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Existence of positive solutions of Lidstone boundary value problems

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Abstract

We show that there exists at least one positive solution for Lidstone boundary value problem

$$(-1)^n u^{(2n)}(t) = f\big(t, u(t), u''(t), \dots, u^{(2(n-1))}(t)\big), \quad 0 < t < 1,$$
$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, \dots, n-1,$$

under some suitable conditions. Our proof is based upon global bifurcation techniques.

Keywords: Lidstone boundary value problem; Positive solution; Bifurcation method

1. Introduction

In this paper we consider the existence of positive solutions of Lidstone boundary value problem

$$(-1)^n u^{(2n)}(t) = f\big(t, u(t), u''(t), \dots, u^{(2(n-1))}(t)\big), \quad 0 < t < 1, \tag{1}$$
$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, \dots, n-1, \tag{2}$$

where f is continuous, $n \geqslant 1$.

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The Lidstone boundary value problem arises in many different areas of applied mathematics and physics. In particular, if $n = 2$, the problem (1) and (2) describes the deformation of an elastic beam whose both ends are simply supported. Recently many authors have studied the existence and multiplicity of positive solutions of this problem, see [3, 5–11,13]. But all of these results are based upon the upper and lower solution method or Leray–Schauder continuation theorem and topological degree, and many results require that the nonlinearity f do not depend on any derivatives of u, see [13]. In this paper, we establish an existence result of positive solution for the problem (1) and (2) by using global bifurcation techniques. This is a novel approach, which is different from the approaches employed in previous papers. By discussing the behavior of positive solution branches of the equations with parameters, we can determine the exact number of positive solutions.

In this paper we will use the following notations. Let

$$\mathbb{R}_+ = [0, \infty), \qquad \mathbb{R}_- = (-\infty, 0],$$
$$U = (u_0, u_1, \dots, u_{n-1}) \in \mathbb{R}^n, \qquad |U| = \sqrt{u_0^2 + u_1^2 + \cdots + u_{n-1}^2},$$
$$\mathbb{R}_i^n = \prod_{i=0}^{n-1} (-1)^i \mathbb{R}_+,$$

where

$$(-1)^i \mathbb{R}_+ = \begin{cases} \mathbb{R}_+, & i \text{ is even}, \\ \mathbb{R}_-, & i \text{ is odd}. \end{cases}$$

We make the following assumptions:

(H1) $f : [0, 1] \times \mathbb{R}_i^n \to \mathbb{R}_+$ is continuous and there exist $A = (a_0, a_1, \dots, a_{n-1})$, $B = (b_0, b_1, \dots, b_{n-1}) \in \mathbb{R}_+^n \setminus \{(0, 0, \dots, 0)\}$ such that

$$f(t, U) = \sum_{i=0}^{n-1} (-1)^i a_i u^{(2i)}(t) + o\big(|U|\big), \quad |U| \to 0, \tag{3}$$

$$f(t, U) = \sum_{i=0}^{n-1} (-1)^i b_i u^{(2i)}(t) + o\big(|U|\big), \quad |U| \to \infty, \tag{4}$$

uniformly in $t \in [0, 1]$.

(H2) $f(t, U) > 0$ for any $t \in [0, 1]$ and $U \not\equiv 0$.

(H3) There exists $C = (c_0, c_1, \dots, c_{n-1}) \in \mathbb{R}_+^n \setminus \{(0, 0, \dots, 0)\}$ such that

$$f(t, U) \geqslant \sum_{i=0}^{n-1} (-1)^i c_i u^{(2i)}(t), \quad (t, U) \in [0, 1] \times \mathbb{R}_i^n.$$

In this paper, we give some conditions on A, B, which ensure that (1) and (2) has at least one positive solution.

2. Preliminaries

Firstly we state the Dancer's results on the global bifurcation branches for positive mapping (see [4]), which play a very important role in the proof of our main results.

Suppose that E is a real Banach space with norm $\|\cdot\|$. Let K be a cone in E. A nonlinear mapping $T:[0,\infty)\times K\to E$ is said to be positive if $T([0,\infty)\times K)\subseteq K$. It is said to be K-completely continuous if T is continuous and maps bounded subsets of $[0,\infty)\times K$ to precompact subset of E. Finally, we say that a positive linear operator V on E is a linear minorant for T if $T(\lambda,u)\geqslant \lambda V(u)$ for $(\lambda,u)\in[0,\infty)\times K$. If N is a continuous linear operator on E, we denote $r(N)$ the spectrum radius of N. Define

$$C_K(N)=\bigl\{\lambda\in[0,\infty)\colon \text{ there exists } x\in K \text{ with } \|x\|=1 \text{ and } x=\lambda Nx\bigr\}.$$

Lemma 1. *Assume that*

(i) *K has nonempty interior and $E=\overline{K-K}$.*
(ii) *$T:[0,\infty)\times K\to E$ is K-completely continuous and positive, $T(\lambda,0)=0$ for $\lambda\in\mathbb{R}$, $T(0,u)=0$ for $u\in K$ and*

$$T(\lambda,u)=\lambda Nu+F(\lambda,u),$$

where $N:E\to E$ is a strongly positive linear compact operator on E with $r(N)>0$, $F:[0,\infty)\times K\to E$ satisfies $\|F(\lambda,u)\|=o(\|u\|)$ as $\|u\|\to 0$ locally uniformly in λ.

Then there exists an unbounded connected subset $\mathcal{L}$ of

$$\mathcal{D}_K(T)=\bigl\{(\lambda,u)\in[0,\infty)\times K\colon\ u=T(\lambda,u),\ u\neq 0\bigr\}\cap\bigl\{\bigl(r(N)^{-1},0\bigr)\bigr\}$$

such that $(r(N)^{-1},0)\in\mathcal{L}$. Moreover, if T has a linear minorant V and there exists a

$$(\mu,y)\in(0,\infty)\times K$$

such that $\|y\|=1$ and $\mu Vy\geqslant y$, then $\mathcal{L}$ can be chosen in

$$\mathcal{D}_K(T)\cap\bigl([0,\mu]\times K\bigr).$$

Proof. See Dancer [4]. □

Let $G_n(t,s)$ is the Green's function of homogeneous boundary value problem

$$u^{(2n)}(t)=0,\quad 0\leqslant t\leqslant 1,$$
$$u^{(2i)}(0)=u^{(2i)}(1)=0,\quad i=0,1,\dots,n-1.$$

By [2], the Green's function $G_n(t,s)$ can be expressed as

$$G_i(t,s)=\int_0^1 G(t,\tau)G_{i-1}(\tau,s)\,d\tau,\quad i=2,3,\dots,n,$$

where

$$G_1(t,s)=G(t,s)=\begin{cases} t(1-s), & 0\leqslant t\leqslant s\leqslant 1,\\ s(1-t), & 0\leqslant s\leqslant t\leqslant 1.\end{cases} \tag{5}$$

It is clear that

$$t(1-t)G(s,s)\leqslant G(t,s)\leqslant t(1-t) \tag{6}$$

for any $0\leqslant s\leqslant 1$, and

$$G_n(t,s)>0, \quad (t,s)\in(0,1)\times(0,1).$$

Define the operator $S: C[0,1]\to C[0,1]$ as follows:

$$(Su)(t)=\int_0^1 G_n(t,s)f\bigl(s,U(s)\bigr)\,ds, \quad t\in[0,1].$$

It is well known that the solutions of BVP (1) and (2) are equivalent to the fixed points of S.

Let

(H4) $D=(d_0,d_1,\ldots,d_{n-1})\in\mathbb{R}_+^n\setminus\{(0,0,\ldots,0)\}$.

Definition 1. We say λ is a generalized eigenvalue of linear problem

$$(-1)^n u^{(2n)}(t)=\lambda\sum_{i=0}^{n-1}(-1)^i d_i u^{(2i)}(t), \quad 0<t<1, \tag{7}$$

$$u^{(2i)}(0)=u^{(2i)}(1)=0, \quad i=0,1,\ldots,n-1, \tag{8}$$

if (7) and (8) has a nontrivial solution.

Lemma 2. *Assume that* (H4) *holds. Then the generalized eigenvalues of* (7) *and* (8) *are given by*

$$0<\lambda_1(D)<\lambda_2(D)<\cdots<\lambda_n(D)<\cdots, \tag{9}$$

where

$$\lambda_k(D)=\frac{(k\pi)^{2n}}{\sum_{i=0}^{n-1}d_i(k\pi)^{2i}}, \quad k\in\mathbb{N}. \tag{10}$$

The generalized eigenfunction corresponding to $\lambda_k(D)$ is

$$\varphi_k(t)=\sin k\pi t. \tag{11}$$

Proof. First of all we note that λ is a generalized eigenvalue of (7) and (8) if and only if

$$\frac{\lambda\sum_{i=0}^{n-1}d_i(k\pi)^{2i}}{(k\pi)^{2n}}=1$$

for $k\in\mathbb{N}$, so (10) is true.

Now we define a function

$$R(x)=\frac{x^{2n}}{\sum_{i=0}^{n-1} d_i x^{2i}},\quad x\in[\pi,\infty).$$

Since

$$R'(x)=\frac{x^{2n-1}\sum_{i=0}^{n-1}2(n-i)d_i x^{2i}}{[\sum_{i=0}^{n-1} d_i x^{2i}]^2}>0,$$

and

$$\lambda_k=R(k\pi),$$

then $\{\lambda_k(D)\}$ is strictly increasing.

Nextly let u be a nontrivial solution of

$$(-1)^n u^{(2n)}(t)=\lambda_k(D)\sum_{i=0}^{n-1}(-1)^i d_i u^{(2i)}(t),\quad 0<t<1, \tag{12}$$

$$u^{(2i)}(0)=u^{(2i)}(1)=0,\quad i=0,1,\dots,n-1, \tag{13}$$

and we denote $L_0u=-u''$ with $\mathcal{D}(L_0)=\{u\in C^2[0,1]\colon\ u(0)=u(1)=0\}$; then there exist n complex numbers r_i such that

$$\begin{aligned}&(-1)^n u^{(2n)}(t)-\lambda_k(D)\sum_{i=0}^{n-1}(-1)^i d_i u^{(2i)}(t)\\ &\quad=(L_0+r_0I)(L_0+r_1I)\cdots(L_0+r_{n-1}I)u(t)=0.\end{aligned}$$

Hence there must exist i, $0\leqslant i\leqslant n-1$, such that $(L_0+r_i)u=0$, which implies that

$$r_i=(j\pi)^2$$

for some $j\in\mathbb{N}$, and consequently

$$u(t)=\sin j\pi t \tag{14}$$

is a nontrivial solution.

By substituting (14) into (12), we have

$$\lambda_k(D)=\frac{(j\pi)^{2n}}{\sum_{i=0}^{n-1} d_i (j\pi)^{2i}},$$

which implies $j=k$ and accordingly (11) holds. □

Corollary 1. *Assume that* (H4) *holds. If either*

$$\frac{\sum_{i=0}^{n-1} d_i\pi^{2i}}{\pi^{2n}}<1\quad\textit{or}\quad\frac{\sum_{i=0}^{n-1} d_i\pi^{2i}}{\pi^{2n}}>1,$$

then

$$\lambda_1(D)>1\quad\textit{or}\quad\lambda_1(D)<1.$$

Proof. From (10) we easily see that

$$\lambda_1(D)\frac{\sum_{i=0}^{n-1} d_i \pi^{2i}}{\pi^{2n}} = 1,$$

then Corollary 1 naturally holds. □

3. Main results

Theorem 1. *Assume* (H1)–(H3) *hold. If either*

$$\lambda_1(B) < 1 < \lambda_1(A) \tag{15}$$

or

$$\lambda_1(A) < 1 < \lambda_1(B), \tag{16}$$

then (1) *and* (2) *have at least one positive solution.*

Corollary 2. *Assume that* (H1)–(H3) *hold. If either*

$$\frac{\sum_{i=0}^{n-1} b_i(\pi)^{2i}}{(\pi)^{2n}} > 1 \quad \textit{and} \quad \frac{\sum_{i=0}^{n-1} a_i(\pi)^{2i}}{(\pi)^{2n}} < 1 \tag{17}$$

or

$$\frac{\sum_{i=0}^{n-1} a_i(\pi)^{2i}}{(\pi)^{2n}} > 1 \quad \textit{and} \quad \frac{\sum_{i=0}^{n-1} b_i(\pi)^{2i}}{(\pi)^{2n}} < 1, \tag{18}$$

then (1) *and* (2) *have at least one positive solution.*

It is easy to know that if $\lambda_1(A) = 1 = \lambda_1(B)$, then the existence of positive solution of the problem (1) and (2) cannot be guaranteed.

Now let

$$e(t) := \sin \pi t, \quad t \in [0, 1],$$

and we denote E is such a Banach space that its every element $u \in C^{2n-2}[0, 1]$ satisfying

$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, \dots, n-1,$$

and there exists a constant $\gamma \in (0, \infty)$ such that

$$-\gamma e(t) \leqslant (-1)^{n-1} u^{(2(n-1))}(t) \leqslant \gamma e(t), \quad t \in [0, 1]. \tag{19}$$

Now for any $u \in E$, we have that

$$(-1)^i u^{(2i)}(t) = \int_0^1 G(t, s)(-1)^{i+1} u^{(2(i+1))}(s)\, ds, \tag{20}$$

and

$$\frac{1}{\pi^2} e(t) = \int_0^1 G(t, s) e(s)\, ds, \tag{21}$$

where $G(t,s)$ is given in (5). Using (20), we have that

$$-\frac{\gamma}{\pi^{2(n-1-i)}}e(t) \leqslant (-1)^i u^{(2i)}(t) \leqslant \frac{\gamma}{\pi^{2(n-1-i)}}e(t) \tag{22}$$

for any $t \in [0,1]$, $i = 0, 1, \ldots, n-1$.

Since $\frac{\gamma}{\pi^{2i}} \leqslant \gamma$ for any $0 \leqslant i \leqslant n-1$, then we can define the norm in E by

$$\|u\|_E := \inf\bigl\{\gamma\colon -\gamma e(t) \leqslant (-1)^{n-1} u^{(2(n-1))}(t) \leqslant \gamma e(t),\ t \in [0,1]\bigr\}.$$

It is easy to check that $(E, \|\cdot\|_E)$ is a Banach space. Let

$$K := \bigl\{u \in E\colon (-1)^i u^{(2i)}(t) \geqslant 0,\ t \in [0,1],\ i = 0, 1, \ldots, n-1\bigr\},$$

then K is normal and has a nonempty interior, moreover $E = \overline{K - K}$.

Let $Y = C[0,1]$ with the norm

$$\|u\|_\infty = \max_{t \in [0,1]} \bigl|u(t)\bigr|.$$

Define the operator $L : \mathcal{D}(L) \to Y$ by setting

$$Lu := (-1)^n u^{(2n)}(t), \quad u \in \mathcal{D}(L), \tag{23}$$

where

$$\mathcal{D}(L) = \bigl\{u \in C^{2n}[0,1]\colon u^{(2i)}(0) = u^{(2i)}(1) = 0,\ i = 0, 1, \ldots, n-1\bigr\}.$$

We can verify that $L^{-1} : Y \to E$ is compact.

Lemma 3. *Let $h \in Y$ with $h \geqslant 0$ and $h(t_0) > 0$ for some $t_0 \in [0,1]$, and*

$$Lu - h = 0,$$

then $u \in \operatorname{int} K$.

Proof. Using (22), we only need to show that there exist constants $r_1, r_2 > 0$ such that

$$r_1 e(t) \leqslant (-1)^{n-1} u^{(2(n-1))}(t) \leqslant r_2 e(t), \quad t \in [0,1]. \tag{24}$$

In fact, from (6) and (20) we have that

$$\begin{aligned}(-1)^{n-1} u^{(2(n-1))}(t) &= \int_0^1 G(t,s)(-1)^n u^{(2n)}(s)\,ds = \int_0^1 G(t,s)h(s)\,ds \\ &\leqslant t(1-t)\int_0^1 h(s)\,ds \leqslant t(1-t)\|h\|_\infty,\end{aligned}$$

$$(-1)^{n-1} u^{(2(n-1))}(t) = \int_0^1 G(t,s)h(s)\,ds \geqslant t(1-t)\int_0^1 G(s,s)h(s)\,ds.$$

Since there exist constants $c_1, c_2 > 0$ such that

$$c_1 \sin \pi t \leqslant t(1-t) \leqslant c_2 \sin \pi t, \quad 0 \leqslant t \leqslant 1,$$

then (24) naturally holds. $\square$

Proof of Theorem 1. Let $g, h \in C([0,1] \times \mathbb{R}_i^n, \mathbb{R})$ be such that

$$f(t,U) = \sum_{i=0}^{n-1} (-1)^i a_i u^{(2i)}(t) + g(t,U), \tag{25}$$

$$f(t,U) = \sum_{i=0}^{n-1} (-1)^i b_i u^{(2i)}(t) + h(t,U), \tag{26}$$

which implies by (H1) that

$$\lim_{|U| \to 0} \frac{g(t,U)}{|U|} = 0 \quad \text{and} \quad \lim_{|U| \to \infty} \frac{h(t,U)}{|U|} = 0 \tag{27}$$

uniformly in $t \in [0,1]$.

Define a function

$$\tilde{h}(r) = \max\bigl\{\bigl|h(t,U)\bigr|: \ 0 \leqslant |U| \leqslant r, \ t \in [0,1]\bigr\},$$

then $\tilde{h}$ is nondecreasing and

$$\lim_{r \to \infty} \frac{\tilde{h}(r)}{r} = 0. \tag{28}$$

Let us consider

$$Lu = \lambda \sum_{i=0}^{n-1} (-1)^i a_i u^{(2i)}(t) + \lambda g(t,U) \tag{29}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

It is easy to check that (29) is equivalent to the following integral equation:

$$\begin{aligned} u(t) = {}& \lambda \int_0^1 G_n(t,s) \sum_{i=0}^{n-1} (-1)^i a_i u^{(2i)}(s)\, ds + \lambda \int_0^1 G_n(t,s) g\bigl(s, U(s)\bigr)\, ds \\ =:{}& T(\lambda, u)(t). \end{aligned}$$

Now we define $N : E \to E$ by

$$Nu(t) := \int_0^1 G_n(t,s) \sum_{i=0}^{n-1} (-1)^i a_i u^{(2i)}(s)\, ds.$$

It is easy to check that N is a strongly positive linear operator on E, and is completely continuous. From Lemma 2 and [1, Theorem 3.2], we have

$$r(N) = \bigl[\lambda_1(A)\bigr]^{-1}.$$

Nextly we define $F:[0,\infty)\times E\to E$ by

$$F(\lambda,u):=\lambda\int_0^1 G_n(t,s)g\big(s,U(s)\big)\,ds.$$

From the definition of the norm of Banach space E, we can see that

$$\|u\|_\infty\leqslant\|u''\|_\infty\leqslant\cdots\leqslant\|u^{(2(n-1))}\|_\infty\leqslant\|u\|_E. \tag{30}$$

Combining with (28), we have

$$\big\|F(\lambda,u)\big\|_E=o\big(\|u\|_E\big)$$

locally uniformly in λ.

From (H2) and Lemma 3, we know that if (λ,u) with $\lambda>0$ is a nontrivial solution of (29), then

$$u\in\operatorname{int}K.$$

Combining with Lemma 1, there exists an unbounded connected subset $\mathcal{L}$ of the set

$$\big\{(\lambda,u)\in(0,\infty)\times K:\ u=T(\lambda,u),\ u\in\operatorname{int}K\big\}\cup\big\{\big(\lambda_1(A),0\big)\big\}$$

such that $(\lambda_1(A),0)\in\mathcal{L}$.

It is clear that any solution of (29) of the form $(1,u)$ yields a solution u of the problem (1) and (2). We show that $\mathcal{L}$ crosses the hyperplane $\{1\}\times E$ in $\mathbb{R}\times E$. Using (15) or (16), it is enough to show that $\mathcal{L}$ joins $(\lambda_1(A),0)$ to $(\lambda_1(B),\infty)$.

Let $(\mu_m,y_m)\in\mathcal{L}$ satisfy

$$\mu_m+\|y_m\|_E\to\infty. \tag{31}$$

We note that $\mu_m>0$ for all $m\in\mathbb{N}$ since $(0,0)$ is the only solution of (29) for $\lambda=0$ and $\mathcal{L}\cap(\{0\}\times E)=\emptyset$.

Case 1. $\lambda_1(B)<1<\lambda_1(A)$. In this case, we show that

$$\big(\lambda_1(B),\lambda_1(A)\big)\subseteq\big\{\lambda\in\mathbb{R}:\ \exists(\lambda,u)\in\mathcal{L}\big\}.$$

Step 1. We show that there exists a constant $M>0$ such that $\mu_m\in(0,M]$ for all m. In fact, by Lemma 1 we only need to show that T has a linear minorant V and there exists a $(\mu,y)\in(0,\infty)\times K$ such that $\|y\|_E=1$ and $\mu Vy\geqslant y$.

By (H3), there exists $C=(c_0,c_1,\dots,c_{n-1})\in\mathbb{R}^n_+\setminus\{(0,0,\dots,0)\}$ such that

$$f(t,U)\geqslant\sum_{i=0}^{n-1}(-1)^i c_i u^{(2i)}(t),\qquad (t,U)\in[0,1]\times\mathbb{R}^n_i. \tag{32}$$

For $u\in E$, let

$$Vu(t):=\int_0^1 G_n(t,s)\sum_{i=0}^{n-1}(-1)^i c_i u^{(2i)}(s)\,ds,$$

then V is a linear minorant of T. Now we choose $y(t)=\frac{e(t)}{\pi^{2(n-1)}}$, then $y\in K$ and $\|y\|_E=1$. Moreover,

$$\left[\frac{\sum_{i=0}^{n-1}c_i\pi^{2i}}{\pi^{2n}}\right]^{-1}Vy(t)=y(t). \tag{33}$$

From Lemma 1 we have that

$$|\mu_m|\leqslant\left[\frac{\sum_{i=0}^{n-1}c_i\pi^{2i}}{\pi^{2n}}\right]^{-1}.$$

Step 2. We show that $\mathcal{L}$ joins $(\lambda_1(A),0)$ to $(\lambda_1(B),\infty)$. From (31) and the result of Step 1 we have that $\|y_m\|_E\to\infty$. We divide the equation

$$Ly_m=\mu_m\sum_{i=0}^{n-1}(-1)^i b_i y_m^{(2i)}+\mu_m h\big(t,Y_m(t)\big) \tag{34}$$

by $\|y_m\|_E$ and set $\bar y_m=\frac{y_m}{\|y_m\|_E}$. Since $\bar y_m$ is bounded in Banach space E, then there exists a subsequence, which we still denote $\bar y_m$ such that $\bar y_m\to\bar y$ for some $\bar y\in E$ with $\|\bar y\|_E=1$. Since $\tilde h$ is nondecreasing, then we easily obtain from (30) that

$$\frac{|h(y_m(t))|}{\|y_m\|_E}\leqslant\frac{\tilde h(|y_m(t)|)}{\|y_m\|_E}\leqslant\frac{\tilde h(\|y_m\|_\infty)}{\|y_m\|_E}\leqslant\frac{\tilde h(\|y_m\|_E)}{\|y_m\|_E}.$$

Using (28), we have

$$\lim_{m\to\infty}\frac{|h(y_m)|}{\|y_m\|_E}=0.$$

Hence

$$\bar y(t):=\bar\mu\int_0^1 G_n(t,s)\sum_{i=0}^{n-1}(-1)^i b_i u^{(2i)}(s)\,ds,$$

where $\bar\mu:=\lim_{m\to\infty}\mu_m$, again choosing a subsequence and relabelling it if necessary. Obviously $\bar\mu\neq 0$. In fact, if $\bar\mu=0$, the $\bar y\equiv 0$, it is contrary to the fact $\|\bar y\|_E=1$. Therefore

$$L\bar y=\bar\mu\sum_{i=0}^{n-1}(-1)^i b_i\bar y^{(2i)},$$

which implies that $\bar\mu=\lambda_1(B)$ by Lemma 2. Hence $\mathcal{L}$ joins $(\lambda_1(A),0)$ to $(\lambda_1(B),\infty)$.

Case 2. $\lambda_1(A)<1<\lambda_1(B)$. Let $(\mu_m,y_m)\in\mathcal{L}$ such that

$$\lim_{m\to\infty}\big(\mu_m+\|y_m\|_E\big)=\infty.$$

Now if there exists a constant $M>0$ such that

$$\mu_m\in(0,M],\quad m\in\mathbb{N},$$

then applying the similar argument used in Step 2 of Case 1, after taking a subsequence and relabelling it if necessary, we have that

$$(\mu_m, y_m) \to \big(\lambda_1(B), \infty\big), \quad m \to \infty.$$

Hence $\mathcal{L}$ joins $(\lambda_1(A), 0)$ to $(\lambda_1(B), \infty)$ and the result follows.
If

$$\lim_{m\to\infty} \mu_m = \infty,$$

then we must have that

$$\big(\lambda_1(A), \lambda_1(B)\big) \subseteq \big\{\lambda \in (0,\infty)\colon\ (\lambda, u) \in \mathcal{L}\big\},$$

and moreover,

$$\big(\{1\} \times E\big) \cap \mathcal{L} \neq \emptyset,$$

which implies that there exists a $u \in E$ such that $(1, u) \in \mathcal{L}$ is a solution of (29), naturally u is a positive solution of (1) and (2).

The proof of Theorem 1 is completed. □

Remark. Schaaf and Schmitt [12] in 1992 considered the asymptotic behavior of positive solution branches of elliptic problems. Similarly, under some suitable conditions, we can prove the following eigenvalue problem:

$$(-1)^n u^{(2n)}(t) = \lambda f\big(t, u(t), u''(t), \ldots, u^{(2(n-1))}(t)\big), \quad 0 < t < 1,$$
$$u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1, \ldots, n-1,$$

has a connected C^1 component in solution set. By discussed the perturbation of branching curve at $\lambda = 1$, we can determine the exact number of positive solutions for problem (1) and (2), or obtain the existence results of infinitely many positive solutions after adding certain conditions.

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