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. We shall prove that (u_n) is bounded by way of contradiction. Assume that (u_n) is unbounded and put $v_n = u_n/\|u_n\|$. Passing to a subsequence if necessary, we may assume that

$$\begin{gathered} \|u_n\| \to \infty, \\ v_n \rightharpoonup v \text{ weakly in } H_0^1(\Omega), \\ v_n \to v \text{ in } L^2(\Omega) \end{gathered} \tag{9}$$

as $n \to \infty$. By (4) there exists a real constant γ satisfying

$$\limsup_{|t|\to+\infty} \frac{g(t)}{t} < \gamma < \lambda_2 - \lambda_1. \tag{10}$$

Moreover, it follows from (4) that for every $\varepsilon > 0$ there exists $M > 0$ such that

$$-\varepsilon \le \frac{g(t)}{t} \le \gamma$$

for all $|t| \ge M$. choose $\eta \in C(R, R)$ such that $0 \le \eta \le 1$, $\eta(t) = 1$ for all $|t| \le M$, and $\eta(t) = 0$ for $|t| \ge 2M$. Set

$$f_n(x) = \begin{cases} (1-\eta(u_n))g(u_n)/u_n & u_n \ne 0, \\ 0 & u_n = 0. \end{cases}$$

Then one has

$$-\varepsilon \leq f_n(x) \leq \gamma$$

for a.e. $x \in \Omega$. Thus without loss of generality we may also assume that

$$f_n \rightharpoonup f \text{ weakly}^* \text{ in } L^\infty(\Omega)$$

as $n \to \infty$, which implies that $f_n \rightharpoonup f$ weakly in $L^2(\Omega)$ as $n \to \infty$. Now the fact that

$$-\varepsilon \leq f(x) \leq \gamma \tag{11}$$

for a.e. $x \in \Omega$ follows from the weak closedness of the convex subset K of $L^2(\Omega)$ given by

$$K = \{s \in L^2(\Omega) \mid -\varepsilon \leq s(x) \leq \gamma \text{ for a.e. } x \in \Omega\}.$$

By Mazur's Theorem (see e.g. Theorem V.1.2 in [19]), we only need to prove that the convex set K is closed, which follows from Theorem 3.1.2 in [18]. Moreover, we have

$$\frac{g(u_n)}{\|u_n\|} \rightharpoonup fv \quad \text{weakly in } L^2(\Omega) \tag{12}$$

as $n \to \infty$. In fact, (12) follows from

$$\frac{(1-\eta(u_n))g(u_n)}{\|u_n\|} = f_n v_n \rightharpoonup fv \quad \text{weakly in } L^2(\Omega)$$

and

$$\frac{\eta(u_n)g(u_n)}{\|u_n\|} \to 0 \quad \text{in } L^2(\Omega),$$

which can be proved by some simple calculations. From (12), the assumption that $\varphi'(u_n) \to 0$ as $n \to \infty$, and the fact that

$$\left\langle \varphi'(u_n), \frac{w}{\|u_n\|} \right\rangle = \int_\Omega \nabla v_n \nabla w \, dx - \lambda_1 \int_\Omega v_n w \, dx - \int_\Omega \frac{g(u_n)}{\|u_n\|} w \, dx + \int_\Omega \frac{h}{\|u_n\|} w \, dx$$

for all n and every $w \in H_0^1(\Omega)$, we obtain

$$\int_\Omega \nabla v \nabla w \, dx - \lambda_1 \int_\Omega vw \, dx = \int_\Omega fvw \, dx$$

for all $w \in H_0^1(\Omega)$. Write $v = a\psi(x) + v_0$, where $a \in R$ and $v_0 \in (\text{span}\{\psi(x)\})^\perp$. It is obvious that $\|v_0\| \geq \sqrt{\lambda_2}\|v_0\|_{L_2}$ for v_0 as above. Then we have

$$a\int_\Omega fv\psi\,dx = 0, \qquad \int_\Omega |\nabla v_0|^2\,dx - \lambda_1\int_\Omega |v_0|^2\,dx = \int_\Omega fvv_0\,dx. \tag{13}$$

Thus it follows from (11) and (13) that

$$\begin{aligned}\gamma\int_\Omega |v_0|^2\,dx &\geq \int_\Omega f|v_0|^2\,dx\\ &= \int_\Omega |\nabla v_0|^2\,dx - \lambda_1\int_\Omega |v_0|^2\,dx - a\int_\Omega fv_0\psi\,dx\\ &= \int_\Omega |\nabla v_0|^2\,dx - \lambda_1\int_\Omega |v_0|^2\,dx + a^2\int_\Omega f|\psi|^2\,dx\\ &\geq (\lambda_2-\lambda_1)\int_\Omega |v_0|^2\,dx - a^2\varepsilon\int_\Omega |\psi|^2\,dx,\end{aligned}$$

which implies that

$$(\lambda_2-\lambda_1-\gamma)\int_\Omega |v_0|^2\,dx \leq a^2\varepsilon\int_\Omega |\psi|^2\,dx.$$

By (10) and the arbitrariness of ε, we have that

$$v_0 = 0. \tag{14}$$

Hence one has

$$v = a\psi(x).$$

From (9), (12)–(14), and that

$$\begin{aligned}\left\langle\varphi'(u_n), \frac{v_n}{\|u_n\|}\right\rangle &= \int_\Omega |\nabla v_n|^2\,dx - \lambda_1\int_\Omega |v_n|^2\,dx\\ &\quad - \int_\Omega \frac{g(u_n)}{\|u_n\|}v_n\,dx + \int_\Omega \frac{h}{\|u_n\|}v_n\,dx\end{aligned}$$

for all n we obtain

$$\int_\Omega |\nabla v_n|^2\,dx \to \lambda_1\int_\Omega |v|^2\,dx + \int_\Omega f|v|^2\,dx = \int_\Omega |\nabla v|^2\,dx$$

as $n \to \infty$, which implies that $v_n \to v \in H_0^1(\Omega)$ by the uniform convexity of $H_0^1(\Omega)$. Noting that $\|v_n\| = 1$ we have $v \neq 0$. Hence $a \neq 0$. Without loss of generality we may assume that $a > 0$. Hence we have

$$\begin{aligned}\left|\int_\Omega v_n^+\,dx - \int_\Omega a\psi\,dx\right| &= \left|\int_\Omega v_n^+\,dx - \int_\Omega a\psi^+\,dx\right| \leq \|v_n^+ - a\psi^+\|_{L^1(\Omega)}\\ &\leq \|v_n - a\psi\|_{L^1(\Omega)} \leq C\|v_n - a\psi\| \to 0\end{aligned}$$

as $n \to \infty$ and

$$\left|\int_\Omega v_n^- \, dx\right| = \left|\int_\Omega v_n^- \, dx - \int_\Omega a\psi^- \, dx\right| \le \|v_n^- - a\psi^-\|_{L^1(\Omega)}$$
$$\le \|v_n - a\psi\|_{L^1(\Omega)} \le C\|v_n - a\psi\| \to 0$$

as $n \to \infty$ by Sobolev's inequality (8), which implies that

$$\lim_{n\to\infty} \int_\Omega v_n^+ \, dx = \int_\Omega a\psi \, dx, \qquad \lim_{n\to\infty} \int_\Omega v_n^- \, dx = 0. \tag{15}$$

It follows from (5) that $\underline{F}(+\infty) > -\infty$ and $\overline{F}(-\infty) < +\infty$. Set

$$C_\varepsilon = \begin{cases} \underline{F}(+\infty) - \varepsilon & \underline{F}(+\infty) < +\infty, \\ 1/\varepsilon & \underline{F}(+\infty) = +\infty, \end{cases}$$

and

$$D_\varepsilon = \begin{cases} \overline{F}(-\infty) + \varepsilon & \overline{F}(-\infty) > -\infty, \\ -1/\varepsilon & \overline{F}(-\infty) = -\infty. \end{cases}$$

Then there exists $M > 0$ such that $F(t) \ge C_\varepsilon$ for $t \ge M$ and $F(t) \le D_\varepsilon$ for $t \le -M$. Thus one has

$$F(t)t \ge \begin{cases} C_\varepsilon t - C_1 & t \ge 0 \\ D_\varepsilon t - C_1 & t \le 0, \end{cases}$$

where $C_1 = (|C_\varepsilon| + |D_\varepsilon|)M + \max_{|t|\le M} |F(t)t|$. Hence we have

$$\int_\Omega F(u_n)v_n \, dx = \int_\Omega F(u_n^+)v_n^+ \, dx + \int_\Omega F(-u_n^-)(-v_n^-) \, dx$$
$$\ge C_\varepsilon \int_\Omega v_n^+ \, dx + D_\varepsilon \int_\Omega (-v_n^-) \, dx - \frac{2C_1}{\|u_n\|}$$

for all n. Letting $n \to \infty$, one has

$$\liminf_{n\to\infty} \int_\Omega F(u_n)v_n \, dx \ge C_\varepsilon \int_\Omega a\psi \, dx$$

by (15). It follows from the arbitrariness of ε that

$$\liminf_{n\to\infty} \int_\Omega F(u_n)v_n \, dx \ge a\underline{F}(+\infty) \int_\Omega \psi \, dx. \tag{16}$$

But from (5) and the fact that

$$\langle \varphi'(u_n), v_n\rangle - \frac{2\varphi(u_n)}{\|u_n\|} = \int_\Omega F(u_n)v_n \, dx - \int_\Omega hv_n \, dx$$

for all n, we obtain

$$\lim_{n\to\infty} \int_\Omega F(u_n)v_n \, dx = \int_\Omega hv \, dx = a\int_\Omega h\psi \, dx < a\underline{F}(+\infty) \int_\Omega \psi \, dx,$$

which contradicts (16). Hence (u_n) is bounded, which implies that the (PS) condition is satisfied by the subcritical growth of g.

LEMMA 2. *Suppose that* $g \in C(R, R)$ *such that* (4) *holds. Then one has*

$$\liminf_{t\to+\infty} \frac{G(t)}{t} \geq \underline{F}(+\infty), \qquad \limsup_{t\to-\infty} \frac{G(t)}{t} \leq \overline{F}(-\infty).$$

Proof. We consider the first inequality. Without loss of generality we may assume that $\underline{F}(+\infty) > -\infty$. For $\varepsilon > 0$, let

$$C_\varepsilon = \begin{cases} \underline{F}(+\infty) - \varepsilon & \underline{F}(+\infty) < +\infty \\ 1/\varepsilon & \underline{F}(+\infty) = +\infty. \end{cases}$$

Then there exists $M > 0$ such that

$$F(t) \geq C_\varepsilon$$

for all $t \geq M$; that is,

$$\frac{d}{dt}\left(-\frac{G(t)}{t^2}\right) = \frac{F(t)}{t^2} \geq \frac{C_\varepsilon}{t^2} = \frac{d}{dt}\left(-\frac{C_\varepsilon}{t}\right)$$

for all $t \geq M$. Integrating two sides over $[t, s]$ and noting that

$$\liminf_{s\to+\infty} \frac{G(s)}{s^2} \geq 0$$

by (4), one obtains

$$\frac{G(t)}{t^2} \geq \frac{C_\varepsilon}{t}$$

for all $t \geq M$. Hence one has

$$\liminf_{t\to+\infty} \frac{G(t)}{t} \geq C_\varepsilon.$$

By the arbitrariness of ε we complete our proof for the first inequality. The second one is similar, so we omit its proof.

LEMMA 3. *Assume that* (2) *and* (7) *hold. Then the function* φ, *where* $k > 1$, *satisfies the* (*PS*) *condition.*

Proof. Suppose that (u_n) is a (PS) sequence for φ in $H_0^1(\Omega)$. Then (u_n) is bounded. In fact, if not, then (u_n) has a subsequence, say (u_n), such that

$$\|u_n\| \to \infty$$

as $n \to \infty$. Set

$$W_1 = \mathrm{Ker}(\lambda_1 + \Delta) \oplus \cdots \oplus \mathrm{Ker}(\lambda_{k-1} + \Delta),$$

$W = \mathrm{Ker}(\lambda_k + \Delta)$, and $W_2 = (W_1 + W)^\perp$. Write u_n in the form

$$u_n = w_{n1} + w_n + w_{n2},$$

where $w_{n1} \in W_1$, $w_n \in W$, and $w_{n2} \in W_2$. It follows from (2) that for every $\varepsilon > 0$ there exists $M > 0$ such that

$$|g(t)| < \varepsilon|t|$$

for all $|t| \geq M$, which implies that

$$|g(t)| < \varepsilon|t| + C_M \tag{17}$$

for all $t \in R$, where $C_M = \max_{|t|\leq M} |g(t)|$. By (17), the Hölder inequality, and Sobolev's inequality (8) we have

$$\begin{aligned}
-\|w_{n1}\| &\leq \langle \varphi'(u_n), w_{n1}\rangle \\
&\leq \int_\Omega |\nabla w_{n1}|^2\, dx - \lambda_k \int_\Omega |w_{n1}|^2\, dx + \varepsilon \int_\Omega |u_n w_{n1}|\, dx \\
&\quad + C_M \int_\Omega |w_{n1}|\, dx + \int_\Omega h w_{n1}\, dx \\
&\leq \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|w_{n1}\|^2 + \varepsilon \|u_n\|_{L^2(\Omega)} \|w_{n1}\|_{L^2(\Omega)} \\
&\quad + \|C_M + |h|\|_{L^2(\Omega)} \|w_{n1}\|_{L^2(\Omega)} \\
&\leq \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|w_{n1}\|^2 + C^2 \varepsilon \|u_n\| \|w_{n1}\| \\
&\quad + C\|C_M + |h|\|_{L^2(\Omega)} \|w_{n1}\|
\end{aligned}$$

for large n. Thus we obtain

$$\limsup_{n\to\infty} \frac{\|w_{n1}\|}{\|u_n\|} \leq \frac{C^2 \lambda_{k-1} \varepsilon}{\lambda_k - \lambda_{k-1}}.$$

By the arbitrariness of ε, one has that $w_{n1}/\|u_n\| \to 0$ as $n \to \infty$. In a similar way one obtains that $w_{n2}/\|u_n\| \to 0$ as $n \to \infty$. By the finite dimensionality of W, $(w_n/\|u_n\|)$ has a convergent subsequence, say $(w_n/\|u_n\|)$, such that $w_n/\|u_n\| \to v \in W$ as $n \to \infty$. It follows that

$$v_n \triangleq \frac{u_n}{\|u_n\|} \to v \qquad \text{in } H_0^1(\Omega) \tag{18}$$

as $n \to \infty$, which implies that $v \neq 0$. In a way similar to that used in proving (16) we have

$$\liminf_{n\to\infty} \int_\Omega F(u_n) v_n\, dx \geq \underline{F}(+\infty) \int_\Omega v^+\, dx - \overline{F}(-\infty) \int_\Omega v^-\, dx. \tag{19}$$

But from (7) and

$$\langle \varphi'(u_n), u_n\rangle - 2\varphi(u_n) = \int_\Omega F(u_n) u_n\, dx - \int_\Omega h u_n\, dx$$

for all n, we obtain

$$\lim_{n\to\infty}\int_\Omega F(u_n)v_n\,dx = \int_\Omega hv\,dx < \underline{F}(+\infty)\int_\Omega v^+\,dx - \overline{F}(-\infty)\int_\Omega v^-\,dx,$$

which contradicts (19). Hence (u_n) is bounded. Therefore φ satisfies the (PS) condition.

Proof of Theorem 1. By Lemma 1 and the Saddle Point Theorem (see [20]) we only need to prove that

$$\varphi(a\psi) \to -\infty \tag{20}$$

as $|a| \to \infty$ in R and

$$\varphi(u) \to +\infty \tag{21}$$

as $\|u\| \to \infty$ in $H_1^\perp$, where

$$H_1 = \{a\psi \mid a \in R\}.$$

If (20) does not hold, there exist a real sequence (a_n) and a real constant C_0 such that $|a_n| \to \infty$ as $n \to \infty$ and $\varphi(a_n\psi) \ge C_0$ for all n. Without loss of generality we may assume that $a_n \to +\infty$ as $n \to \infty$. It follows that

$$\liminf_{n\to\infty}\frac{1}{a_n}\varphi(a_n\psi) \ge 0. \tag{22}$$

From the proof of Lemma 2 we obtain that

$$\frac{G(a_n\psi(x))}{a_n} \ge C_\varepsilon\psi(x)$$

for all n and a.e. $x \in \Omega$. Hence by the Lebesgue–Fatou Lemma, Lemma 2, and (5), we have

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{a_n}\varphi(a_n\psi) &= -\liminf_{n\to\infty}\int_\Omega\frac{G(a_n\psi)}{a_n}\,dx + \int_\Omega h\psi\,dx\\ &\le -\underline{F}(+\infty)\int_\Omega\psi\,dx + \int_\Omega h\psi\,dx\\ &< 0,\end{aligned}$$

which contradicts (22).

Now we prove (21). By (3) there exists $M > 0$ such that $|g(t)| \le \gamma|t|$ for all $|t| \ge M$, where γ is a constant satisfying $\limsup_{|t|\to+\infty} g(t)/t < \gamma < \lambda_2 - \lambda_1$. Thus we have

$$|g(t)| \le \gamma|t| + C_M$$

for all $t \in R$, where $C_M = \sup_{|t| \le M} |g(t)|$. Hence we obtain

$$|G(t)| \le \frac{1}{2}\gamma |t|^2 + C_M |t|$$

for all $t \in R$. By Sobolev's inequality (8) one has

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\lambda_1 \int_\Omega |u|^2\,dx - \int_\Omega G(u)\,dx + \int_\Omega hu\,dx \\
&\ge \frac{\lambda_2 - \lambda_1 - \gamma}{2\lambda_2} \int_\Omega |\nabla u|^2\,dx - \|C_M + |h|\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\
&\ge \frac{\lambda_2 - \lambda_1 - \gamma}{2\lambda_2} \|u\|^2 - C\|C_M + |h|\|_{L^2(\Omega)} \|u\|
\end{aligned}$$

for all $u \in H_1^\perp$, which implies (21). Now by (20), (21), and the Saddle Point Theorem we complete our proof.

Proof of Theorem 2. By Lemma 3 and the Saddle Point Theorem one only needs to prove that

$$\varphi(u) \to +\infty \tag{23}$$

as $\|u\| \to \infty$ in W_2 and

$$\varphi(u) \to -\infty \tag{24}$$

as $\|u\| \to \infty$ in $W_1 + W$, where

$$W_1 = \operatorname{Ker}(\lambda_1 + \Delta) \oplus \cdots \oplus \operatorname{Ker}(\lambda_{k-1} + \Delta),$$

$W = \operatorname{Ker}(\lambda_k + \Delta)$, and $W_2 = (W_1 + W)^\perp$. It follows from (17) that

$$|G(t)| \le \frac{1}{2}\varepsilon |t|^2 + C_M |t| \tag{25}$$

for all $t \in R$. In a manner similar to the proof of (21) one can prove (23).

Now we prove (24). If (24) does not hold, there exist a sequence (u_n) in $W_1 + W$ and a real constant C_0 such that $\|u_n\| \to \infty$ as $n \to \infty$ and $\varphi(u_n) \ge C_0$ for all n. It follows that

$$\liminf_{n\to\infty} \frac{1}{\|u_n\|}\varphi(u_n) \ge 0, \qquad \liminf_{n\to\infty} \frac{1}{\|u_n\|^2}\varphi(u_n) \ge 0. \tag{26}$$

Write $u_n = w_{n1} + w_n$, where $w_{n1} \in W_1$, $w_n \in W$. In the case that $\liminf_{n\to\infty} \|w_n\|/\|u_n\| > 0$, by (25) we have

$$\begin{aligned}
\varphi(u_n) &\le \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx - \frac{1}{2}\lambda_k \int_\Omega |u_n|^2\,dx + \frac{1}{2}\varepsilon \int_\Omega |u_n|^2\,dx \\
&\quad + C_M \int_\Omega |u_n|\,dx + \int_\Omega h u_n\,dx \\
&\le \frac{1}{2}\int_\Omega |\nabla w_{n1}|^2\,dx - \frac{1}{2}\lambda_k \int_\Omega |w_{n1}|^2\,dx + \frac{1}{2}\varepsilon \int_\Omega |u_n|^2\,dx \\
&\quad + \|C_M + |h|\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega)} \\
&\le \frac{1}{2}\left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|w_{n1}\|^2 + \frac{1}{2}C^2\varepsilon\|u_n\|^2 + C\|C_M + |h|\|_{L^2(\Omega)}\|u_n\|
\end{aligned}$$

for all n, which implies that

$$\limsup_{n\to\infty} \frac{1}{\|u_n\|^2}\varphi(u_n) \le -\frac{\lambda_k - \lambda_{k-1}}{2\lambda_{k-1}}\left(\liminf_{n\to\infty} \frac{\|w_{n1}\|}{\|u_n\|}\right)^2 + \frac{1}{2}C^2\varepsilon.$$

It follows from the arbitrariness of ε that

$$\limsup_{n\to\infty} \frac{1}{\|u_n\|^2}\varphi(u_n) < 0,$$

which contradicts (26). Now we consider the case that (u_n) has a subsequence, say (u_n), such that $w_{n1}/\|u_n\| \to 0$ as $n \to \infty$. Because W is finite-dimensional, without loss of generality we may assume that $w_n/\|u_n\| \to v$ in W as $n \to \infty$. Hence one has

$$v_n \triangleq \frac{u_n}{\|u_n\|} \to v \qquad \text{in } H_0^1(\Omega)$$

as $n \to \infty$. From Lemma 2 we obtain

$$\liminf_{n\to\infty} \frac{1}{\|u_n\|}\int_\Omega G(u_n)\,dx \ge \underline{F}(+\infty)\int_\Omega v^+\,dx - \overline{F}(-\infty)\int_\Omega v^-\,dx$$

in a manner similar to the proof of (19). Noting that

$$\varphi(u_n) \le -\int_\Omega G(u_n)\,dx + \int_\Omega h u_n\,dx$$

for all n, we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{\|u_n\|}\varphi(u_n) &\le -\liminf_{n\to\infty} \frac{1}{\|u_n\|}\int_\Omega G(u_n)\,dx + \int_\Omega hv\,dx \\
&\le -\underline{F}(+\infty)\int_\Omega v^+\,dx + \overline{F}(-\infty)\int_\Omega v^-\,dx + \int_\Omega hv\,dx \\
&< 0,
\end{aligned}$$

which contradicts (26), too. Hence (24) holds. The proof is completed.

Proof of Theorem 3. In a manner similar to the proof of Lemma 3 we can prove that the functional φ, where $k > 1$, satisfies the (PS) condition. As in the proof of Theorem 2 one has

$$\varphi(u) \to +\infty$$

as $\|u\| \to \infty$ in $W + W_2$ and

$$\varphi(u) \to -\infty$$

as $\|u\| \to \infty$ in W_1. By the Saddle Point Theorem, φ has at least one critical point. Therefore Theorem 3 holds.

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