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Maximal functions along surfaces in product spaces

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Abstract

Under certain natural conditions of a measurable radial function $\Gamma:\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\Gamma(y_1, y_2) = \Gamma(|y_1|, |y_2|)$, we show that the maximal function along surface

$$M_\Gamma f(x_1, x_2, x_3) = \sup_{r_1, r_2 > 0} \left\{ \frac{1}{r_1^n r_2^m} \int_{|y_2| \leqslant r_2} \int_{|y_1| \leqslant r_1} \left| f\left(x_1 - y_1, x_2 - y_2, x_3 - \Gamma\left(|y_1|, |y_2|\right)\right)\right| dy_1\, dy_2 \right\}$$

is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for all $p > 1$ and $n, m \geqslant 1$.

Keywords: Maximal function; Hardy–Littlewood maximal function; Product space; Along surface

Introduction

For a surface S in $\mathbb{R}^3$ parametrized as $(s, t, \phi(s, t))$, we define the Hilbert transform and the maximal function along this surface respectively by

$$Hf(x_1, x_2, x_3) = \text{p.v.} \int_{|t| < C_2} \int_{|s| < C_1} f\left(x_1 - s, x_2 - t, x_3 - \phi(s, t)\right) \frac{ds\, dt}{st}$$

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and

$$Mf(x_1,x_2,x_3)=\sup_{0<r_i\leqslant C_i\,(i=1,2)}\left\{\frac{1}{r_1r_2}\int_0^{r_2}\int_0^{r_1}\bigl|f\bigl(x_1-s,x_2-t,x_3-\phi(s,t)\bigr)\bigr|\,ds\,dt\right\}.$$

When $\phi(s,t)=|s|^\alpha|t|^\beta$, $\alpha,\beta>0$, the Hilbert transform Hf (with $C_1=C_2=\infty$) is known to be bounded in $L^p(\mathbb{R}^3)$, $|1/p-1/2|<\epsilon$ for some $\epsilon>0$ (see [5,7,8]). On the other hand, if $\phi(s,t)=|s|^\alpha|t|^\beta$, $\alpha,\beta>0$, $\phi(0,0)=\nabla\phi(0,0)=0$ and ϕ has nonvanishing second order derivatives at the origin, then the maximal function Mf (with $C_1=C_2=\infty$) is bounded in $L^p(\mathbb{R}^3)$ for $1<p<\infty$ (see [1,2]). The author of [3] has obtained the L^p boundedness of Hf and Mf $(1<p<\infty)$ for the three types of surfaces considered below.

Type A. $\phi(s,t)=|s|^\alpha|t|^\beta$, $\alpha,\beta>0$. Here $C_1=C_2=\infty$.

Type B. $\phi(s,t)$ is an even function (with respect to each one of the variables) of class C^2 in a neighborhood of the origin with $D_1^2\phi(0,0)$ and $D_2^2\phi(0,0)\neq 0$, $D_{12}\phi(s,0)\geqslant 0$ if $D_1^2\phi(0,0)>0$ (respectively $D_{12}\phi(s,0)\leqslant 0$ if $D_1^2\phi(0,0)<0$) and a similar condition over $D_{12}\phi(0,t)$. Here $D_i\phi(s,t)$ stands for the derivative of ϕ with respect to the ith variable $(i=1,2)$; $D_i^2\phi(s,t)=D_i(D_i\phi(s,t))$ and $D_{12}\phi(s,t)=D_1(D_2\phi(s,t))$. C_1 and C_2 must be chosen such that $D_i^2\phi(s,t)\geqslant A$ $(i=1,2)$ for some $A>0$ in $0<s\leqslant C_1$, $0<t\leqslant C_2$.

Type C. $\phi(s,t)$ is an even function of class C^2 such that $D_1^2\phi(s,t)$ and $D_{12}\phi(s,0)$ (respectively $D_2^2\phi(s,t)$ and $D_{12}\phi(0,t)$) are nonnegative and nondecreasing in $s>0$ (respectively in $t>0$). In this case C_1 and C_2 must be chosen such that these conditions hold in $0<s\leqslant C_1$, $0<t\leqslant C_2$. For this type, observe that surfaces with a contact of infinite order at the origin are allowed; for example, $\phi(s,t)=s^2t^2(e^{-1/|s|}+e^{-1/|t|})$ for which $C_1=C_2=\infty$.

Inspiring of the work in [3], we would like to study the L^p boundedness of the maximal function $M_\Gamma f$ (as defined in the abstract) in higher dimension, $n,m\geqslant 1$, based on the L^p boundedness of the partial maximal functions (see Theorem 1 below) in lower dimensions. We now describe some definitions and notations. Then we will state the results.

Definitions and notations. Let $\mathbb{R}^+$ stand for $[0,\infty)$. A function $\phi:\mathbb{R}^+\to\mathbb{R}$ is called a type I function if ϕ is strictly increasing on $[0,\infty)$ and ϕ' is increasing on $(0,\infty)$.

A function $\phi:\mathbb{R}^+\to\mathbb{R}$ or $\phi:(0,\infty)\to\mathbb{R}$ is a type II function if ϕ is strictly decreasing on its domain and ϕ' is increasing on $(0,\infty)$.

A function $\phi:\mathbb{R}^+\to\mathbb{R}$ is a type III function if

(i) $\phi(0)=0$ and ϕ is strictly increasing on $[0,\infty)$,
(ii) ϕ' is decreasing on $(0,\infty)$ and
(iii) $t\phi'(t)\geqslant\alpha\,\phi(t)$ for all $t\in(0,\infty)$ and for some fixed $\alpha>0$.

For $f\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})$, $n,m\geqslant 1$, consider the following maximal functions:

$$M_\Gamma^{(1)} f(x_1,x_2,x_3) = \sup_{r_1,r_2>0}\left\{\frac{1}{r_1^n r_2^m}\int\limits_{|y_2|\leqslant r_2}\int\limits_{|y_1|\leqslant r_1}\left|f\big(x_1-y_1,x_2,x_3-\Gamma\big(|y_1|,|y_2|\big)\big)\right|dy_1\,dy_2\right\},$$

$$M_\Gamma^{(2)} f(x_1,x_2,x_3) = \sup_{r_1,r_2>0}\left\{\frac{1}{r_1^n r_2^m}\int\limits_{|y_2|\leqslant r_2}\int\limits_{|y_1|\leqslant r_1}\left|f\big(x_1,x_2-y_2,x_3-\Gamma\big(|y_1|,|y_2|\big)\big)\right|dy_1\,dy_2\right\},$$

$$M_\Gamma^{(1,2)} f(x_1,x_2,x_3) = \sup_{r_1,r_2>0}\left\{\frac{1}{r_1^n r_2^m}\int\limits_{|y_2|\leqslant r_2}\int\limits_{|y_1|\leqslant r_1}\left|f\big(x_1,x_2,x_3-\Gamma\big(|y_1|,|y_2|\big)\big)\right|dy_1\,dy_2\right\}$$

and

$$M_\Gamma f(x_1,x_2,x_3) = \sup_{r_1,r_2>0}\left\{\frac{1}{r_1^n r_2^m}\int\limits_{|y_2|\leqslant r_2}\int\limits_{|y_1|\leqslant r_1}\left|f\big(x_1-y_1,x_2-y_2,x_3-\Gamma\big(|y_1|,|y_2|\big)\big)\right|dy_1\,dy_2\right\},$$

where $x_1, y_1 \in \mathbb{R}^n$, $x_2, y_2 \in \mathbb{R}^m$, $x_3 \in \mathbb{R}$ and $\Gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a measurable function which is radial with respect to both variables $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$, i.e. $\Gamma(y_1, y_2) = \Gamma(|y_1|, |y_2|)$ for all $y_1 \in \mathbb{R}^n$ and $y_2 \in \mathbb{R}^m$.

Denote $h_t(s) = \Gamma(s,t)$ for every fixed $t \geqslant 0$. Similarly, denote $\gamma_s(t) = \Gamma(s,t)$ for every fixed $s \geqslant 0$.

Throughout the rest of this paper, the letter C denotes a positive constant which may vary at each occurrence it appears. However, it does not depend on any essential variable.

Theorem 1. *Suppose the partial maximal functions $M_\Gamma^{(1)} f$, $M_\Gamma^{(2)} f$ and $M_\Gamma^{(1,2)} f$ are bounded in L^p for all $p > 1$. Then the maximal function $M_\Gamma f$ is bounded in L^p for all $p > 1$ and $n, m \geqslant 3$. The conclusion holds for the case $n = 1, 2$ provided that the function $h_t(s) = \Gamma(s,t)$ satisfies one of the following conditions for every fixed $t > 0$:*

(a) *$h_t'(s) > 0$, $h_t''(s) > 0$ and $h_t'(s)/s$ is increasing on $(0,\infty)$,*
(b) *$h_t'(s) < 0$, $h_t''(s) > 0$ and $s\,h_t'(s)$ is increasing on $(0,\infty)$,*
(c) *$h_t'(s) > 0$, $h_t''(s) < 0$ and $s\,h_t'(s)$ is decreasing on $(0,\infty)$, or*
(d) *$\Gamma(s,t) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$, $h_t'(s) > 0$ and $h_t''(s) \geqslant 0$ for all (s,t) in the support of Γ.*

Moreover, the conclusion also holds for the case $m = 1, 2$ if the function $\gamma_s(t) = \Gamma(s,t)$ satisfies one of the similar conditions above for every fixed $s > 0$.

Theorem 2. *Suppose $\Gamma(s,t)$ have continuous first order partial derivatives for all $s, t > 0$. If $h_t(s)$ and $\gamma_s(t)$ are either type I, II or III functions (with the constant α in the definition* *of type III function independent of both variables s and t) for each fixed $t > 0$ and for each fixed $s > 0$, respectively, then the maximal functions*

$$M_{1,2}g(x_3) = \sup_{r_1,r_2>0}\left\{\frac{1}{r_1r_2}\int_0^{r_2}\int_0^{r_1}\left|g\bigl(x_3 - \Gamma(s,t)\bigr)\right| ds\,dt\right\},$$

$$M_1g_1(x_1,x_3) = \sup_{r_1,r_2>0}\left\{\frac{1}{r_1r_2}\int_0^{r_2}\int_0^{r_1}\left|g_1\bigl(x_1 - s, x_3 - \Gamma(s,t)\bigr)\right| ds\,dt\right\}$$

and

$$M_2g_2(x_2,x_3) = \sup_{r_1,r_2>0}\left\{\frac{1}{r_1r_2}\int_0^{r_2}\int_0^{r_1}\left|g_2\bigl(x_2 - t, x_3 - \Gamma(s,t)\bigr)\right| ds\,dt\right\}$$

($x_1, x_2, x_3 \in \mathbb{R}$, $g \in L^p(\mathbb{R})$, and $g_1, g_2 \in L^p(\mathbb{R}^2)$) are bounded in L^p for all $p > 1$.

Corollary 1. *Let $\Gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a C^1 function such that $h_t(s)$ and $\gamma_s(t)$ are functions of type I, II or III for each fixed $t > 0$ and for each fixed $s > 0$, respectively. Then the maximal function $M_\Gamma f(x_1, x_2, x_3)$ (in Theorem 1) is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for all $p > 1$ and $n, m \geqslant 3$. If $h_t(s)$ (respectively $\gamma_s(t)$) is either a type I function which satisfies hypothesis* (a) *of Theorem* 1 *or a type II function satisfying hypothesis* (b) *of Theorem* 1 *for each fixed $t > 0$ (respectively $s > 0$), then the above result also holds for n (respectively m) $= 1$ or 2.*

Corollary 2. *Suppose $\Gamma : [0, c] \times \mathbb{R}^+ \to \mathbb{R}$ or $\Gamma : [0, c] \times (0, \infty) \to \mathbb{R}$ is a bounded function such that $\frac{\partial \Gamma(s,t)}{\partial s} > 0$, $\frac{\partial^2 \Gamma(s,t)}{\partial s^2} \geqslant 0$, $\frac{\partial \Gamma(s,t)}{\partial t} < 0$ and $\frac{\partial^2 \Gamma(s,t)}{\partial t^2} \geqslant 0$ for all $(s,t) \in (0, c) \times (0, \infty)$. Then the maximal function $M_\Gamma f(x_1, x_2, x_3)$ is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for all $p > 1$, $n \geqslant 1$ and $m \geqslant 3$.*

Corollary 3. *Suppose $\Gamma : [0, c] \times [0, d] \to \mathbb{R}$ is a C^1 function such that $\frac{\partial \Gamma(s,t)}{\partial s}, \frac{\partial \Gamma(s,t)}{\partial t} > 0$ and $\frac{\partial^2 \Gamma(s,t)}{\partial s^2}, \frac{\partial^2 \Gamma(s,t)}{\partial t^2} \geqslant 0$ for all $(s,t) \in (0, c) \times (0, d)$. Then the maximal function $M_\Gamma f(x_1, x_2, x_3)$ is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for all $p > 1$ and $n, m \geqslant 1$.*

Examples. (1) Consider $\Gamma(s,t) = s^\alpha t^\beta$, $\alpha, \beta \neq 0$ and $s, t \geqslant 0$ ($s > 0$ if $\alpha < 0$ and similarly $t > 0$ if $\beta < 0$). For each fixed $t > 0$, the function $h_t(s) = \Gamma(s,t)$ is a type I function if $\alpha \geqslant 1$, a type III function if $0 < \alpha < 1$ and a type II function if $\alpha < 0$. Similar conclusion holds for the function $\gamma_s(t) = \Gamma(s,t)$. Thus the maximal function $M_\Gamma f(x_1, x_2, x_3)$ is bounded in $L^p(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R})$ for all $p > 1$ and $n, m \geqslant 3$ if $\alpha, \beta \neq 0$. The result also holds for the case n (respectively m) $= 1$ or 2 if $\alpha \geqslant 2$ (respectively $\beta \geqslant 2$) or $\alpha < 0$ (respectively $\beta < 0$).

Remark 1. The example above also holds for the case n (respectively m) $= 1$ or 2 when $0 < \alpha < 2$ (respectively $0 < \beta < 2$). To see this we only need to verify the decay estimate of $I_1(\zeta_1)$ (respectively $I_2(\zeta_2)$) (see Eqs. (5)–(6) in the proof of Theorem 1) in the case $n=1$ (respectively $m=1$). A proof for this decay estimate is given in [3,6].

(2) Let $\phi_1(r)=r^{\alpha_1}e^{\alpha_2 r}$, $r\geqslant 0$, $\alpha_1\geqslant 2$ and $\alpha_2\geqslant 0$. Then $\phi_1'(r)>0$, $\phi_1''(r)\geqslant 0$ and $\frac{d}{dr}(\frac{\phi_1'(r)}{r})\geqslant 0$. Let $\phi_2(r)=r^{-\beta_1}e^{-\beta_2 r}$, $r>0$, $\beta_1\geqslant 1$ and $\beta_2\geqslant 0$. Note that $\phi_2(r)$ is strictly decreasing on $(0,\infty)$, and both $\phi_2'(r)$ and $r\phi_2'(r)$ are increasing on $(0,\infty)$. Now let $\Gamma(s,t)=\phi(s)\psi(t)$, where ϕ and ψ are either ϕ_1 or ϕ_2 as defined above. Then the maximal function $M_\Gamma f(x_1,x_2,x_3)$ is bounded in $L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})$ for all $p>1$ and $n,m\geqslant 1$.

(3) Let $\Gamma(s,t)=s^2t^2(e^{-1/s}+e^{-1/t})$, $s,t>0$ (surface with a contact of infinite order at the origin). The functions $h_s(t)=\Gamma(s,t)$ and $\gamma_t(s)=\Gamma(s,t)$ both satisfy hypothesis (a) of Theorem 1 for each fixed $s>0$ and each fixed $t>0$, respectively. Therefore, the maximal function $M_\Gamma f(x_1,x_2,x_3)$ is bounded in $L^p(\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R})$ for all $p>1$ and $n,m\geqslant 1$. Note that these problems on surfaces appear as a natural generalization of their analogues on curves (see [6]).

Proof of Theorem 1. We apply Theorem 1 [3] to prove this theorem. Consider the new maximal function

$$\begin{aligned}
&Nf(x_1,x_2,x_3)\\
&\quad=\sup_{j,k\in\mathbb{Z}}\left\{\frac{1}{2^{nk+mj}}\int\limits_{|y_2|\cong 2^j}\int\limits_{|y_1|\cong 2^k} f\big(x_1-y_1,x_2-y_2,x_3-\Gamma\big(|y_1|,|y_2|\big)\big)\,dy_1\,dy_2\right\}\\
&\quad=\sup_{j,k\in\mathbb{Z}}\mu_{j,k}*f(x_1,x_2,x_3).
\end{aligned}$$

Since $Mf(x_1,x_2,x_3)\leqslant C_1N(|f|)(x_1,x_2,x_3)\leqslant C_2Mf(x_1,x_2,x_3)$ for some constants $C_1,C_2>0$, it suffices to prove the results for the maximal function $N(|f|)(x_1,x_2,x_3)$ instead of $Mf(x_1,x_2,x_3)$. We may assume $f\geqslant 0$. Observe that $\mu_{j,k}$ are finite positive Borel measures which are uniformly bounded for all $j,k\in\mathbb{Z}$. By Theorem 1 [3], we need to show that the following inequalities hold for all $j,k\in\mathbb{Z}$ and for some fixed $\alpha,\beta>0$:

$$\big|\hat{\mu}_{j,k}(\zeta_1,\zeta_2,\zeta_3)\big|\leqslant C|2^k\zeta_1|^{-\alpha}|2^j\zeta_2|^{-\beta},\tag{1}$$

$$\big|\Delta^1_{\zeta_1}\hat{\mu}_{j,k}(0,\zeta_2,\zeta_3)\big|\leqslant C|2^k\zeta_1|^{\alpha}|2^j\zeta_2|^{-\beta},\tag{2}$$

$$\big|\Delta^2_{\zeta_2}\hat{\mu}_{j,k}(\zeta_1,0,\zeta_3)\big|\leqslant C|2^k\zeta_1|^{-\alpha}|2^j\zeta_2|^{\beta},\tag{3}$$

$$\big|\Delta^{1,2}_{\zeta_1,\zeta_2}\hat{\mu}_{j,k}(0,0,\zeta_3)\big|\leqslant C|2^k\zeta_1|^{\alpha}|2^j\zeta_2|^{\beta},\tag{4}$$

where

$$\begin{aligned}
&\Delta^1_{h_1}f(x_1,x_2,x_3):=f(x_1+h_1,x_2,x_3)-f(x_1,x_2,x_3),\\
&\Delta^2_{h_2}f(x_1,x_2,x_3):=f(x_1,x_2+h_2,x_3)-f(x_1,x_2,x_3)
\end{aligned}$$

and

$$\Delta^{1,2}_{h_1,h_2}f(x_1,x_2,x_3):=\Delta^1_{h_1}\big(\Delta^2_{h_2}f(x_1,x_2,x_3)\big).$$

The Fourier transform of $\mu_{j,k}$ is

$$\hat{\mu}_{j,k}(\zeta_1,\zeta_2,\zeta_3)=\frac{1}{2^{nk+mj}}\int\limits_{|y_2|\cong 2^j}\int\limits_{|y_1|\cong 2^k} e^{i(\zeta_1\cdot y_1+\zeta_2\cdot y_2+\zeta_3\Gamma(|y_1|,|y_2|))}\,dy_1\,dy_2.$$

The estimates near zero are trivial, because the factors $(e^{i\zeta_1\cdot y_1}-1)$ and $(e^{i\zeta_2\cdot y_2}-1)$ are present in the integrand. Therefore, we only prove inequality (1). Denote

$$I_1(\zeta_1)=\frac{1}{2^{nk}}\int\limits_{|y_1|\cong 2^k} e^{i(\zeta_1\cdot y_1+\zeta_3\Gamma(|y_1|,|y_2|))}\,dy_1$$

and

$$I_2(\zeta_2)=\frac{1}{2^{mj}}\int\limits_{|y_2|\cong 2^j} e^{i(\zeta_2\cdot y_2+\zeta_3\Gamma(|y_1|,|y_2|))}\,dy_2.$$

Then

$$\hat{\mu}_{j,k}(\zeta_1,\zeta_2,\zeta_3)=\frac{1}{2^{mj}}\int\limits_{|y_2|\cong 2^j} e^{i\zeta_2\cdot y_2} I_1(\zeta_1)\,dy_2 \tag{5}$$

$$=\frac{1}{2^{nk}}\int\limits_{|y_1|\cong 2^k} e^{i\zeta_1\cdot y_1} I_2(\zeta_2)\,dy_1. \tag{6}$$

We first obtain the estimates of $I_1(\zeta_1)$ by considering three separate cases: $n=1$, $n=2$ and $n\geqslant 3$.

Case 1. $n=1$. We write

$$I_1(\zeta_1)=\frac{1}{2^k}\int_{2^k}^{2^{k+1}} e^{i\zeta_1 s+i\zeta_3 h_{|y_2|}(s)}\,ds+\frac{1}{2^k}\int_{2^k}^{2^{k+1}} e^{-i\zeta_1 s+i\zeta_3 h_{|y_2|}(s)}\,ds$$
$$\equiv J_1(\zeta_1)+J_2(\zeta_1),$$

where $h_{|y_2|}(s)=\Gamma(s,|y_2|)$. To obtain the estimates of $J_1(\zeta_1)$ and $J_2(\zeta_1)$, we need the following lemma.

Lemma 1 [4]. *Let $\phi_k(t)=2^k\zeta_1 t+\zeta_3 h(2^k t)$, where $\zeta_1,\zeta_3\in\mathbb{R}$, and $k\in\mathbb{Z}$. Let $J_R=\int_1^R e^{i\phi_k(t)}\,dt$ for $1\leqslant R\leqslant 2$. Suppose the function $h(t)$ defined on $(0,\infty)$ satisfies one of the following conditions*:

(e) *$h'(t)>0$, $h''(t)>0$ and $h'(t)/t$ is increasing for all $t>0$,*
(f) *$h'(t)<0$, $h''(t)>0$ and $t\,h'(t)$ is increasing for all $t>0$, or*
(g) *$h'(t)>0$, $h''(t)<0$ and $t\,h'(t)$ is decreasing for all $t>0$.*

Then $J_R\leqslant C|2^k\zeta_1|^{-1/2}$, where C is independent of the particular function $h(t)$.

Proof of Lemma 1. The proof of case (e) is given in [4]. The idea for the proof of the remaining cases is similar to the proof of case (e). For convenience, we present the proof of case (f) in detail. It is enough to prove the lemma when $k=0$. For $k=0$, $\phi_0(t)=\zeta_1 t+\zeta_3 h(t)$, and thus $\phi_0'(t)=\zeta_1+\zeta_3 h'(t)$. We first consider the case $\zeta_3>0$.

If $\zeta_1\leqslant 0$, then $\phi_0'(t)\leqslant\zeta_1$. Thus $|\phi_0'(t)|\geqslant|\zeta_1|$, and the result follows from van der Corput's lemma. If $\zeta_1>0$, then there is a unique t_0 such that $\phi_0'(t_0)=\zeta_1+\zeta_3 h'(t_0)=0$. Let $t_1=\min\{t_0,2\}$, $\delta=|\zeta_1|^{-1/2}$, and decompose $J_R=\int_{A_1}\cdots+\int_{A_2}\cdots+\int_{A_3}\cdots\equiv J_{R,1}+J_{R,2}+J_{R,3}$, where $A_1=[1,R]\cap[t_1-\delta,t_1+\delta]$, $A_2=[1,t_1-\delta]$ and $A_3=[t_1+\delta,R]$. It is clear that $|J_{R,1}|\leqslant 2\delta=2|\zeta_1|^{-1/2}$. Because of the van der Corput's lemma, it is enough to show that $|\phi_0'(t)|\geqslant\frac{1}{2}|\zeta_1|^{1/2}$ if $t\in A_2$ or $t\in A_3$. Now if $t\in A_2$, then $t\leqslant t_1-\delta\leqslant t_0-\delta<t_0$, and

$$\begin{aligned}\phi_0'(t)&=\zeta_1+\zeta_3 h'(t)\frac{t}{t}\leqslant\zeta_1+\zeta_3 h'(t_1)\frac{t_1}{t}\leqslant\zeta_1+\zeta_3 h'(t_0)\frac{t_1}{t}\\&=\zeta_1\left(1-\frac{t_1}{t}\right)\leqslant\zeta_1\frac{(-\delta)}{t}\leqslant\zeta_1\frac{(-\delta)}{2},\end{aligned}$$

whence $|\phi_0'(t)|\geqslant\frac{1}{2}|\zeta_1|^{1/2}$. On the other hand, $A_3=\phi$ unless $t_1=t_0\leqslant 2$. Thus if $t\in A_3$, then $t>t_1=t_0$, and

$$\begin{aligned}\phi_0'(t)&=\zeta_1+\zeta_3 h'(t)\frac{t}{t}\geqslant\zeta_1+\zeta_3 h'(t_0)\frac{t_0}{t}=\zeta_1\left(1-\frac{t_0}{t}\right)\\&\geqslant\zeta_1\frac{\delta}{t}\geqslant|\zeta_1|\frac{\delta}{2}\geqslant\frac{1}{2}|\zeta_1|^{1/2}.\end{aligned}$$

The proof for the case $\zeta_3<0$ is essentially similar to the above proof. We omit the details here. Lemma 1 is proved. □

Remark 2. Note that the constant C in Lemma 1 is independent of the function $h(t)$. In particular, if $h(s)=h_{|y_2|}(s)=\Gamma(s,|y_2|)$, then C is independent of $|y_2|$.

We now obtain the estimates of $J_1(\zeta_1)$. If $h_{|y_2|}(s)$ satisfies hypothesis (a), (b) or (c) of Theorem 1, then by Lemma 1, $J_1(\zeta_1)\leqslant C|2^k\zeta_1|^{-1/2}$, where C is independent of $|y_2|$. If $h_{|y_2|}(s)$ satisfies hypothesis (d), then integrating $J_1(\zeta_1)$ by parts yields

$$J_1(\zeta_1)\leqslant C|2^k\zeta_1|^{-1}\left\{1+\int_{2^k}^{2^{k+1}}|\zeta_3|\left|h'_{|y_2|}(s)\right|ds\right\}.$$

If $|\zeta_3|\leqslant 1$, then the above integral is no greater than $2\|\Gamma\|_\infty$. If $|\zeta_3|>1$, then by a change of variable $s\to|\zeta_3|s$ and by hypothesis (d), the integral above is again dominated by $2\|\Gamma\|_\infty$. In either case, $J_1(\zeta_1)\leqslant C|2^k\zeta_1|^{-1}\leqslant C|2^k\zeta_1|^{-1/2}$. The last inequality follows if $|2^k\zeta_1|>1$. By the same argument, we have $J_2(\zeta_1)\leqslant C|2^k\zeta_1|^{-1/2}$, and consequently

$$I_1(\zeta_1)\leqslant C|2^k\zeta_1|^{-1/2}. \tag{7}$$

Case 2. $n=2$. Note that

$$
\begin{aligned}
I_1(\zeta_1) &= \frac{1}{2^{2k}} \int_{|y_1| \cong 2^k} e^{i(\zeta_1 \cdot y_1 + \zeta_3 \Gamma(|y_1|,|y_2|))}\, dy_1 \\
&= \frac{\omega_n}{2^{2k}} \int_0^{\pi} \int_{2^k}^{2^{k+1}} e^{i(r|\zeta_1|\cos\theta + \zeta_3 h_{|y_2|}(r))} r\, dr\, d\theta \\
&= \omega_n \int_0^{\pi} \int_1^2 e^{i(2^k r|\zeta_1|\cos\theta + \zeta_3 h_{|y_2|}(2^k r))} r\, dr\, d\theta \\
&= \omega_n \left\{ \int_0^{\pi/2-\delta} \int_1^2 \cdots + \int_{\pi/2-\delta}^{\pi/2+\delta} \int_1^2 \cdots + \int_{\pi/2+\delta}^{\pi} \int_1^2 \cdots \right\} \\
&\equiv \omega_n \{J_1 + J_2 + J_3\},
\end{aligned}
$$

where ω_n is a constant depending on n, and $0 < \delta < 1$. This δ will be chosen later. Denote

$$
K = \int_1^2 e^{i(2^k r|\zeta_1|\cos\theta + \zeta_3 h_{|y_2|}(2^k r))} r\, dr = \int_1^2 G'(r) r\, dr,
$$

where

$$
G(r) = \int_1^r e^{i\phi(t)}\, dt \quad \text{and} \quad \phi(t) = 2^k|\zeta_1|(\cos\theta)t + \zeta_3 h_{|y_2|}(2^k t).
$$

If $h_{|y_2|}(2^k t)$ satisfies hypothesis (a), (b), or (c) of Theorem 1, then by an application of Lemma 1 (with ζ_1 being replaced by $|\zeta_1|\cos\theta$), we obtain $|G(r)| \leqslant C|2^k \zeta_1 \cos\theta|^{-1/2}$. Integrating K by parts yields $|K| \leqslant C|G(r)| \leqslant C|2^k\zeta_1\cos\theta|^{-1/2}$. If $h_{|y_2|}(2^k t)$ satisfies hypothesis (d), then by integrating by parts we have

$$
|K| \leqslant C|2^k \zeta_1 \cos\theta|^{-1} \left\{ 1 + \int_{2^k}^{2^{k+1}} |\zeta_3| h'_{|y_2|}(r)\, dr \right\} \leqslant C|2^k\zeta_1\cos\theta|^{-1}.
$$

The last inequality follows since the above integral is dominated by $2\|\Gamma\|_\infty$. In all cases, $|K| \leqslant C|2^k\zeta_1\cos\theta|^{-\alpha}$, where $\alpha = 1$ or $1/2$. Thus

$$
\begin{aligned}
|J_1| &\leqslant C \int_0^{\pi/2-\delta} |2^k\zeta_1\cos\theta|^{-\alpha}\, d\theta \leqslant C(\pi/2-\delta)\big|2^k\zeta_1\cos(\pi/2-\delta)\big|^{-\alpha} \\
&\leqslant C|2^k\zeta_1|^{-\alpha}(\sin\delta)^{-\alpha} \leqslant C|2^k\zeta_1|^{-\alpha}(\delta)^{-\alpha}.
\end{aligned}
$$

The last inequality follows because $\sin\delta \geqslant \frac{2\delta}{\pi}$ for $0 < \delta < 1$. By the same argument, $|J_3| \leqslant C|2^k\zeta_1|^{-\alpha}(\delta)^{-\alpha}$. On the other hand, it is obvious that $|J_2| \leqslant C\delta$. We choose $\delta = |2^k\zeta_1|^{-1/2}$ if $|2^k\zeta_1| > 1$. Then (recall that $\alpha = 1$ or $1/2$)

$$
I_1(\zeta_1) \leqslant \omega_n\{J_1 + J_2 + J_3\} \leqslant C|2^k\zeta_1|^{-1/4} \quad \text{if } |2^k\zeta_1| > 1. \tag{8}
$$

Case 3. $n \geqslant 3$. Note that

$$\begin{aligned} I_1(\zeta_1) &= \frac{1}{2^{nk}} \int\limits_{S^{n-1}} \left(\int\limits_{2^k}^{2^{k+1}} e^{i\{|\zeta_1| r(\zeta_1' \cdot y_1') + \zeta_3 h_{|y_2|}(r)\}} r^{n-1}\, dr \right) d\sigma(y_1') \\ &= \frac{\omega_n}{2^{nk}} \int\limits_{2^k}^{2^{k+1}} e^{i\zeta_3 h_{|y_2|}(r)} r^{n-1} \left(\int\limits_{-1}^{1} (1-s^2)^{(n-3)/2} e^{ir|\zeta_1| s}\, ds \right) dr \\ &\equiv \frac{\omega_n}{2^{nk}} \int\limits_{2^k}^{2^{k+1}} e^{i\zeta_3 h_{|y_2|}(r)} r^{n-1} K_1\big(|\zeta_1|\big)\, dr. \end{aligned}$$

$$\text{If } n = 3,\ \text{then } K_1\big(|\zeta_1|\big) = \frac{2\sin(r|\zeta_1|)}{r|\zeta_1|},\ \text{and thus } \big|I_1(\zeta_1)\big| \leqslant C|2^k\zeta_1|^{-1}. \tag{9}$$

$$\text{If } n \geqslant 4,\ \text{then integrating by parts yields } \big|K_1\big(|\zeta_1|\big)\big| \leqslant \frac{C}{r|\zeta_1|}$$

$$\text{so that } \big|I_1(\zeta_1)\big| \leqslant C|2^k\zeta_1|^{-1}. \tag{10}$$

Combining inequalities (7)–(10), we obtain

$$\big|I_1(\zeta_1)\big| \leqslant C|2^k\zeta_1|^{-1/4} \quad \text{if } |2^k\zeta_1| > 1 \text{ and } n \geqslant 1. \tag{11}$$

By symmetry

$$\big|I_2(\zeta_2)\big| \leqslant C|2^j\zeta_2|^{-1/4} \quad \text{if } |2^j\zeta_2| > 1 \text{ and } m \geqslant 1. \tag{12}$$

Inequalities (5) and (11) imply that

$$\big|\hat{\mu}_{j,k}(\zeta_1,\zeta_2,\zeta_3)\big| \leqslant C|2^k\zeta_1|^{-1/4}, \quad n, m \geqslant 1. \tag{13}$$

Similarly, combining inequalities (6) and (12) yields

$$\big|\hat{\mu}_{j,k}(\zeta_1,\zeta_2,\zeta_3)\big| \leqslant C|2^j\zeta_2|^{-1/4}, \quad n, m \geqslant 1, \tag{14}$$

which together with inequality (13) implies that

$$\big|\hat{\mu}_{j,k}(\zeta_1,\zeta_2,\zeta_3)\big| \leqslant C|2^k\zeta_1|^{-1/8}|2^j\zeta_2|^{-1/8}, \quad n, m \geqslant 1. \tag{15}$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. It suffices to show that these maximal functions are controlled by the Hardy–Littlewood maximal functions. We first consider the maximal function $M_{1,2}g(x_3)$. Suppose $h_t(s)$ is a type I function for every fixed $t > 0$. We may assume $g \geqslant 0$. Since $h_t(s)$ is strictly increasing on $[0, \infty)$ for each fixed $t > 0$, $h_t'(s) > 0$ on $(0, \infty)$. By the inverse function theorem, h_t^{-1} exists and is a C^1 function for each fixed $t > 0$. By a change of variable $w_t = h_t(s)$, we have

$$\begin{aligned}\frac{1}{r_1}\int_0^{r_1} g\bigl(x_3-\Gamma(s,t)\bigr)\,ds &= \frac{1}{r_1}\int_0^{r_1} g\bigl(x_3-h_t(s)\bigr)\,ds\\ &= \frac{1}{r_1}\int_{h_t(0)}^{h_t(r_1)} g(x_3-w_t)\frac{1}{h_t'(h_t^{-1}(w_t))}\,dw_t\\ &= \frac{1}{r_1}\int_{h_t(0)}^{h_t(r_1)} g(x_3-w_t)\bigl(h_t^{-1}\bigr)'(w_t)\,dw_t\\ &= g*\psi_{r_1}(x_3),\end{aligned}$$

where

$$\psi_{r_1}(w_t)=\chi_{[h_t(0),h_t(r_1)]}(w_t)\frac{(h_t^{-1})'(w_t)}{r_1}$$

is decreasing on $[h_t(0),h_t(r_1)]$. Note also that $\int_{\mathbb{R}}\psi_{r_1}(w_t)\,dw_t=1$ for all $r_1>0$ and for each fixed $t>0$. Therefore, $g*\psi_{r_1}(x_3)\leqslant M^Hg(x_3)$ for all $r_1>0$ and for each fixed $t>0$. Here $M^Hg(x_3)$ stands for the Hardy–Littlewood maximal function. It follows that $M_{1,2}g(x_3)\leqslant M^Hg(x_3)$.

The proof for the case that $h_t(s)$ is a type II function is essentially the same. Now suppose $h_t(s)$ is a type III function (with $sh_t'(s)\geqslant\alpha h_t(s)$ for all $s>0$ and some fixed $\alpha>0$ independent of s and t). We have

$$\begin{aligned}\frac{1}{r_1}\int_0^{r_1} g\bigl(x_3-\Gamma(s,t)\bigr)\,ds &= \frac{1}{r_1}\int_{h_t(0)}^{h_t(r_1)} g(x_3-w_t)\frac{1}{h_t'(h_t^{-1}(w_t))}\,dw_t\\ &\leqslant \frac{1}{r_1}\int_0^{h_t(r_1)} g(x_3-w_t)\frac{1}{h_t'(r_1)}\,dw_t\\ &\leqslant \frac{C}{h_t(r_1)}\int_0^{h_t(r_1)} g(x_3-w_t)\,dw_t\\ &\leqslant CM^Hg(x_3),\end{aligned}$$

where C is independent of r_1 and t. It follows that $M_{1,2}g(x_3)\leqslant M^Hg(x_3)$.

We now consider the maximal function $M_1g_1(x_1,x_3)$. Using the above result, it is clear that $M_1g_1(x_1,x_3)\leqslant CM_1^H\circ M_2^Hg_1(x_1,x_3)$, where M_i^H $(i=1,2)$ denotes the Hardy–Littlewood maximal function acting on the ith variable. By symmetry, it follows that $M_2g_2(x_2,x_3)\leqslant CM_1^H\circ M_2^Hg_2(x_2,x_3)$. Theorem 2 is proved. □

Proof of the corollaries. By an application of Theorem 1, we only need to prove the L^p boundedness of the partial maximal functions $M_\Gamma^{(1)}f(x_1,x_2,x_3)$, $M_\Gamma^{(2)}f(x_1,x_2,x_3)$ and $M_\Gamma^{(1,2)}f(x_1,x_2,x_3)$. If $n=m=1$, then by Theorem 2, we have $M_\Gamma^{(1,2)}f(x_1,x_2,x_3)\leqslant$

$CM_3^H f(x_1, x_2, x_3)$, $M_\Gamma^{(1)} f(x_1, x_2, x_3) \leqslant CM_1^H \circ M_3^H f(x_1, x_2, x_3)$ and $M_\Gamma^{(2)} f(x_1, x_2, x_3) \leqslant CM_2^H \circ M_3^H f(x_1, x_2, x_3)$. Therefore these maximal functions are bounded in L^p. If $n \geqslant 2$ or $m \geqslant 2$, the result follows from the method of rotations and an application of Theorem 2. □

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