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A hybrid inexact Logarithmic–Quadratic Proximal method for nonlinear complementarity problems

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Abstract

Inspired by the Logarithmic–Quadratic Proximal method [A. Auslender, M. Teboulle, S. Ben-Tiba, A logarithmic–quadratic proximal method for variational inequalities, Comput. Optim. Appl. 12 (1999) 31–40], we present a new prediction–correction method for solving the nonlinear complementarity problems. In our method, an intermediate point is produced by approximately solving a nonlinear equation system based on the Logarithmic–Quadratic Proximal method; and the new iterate is obtained by convex combination of the previous point and the one generated by the improved extragradient method at each iteration. The proposed method allows for constant relative errors and this yields a more practical Logarithmic–Quadratic Proximal type method. The global convergence is established under mild conditions. Preliminary numerical results indicate that the method is effective for large-scale nonlinear complementarity problems.

Keywords: Logarithmic–Quadratic Proximal method; Monotone mapping; Nonlinear complementarity problem

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1. Introduction

We consider the nonlinear complementarity problem (NCP for abbreviation): find a vector $x \in R^n$ such that

$$x \geqslant 0, \quad F(x) \geqslant 0 \quad \text{and} \quad x^\top F(x) = 0, \tag{1}$$

where F maps from R^n into itself. Throughout this paper, we assume that F is continuous and monotone; and the solution set of (1), denoted by Ω^*, is nonempty. It is well known that NCP is equivalent to the following variational inequality problem:

$$\text{Find} \quad x^* \geqslant 0 \quad \text{such that} \quad (x - x^*)^\top F(x^*) \geqslant 0, \quad \forall x \geqslant 0. \tag{2}$$

NCP has received a lot of attention due to its various applications in operations research, economic equilibrium and engineering design [7,10,14]. Many numerical methods for solving NCP have been developed, e.g., see [8,13,15,18]. Among them is a class of iterative methods based on the theory of maximal monotone operators and variational inequalities.

Let $R^n_+ := \{x \in R^n \mid x \geqslant 0\}$ and the maximal monotone operator (see definition in [16])

$$T(x) := F(x) + N_{R^n_+}(x), \tag{3}$$

where $N_{R^n_+}(\cdot)$ is the normal cone operator to R^n_+, i.e.,

$$N_{R^n_+}(x) := \begin{cases} \{z \in R^n \mid \langle x' - x, z\rangle \leqslant 0, \ \forall x' \in R^n_+\} & \text{if } x \in R^n_+, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4}$$

Then solving (1) is equivalent to finding a root of $T(x)$. The proximal point algorithm (PPA), see, e.g., [9,12,16], is a classical approach to finding a zero point of $T(x)$. In particular, for given $x^k \in R^n_+$ and $\beta_k \geqslant \beta > 0$, the new iterate $x^{k+1} \in R^n_+$ generated by PPA is the solution of the following inclusion:

$$\text{(PPA)} \quad 0 \in \beta_k T(x) + \left(x - x^k\right). \tag{5}$$

In order to obtain the new point x^{k+1}, we often need to solve a variational inequality because in view of (3) and (4) the subproblem (5) of PPA is equivalent to the following variational inequality problem:

$$\text{Find} \quad x \in R^n_+, \qquad (x' - x)^\top \left(x - x^k + \beta_k F(x)\right) \geqslant 0, \quad \forall x' \in R^n_+. \tag{6}$$

In most cases, it is not an easy thing to deal with the problem (6).

Lately, a number of papers have concentrated on generalization of PPA by replacing the linear term $(x - x^k)$ with some nonlinear functions $r(x, x^k)$ (see [1,4–6,19]). Thus some "interior point" proximal methods have been developed for variational inequality problems. Recently, Auslender et al. [2] presented an inexact Logarithmic–Quadratic Proximal (LQP for abbreviation) method: for given $x^k \in R^n_{++} := \operatorname{int} R^n_+$, $\beta_k \geqslant \beta > 0$ and $\xi^k \in R^n$, the new iterate $x^{k+1} \in R^n_{++}$ is the solution of the following inclusion:

$$\text{(LQP)} \quad \xi^k \in \beta_k T(x) + \nabla_x D\left(x, x^k\right), \tag{7}$$

where

$$D\left(x, x^k\right) = \begin{cases} \frac{1}{2}\|x - x^k\|^2 + \mu \sum_{j=1}^n \left((x_j^k)^2 \log \frac{x_j^k}{x_j} + x_j x_j^k - (x_j^k)^2\right) & \text{if } x \in R^n_{++}, \\ +\infty & \text{otherwise} \end{cases}$$

with $\mu \in (0,1)$ which is a given constant. Notice that

$$\nabla_x D(x, x^k) = (x - x^k) + \mu(x^k - X_k^2 x^{-1}), \tag{8}$$

where $X_k = \operatorname{diag}(x_1^k, x_2^k, \ldots, x_n^k)$ and x^{-1} is an n-vector whose jth element is $1/x_j$. Since the iterate x^{k+1} lies in the interior set of R_+^n, based on (3)–(4), solving the subproblem (7)–(8) of LQP method is equivalent to solving the following system of nonlinear equations:

$$\beta_k F(x) + x - (1-\mu)x^k - \mu X_k^2 x^{-1} = \xi^k. \tag{9}$$

Then we only have to consider the nonlinear equation system (9) to get the positive solution of the subproblem (7). To ensure convergence of the inexact LQP method, Auslender et al. [2] supposed that

$$\sum_{k=0}^{\infty} \|\xi^k\| < +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} \langle \xi^k, x^k \rangle < +\infty. \tag{10}$$

Similar criteria were used, for example, in [16]. Such criteria are somewhat undesirable because they are assumptions on the whole generated sequence $\{x^k\}$ and the error sequence $\{\xi^k\}$. Now, it is therefore worthwhile to develop new algorithms which admit less stringent requirements on solving the subproblems (7), namely (9).

Inspired by He [11] and Solodov [17], we propose a new hybrid inexact Logarithmic–Quadratic Proximal method for nonlinear complementarity problems. At each iteration, an intermediate point which can be denoted as the predictor is obtained by solving the nonlinear equation system (9) under suitable inexact criterion; and then the new iterate is computed via convex combination of the previous point and the one generated by the improved extragradient method [11]. In particular, the restriction on ξ^k allows for constant relative error. And we shall prove that the proposed method is globally convergent under mild conditions.

This paper is organized as follows. In Section 2, we present the method and give some remarks; then we summarize some preliminaries on projection operator and nonlinear complementarity problems. In Section 3, some useful lemmas are obtained. In Section 4, we first handle the problem of choosing the optimal step size in the correction step and then prove global convergence of the new method. In Section 5, some preliminary numerical experiments are given to indicate that the proposed method is effective for large-scale nonlinear complementarity problems. Finally, some conclusions are drawn in Section 6.

Throughout the paper we use the Euclidean norm which will be denoted by $\|\cdot\|$.

2. Algorithm and remarks

First we present the basic framework of the new method.

Algorithm: A hybrid inexact LQP method for NCP

$\mu \in (0,1)$, $\eta \in (0,1)$, $\sigma \in (0,1)$ are given.

Step 1. Prediction part

For given $x^k \in R_{++}^n := \operatorname{int} R_+^n$, find $\beta_k > 0$ and $\tilde{x}^k$ such that if there holds

$$\beta_k F(\tilde{x}^k) + \tilde{x}^k - (1-\mu)x^k - \mu X_k^2 (\tilde{x}^k)^{-1} = \xi^k, \tag{11}$$

then $\tilde{x}^k \in R^n_{++}$ and

$$\|\xi^k\| \leqslant \eta\sqrt{1-\mu^2}\|x^k - \tilde{x}^k\|. \tag{12}$$

Step 2. Correction part

Compute the new iterate x^{k+1} via

$$x^{k+1} = (1-\sigma)x^k + \sigma P_{R^n_+}\left[x^k - \frac{\alpha}{1+\mu}\beta_k F(\tilde{x}^k)\right], \tag{13}$$

where $P_{R^n_+}$ denote the projection operator onto R^n_+, and α is the step length that will be specified later.

The following is the existence result of the prediction step.

Proposition 2.1. *For each* $\beta_k > 0$, $\xi^k \in R^n$, $x^k \in \operatorname{int} R^n_+$, *there exists a unique* $x^{k+1} \in \operatorname{int} R^n_+$ *satisfying* (11).

Proof. See the proof in Section 3 in [2]. □

There is a remark on the accuracy criterion (12).

Remark 2.1. Compared to (10), the new restriction on ξ^k (12) is relaxed and more practical. More specifically, the relative error $\frac{\xi^k}{\|x^k - \tilde{x}^k\|}$ can be fixed at the constant $\eta\sqrt{1-\mu^2}$.

Then we turn to consider the correction step (13). As far as the formula (13) is concerned, $\sigma \in (0, 1)$ assures that the new iterates lie in the interior of R^n_+. So it is convenient to use the inexact Logarithmic–Quadratic Proximal method in the next iteration. We will prove the convergence of the proposed method in the following sections.

We close this section with some definitions and basic properties which will be useful in subsequent analysis. In (13), the projection operator $P_{R^n_+}$ is defined as

$$P_{R^n_+}(z) = \operatorname{argmin}\{\|z - x\| \mid x \in R^n_+\}, \quad \forall z \in R^n.$$

From the above definition, it follows that

$$\{z - P_{R^n_+}(z)\}^\top \{x - P_{R^n_+}(z)\} \leqslant 0, \quad \forall z \in R^n,\ \forall x \in R^n_+. \tag{14}$$

Consequently, we have

$$\|P_{R^n_+}(y) - P_{R^n_+}(z)\| \leqslant \|y - z\|, \quad \forall y, z \in R^n, \tag{15}$$

and

$$\|P_{R^n_+}(y) - x\|^2 \leqslant \|y - x\|^2 - \|y - P_{R^n_+}(y)\|^2, \quad \forall x \in R^n_+,\ y \in R^n. \tag{16}$$

These properties can be seen in [3] for details. A function $F : R^n_+ \to R^n$ is said to be a monotone mapping with respect to R^n_+ if

$$(x - y)^\top \big(F(x) - F(y)\big) \geqslant 0, \quad \forall x, y \in R^n_+. \tag{17}$$

3. Some lemmas

In this section we prove some lemmas that are crucial in subsequent analysis.

The following proposition is similar to Lemma 2 in [2]. For completeness, we give the proof.

Lemma 3.1. *For given $x^k > 0$ and $\beta_k > 0$, let $\tilde{x}^k$ be obtained by prediction part* (11)–(12)*, then for any $x \geqslant 0$, we have*

$$\left(\tilde{x}^k - x\right)^\top \left(\xi^k - \beta_k F\left(\tilde{x}^k\right)\right) \geqslant \frac{1+\mu}{2}\left(\left\|\tilde{x}^k - x\right\|^2 - \left\|x^k - x\right\|^2\right) + \frac{1-\mu}{2}\left\|x^k - \tilde{x}^k\right\|^2. \tag{18}$$

Proof. We prove (18) element-wise for $j = 1, \ldots, n$. Note that $\tilde{x}_j^k > 0$. Then we have

$$\left(x_j^k\right)^2 / \tilde{x}_j^k \geqslant 2x_j^k - \tilde{x}_j^k \quad \text{and} \quad x_j\left(x_j^k\right)^2 / \tilde{x}_j^k \geqslant x_j\left(2x_j^k - \tilde{x}_j^k\right).$$

It follows from (11) that

$$\begin{aligned}
&\left(\tilde{x}_j^k - x_j\right)\left(\xi_j^k - \beta_k F_j\left(\tilde{x}^k\right)\right) \\
&\quad = \left(\tilde{x}_j^k - x_j\right)\left(\tilde{x}_j^k - (1-\mu)x_j^k - \mu\left(x_j^k\right)^2/\tilde{x}_j^k\right) \\
&\quad \geqslant \left(\tilde{x}_j^k\right)^2 - (1-\mu)\tilde{x}_j^k x_j^k - \mu\left(x_j^k\right)^2 - \tilde{x}_j^k x_j + (1-\mu)x_j^k x_j + \mu x_j\left(2x_j^k - \tilde{x}_j^k\right) \\
&\quad = \left(\tilde{x}_j^k\right)^2 - (1-\mu)\tilde{x}_j^k x_j^k - \mu\left(x_j^k\right)^2 - (1+\mu)\tilde{x}_j^k x_j + (1+\mu)x_j^k x_j \\
&\quad = \frac{1+\mu}{2}\left(\left(\tilde{x}_j^k - x_j\right)^2 - \left(x_j^k - x_j\right)^2\right) + \frac{1-\mu}{2}\left(x_j^k - \tilde{x}_j^k\right)^2.
\end{aligned}$$

Thus (18) holds and the proof is completed. □

The following lemma is another important property for convergence analysis.

Lemma 3.2. *For given $x^k > 0$ and $\beta_k > 0$, let $\tilde{x}^k$ be obtained by prediction part* (11)–(12)*, then for any $x \geqslant 0$, we have*

$$\left(x - \tilde{x}^k\right)^\top \left(\beta_k F\left(\tilde{x}^k\right) - \xi^k\right) \geqslant \left(x^k - \tilde{x}^k\right)^\top \left((1+\mu)x - \left(\mu x^k + \tilde{x}^k\right)\right). \tag{19}$$

Proof. By a manipulation, we have

$$\begin{aligned}
&\frac{1+\mu}{2}\left(\left\|\tilde{x}^k - x\right\|^2 - \left\|x^k - x\right\|^2\right) + \frac{1-\mu}{2}\left\|x^k - \tilde{x}^k\right\|^2 \\
&\quad = (1+\mu)x^\top x^k - (1+\mu)x^\top \tilde{x}^k - (1-\mu)\left(\tilde{x}^k\right)^\top x^k - \mu\left\|x^k\right\|^2 + \left\|\tilde{x}^k\right\|^2 \\
&\quad = (1+\mu)x^\top\left(x^k - \tilde{x}^k\right) - \left(x^k - \tilde{x}^k\right)^\top\left(\mu x^k + \tilde{x}^k\right) \\
&\quad = \left(x^k - \tilde{x}^k\right)^\top\left((1+\mu)x - \left(\mu x^k + \tilde{x}^k\right)\right).
\end{aligned}$$

The assertion follows from (18) and the above identity immediately. □

4. Main results

The following lemma establishes the Fejér monotonicity properties of $\{x^k\}$ generated by our hybrid inexact method.

Lemma 4.1. *For given $x^k > 0$ and $\beta_k > 0$, let $\tilde{x}^k$ be obtained by prediction part* (11)–(12). *If condition* (12) *is satisfied and the new iterate x^{k+1} is given by* (13), *then for any $\alpha > 0$ and $x^* \in \Omega^*$, we have*

$$\Theta_k(\alpha) := \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 \geqslant \Phi_k(\alpha), \tag{20}$$

where

$$\Phi_k(\alpha) := \sigma\big(2\alpha\varphi_k - \alpha^2\|d^k\|^2\big), \tag{21}$$

with

$$\varphi_k := \frac{1}{1+\mu}\big(x^k - \tilde{x}^k\big)^\top\big(x^k - \tilde{x}^k + \xi^k\big) \tag{22}$$

and

$$d^k := \big(x^k - \tilde{x}^k\big) + \frac{1}{1+\mu}\xi^k. \tag{23}$$

Proof. First, since $x^{k+1} \geqslant 0$, setting $x = x^{k+1}$ in (18), we have

$$\begin{aligned}&\big(\tilde{x}^k - x^{k+1}\big)^\top \frac{1}{1+\mu}\big(\xi^k - \beta_k F\big(\tilde{x}^k\big)\big)\\ &\quad\geqslant \frac{1}{2}\big(\|\tilde{x}^k - x^{k+1}\|^2 - \|x^k - x^{k+1}\|^2\big) + \frac{1-\mu}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2.\end{aligned} \tag{24}$$

Notice that the following is an identity:

$$\big(\tilde{x}^k - x^{k+1}\big)^\top\big(x^k - \tilde{x}^k\big) = -\frac{1}{2}\|x^k - \tilde{x}^k\|^2 + \frac{1}{2}\|x^k - x^{k+1}\|^2 - \frac{1}{2}\|\tilde{x}^k - x^{k+1}\|^2. \tag{25}$$

Adding (24) and (25), according to (23), we have the following:

$$\big(\tilde{x}^k - x^{k+1}\big)^\top\left(d^k - \frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\right) \geqslant -\frac{\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2. \tag{26}$$

Using the monotonicity of F, i.e., (17), we have

$$\big(\tilde{x}^k - x^*\big)^\top \frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big) \geqslant \big(\tilde{x}^k - x^*\big)^\top \frac{1}{1+\mu}\beta_k F(x^*) \geqslant 0. \tag{27}$$

The right inequality in (27) follows from (2) since $\tilde{x}^k > 0$ and $x^* \in \Omega^*$.

It follows from (13) and (16) that

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &= \left\|(1-\sigma)x^k + \sigma P_{R^n_+}\left[x^k - \frac{\alpha}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\right] - x^*\right\|^2\\ &\leqslant \left((1-\sigma)\|x^k - x^*\| + \sigma\left\|P_{R^n_+}\left[x^k - \frac{\alpha}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\right] - x^*\right\|\right)^2\\ &\leqslant (1-\sigma)\|x^k - x^*\|^2 + \sigma\left\|P_{R^n_+}\left[x^k - \frac{\alpha}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\right] - x^*\right\|^2\\ &\overset{(16)}{\leqslant} \|x^k - x^*\|^2 + \sigma\left(-2\alpha\big(x^k - x^*\big)^\top \frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\right)\\ &\qquad + \sigma\left(-\|x^k - x^{k+1}\|^2 + 2\alpha\big(x^k - x^{k+1}\big)\frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\right).\end{aligned} \tag{28}$$

In view of (28) and the notation of $\Theta_k(\alpha)$, we have

$$\begin{aligned}
\Theta_k(\alpha) &\geqslant 2\sigma\alpha\big(x^k-x^*\big)^\top\frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)+\sigma\big\|x^k-x^{k+1}\big\|^2\\
&\quad -2\sigma\alpha\big(x^k-x^{k+1}\big)^\top\frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\\
&\overset{(23)}{=} \sigma\big\|x^k-x^{k+1}-\alpha d^k\big\|^2-\sigma\alpha^2\big\|d^k\big\|^2\\
&\quad +2\sigma\alpha\big(x^k-\tilde{x}^k+\tilde{x}^k-x^{k+1}\big)^\top\bigg(d^k-\frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\bigg)\\
&\quad +2\sigma\alpha\big(x^k-\tilde{x}^k+\tilde{x}^k-x^*\big)^\top\frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\\
&\geqslant -\sigma\alpha^2\big\|d^k\big\|^2+2\sigma\alpha\big(x^k-\tilde{x}^k\big)^\top\bigg(d^k-\frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\bigg)\\
&\quad -2\sigma\alpha\frac{\mu}{1+\mu}\big\|x^k-\tilde{x}^k\big\|^2+2\sigma\alpha\big(x^k-\tilde{x}^k\big)^\top\frac{1}{1+\mu}\beta_k F\big(\tilde{x}^k\big)\\
&\geqslant -\sigma\alpha^2\big\|d^k\big\|^2+2\sigma\alpha\big(x^k-\tilde{x}^k\big)^\top\frac{1}{1+\mu}\big(x^k-\tilde{x}^k+\xi^k\big)\\
&= -\sigma\alpha^2\big\|d^k\big\|^2+2\sigma\alpha\varphi_k
\end{aligned}\tag{29}$$

using (26) and (27) in the second inequality and the definition of φ_k in the last equation. According to the notation of $\Phi_k(\alpha)$, notice that the assertion of the lemma is obtained immediately from (29). □

Since $\Phi_k(\alpha)$ (21) is a concave quadratic function of α, it reaches its maximum at

$$\alpha_k^*=\frac{\varphi_k}{\|d^k\|^2}\quad\text{with } \Phi_k\big(\alpha_k^*\big)=\sigma\alpha_k^*\varphi_k. \tag{30}$$

Note that under condition (12) we have

$$\begin{aligned}
2\varphi_k &\overset{(22)}{=} \frac{2}{1+\mu}\big\|x^k-\tilde{x}^k\big\|^2+\frac{2}{1+\mu}\big(x^k-\tilde{x}^k\big)^\top\xi^k\\
&= \big\|x^k-\tilde{x}^k\big\|^2+\frac{2}{1+\mu}\big(x^k-\tilde{x}^k\big)^\top\xi^k+\frac{1-\mu}{1+\mu}\big\|x^k-\tilde{x}^k\big\|^2\\
&\overset{(12)}{\geqslant} \big\|x^k-\tilde{x}^k\big\|^2+\frac{2}{1+\mu}\big(x^k-\tilde{x}^k\big)^\top\xi^k+\frac{1}{(1+\mu)^2}\big\|\xi^k\big\|^2\\
&\overset{(23)}{=} \big\|d^k\big\|^2
\end{aligned}\tag{31}$$

and

$$\begin{aligned}
\varphi_k &\overset{(22)}{\geqslant} \frac{1}{1+\mu}\big(\big\|x^k-\tilde{x}^k\big\|^2-\big\|x^k-\tilde{x}^k\big\|\cdot\big\|\xi^k\big\|\big)\\
&\overset{(12)}{\geqslant} \frac{1}{1+\mu}\big(\big\|x^k-\tilde{x}^k\big\|^2-\eta\sqrt{1-\mu^2}\big\|x^k-\tilde{x}^k\big\|^2\big)\\
&= \frac{1-\eta\sqrt{1-\mu^2}}{1+\mu}\big\|x^k-\tilde{x}^k\big\|^2.
\end{aligned}\tag{32}$$

Therefore, it follows from (30) and (31) that

$$\alpha_k^* \geqslant \frac{1}{2}. \tag{33}$$

And using (30), (32) and (33), we obtain

$$\Phi_k(\alpha_k^*) \geqslant \sigma \frac{1-\eta\sqrt{1-\mu^2}}{2(1+\mu)} \|x^k - \tilde{x}^k\|^2. \tag{34}$$

For fast convergence, we propose a relaxation factor $\gamma \in [1, 2)$ (its better value is close to 2) and set the step length α in (13) by $\alpha = \gamma\alpha_k^*$. Our recommended correction form is

$$x^{k+1} = (1-\sigma)x^k + \sigma P_{R_+^n}\left[x^k - \frac{\gamma\alpha_k^*}{1+\mu}\beta_k F(\tilde{x}^k)\right], \quad \sigma \in (0, 1]. \tag{35}$$

By simple manipulations we obtain

$$\begin{aligned}\Phi_k(\gamma\alpha_k^*) &\overset{(21)}{=} \sigma\{2\gamma\alpha_k^*\varphi_k - (\gamma^2\alpha_k^*)(\alpha_k^*\|d^k\|^2)\} \\ &\overset{(30)}{=} \sigma(2\gamma-\gamma^2)\alpha_k^*\varphi_k \\ &\overset{(30)}{=} \gamma(2-\gamma)\Phi_k(\alpha_k^*).\end{aligned} \tag{36}$$

It follows from Lemma 4.1 and (34) that

$$\|x^{k+1} - x^*\|^2 \leqslant \|x^k - x^*\|^2 - \sigma\frac{\gamma(2-\gamma)(1-\eta\sqrt{1-\mu^2})}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2. \tag{37}$$

From (37), it can be seen that $\{x^k\}$ is a bounded sequence and

$$\lim_{k\to\infty}\|x^k - \tilde{x}^k\| = 0. \tag{38}$$

Hence, $\{\tilde{x}^k\}$ is also bounded. Let $c := \sigma\frac{\gamma(2-\gamma)(1-\eta\sqrt{1-\mu^2})}{2(1+\mu)}$. (It is obvious that $c > 0$.) Since (37) is true for any $x^* \in \Omega^*$, we have

$$[\operatorname{dist}(x^{k+1}, \Omega^*)]^2 \leqslant [\operatorname{dist}(x^k, \Omega^*)]^2 - c\|x^k - \tilde{x}^k\|^2, \tag{39}$$

where $\operatorname{dist}(x, \Omega^*) := \inf\{\|x - x^*\| \mid x^* \in \Omega^*\}$. Then we conclude that $\{x^k\}$ is Fejér monotone with respect to Ω^*, i.e., the solution set of the NCP. Now, we are ready to prove the convergence of the proposed method.

Theorem 4.1. *If* $\inf_{k=0}^{\infty}\beta_k := \beta > 0$*, then the sequence* $\{x^k\}$ *generated by the proposed method converges to some* x^∞ *which is a solution of the NCP.*

Proof. First, it follows from (19) that

$$(x - \tilde{x}^k)^\top(\beta_k F(\tilde{x}^k) - \xi^k) \geqslant (x^k - \tilde{x}^k)^\top((1+\mu)x - (\mu x^k + \tilde{x}^k)), \quad \forall x \in R_+^n.$$

Since $\lim_{k\to\infty}\|x^k - \tilde{x}^k\| = 0$, $\|\xi^k\| \leqslant \|x^k - \tilde{x}^k\|$ and $\beta_k \geqslant \beta > 0$, we have

$$\lim_{k\to\infty}(x - \tilde{x}^k)^\top F(\tilde{x}^k) \geqslant 0, \quad \forall x \in R_+^n.$$

Because $\{\tilde{x}^k\}$ is bounded, it has at least a cluster point. Let x^∞ be a cluster point of $\{\tilde{x}^k\}$ and the subsequence $\{\tilde{x}^{k_j}\}$ converges to x^∞. It follows that

$$\lim_{j\to\infty}\left(x-\tilde{x}^{k_j}\right)^\top F\left(\tilde{x}^{k_j}\right)\geqslant 0,\quad \forall x\in R_+^n,$$

and consequently

$$\left(x-x^\infty\right)^\top F\left(x^\infty\right)\geqslant 0,\quad \forall x\in R_+^n.$$

This means that x^∞ is a solution of the NCP. Note the inequality (37) is true for all solution points of the NCP, hence, we have

$$\left\|x^{k+1}-x^\infty\right\|^2\leqslant\left\|x^k-x^\infty\right\|^2,\quad \forall k\geqslant 0. \tag{40}$$

Since $\tilde{x}^{k_j}\to x^\infty$ $(j\to\infty)$ and $x^k-\tilde{x}^k\to 0$ $(k\to\infty)$, for any given $\varepsilon>0$, there exists an $l>0$ such that

$$\left\|\tilde{x}^{k_l}-x^\infty\right\|<\varepsilon/2 \quad\text{and}\quad \left\|x^{k_l}-\tilde{x}^{k_l}\right\|<\varepsilon/2. \tag{41}$$

Therefore, for any $k\geqslant k_l$, it follows from (40) and (41) that

$$\left\|x^k-x^\infty\right\|\leqslant\left\|x^{k_l}-x^\infty\right\|\leqslant\left\|x^{k_l}-\tilde{x}^{k_l}\right\|+\left\|\tilde{x}^{k_l}-x^\infty\right\|\leqslant\varepsilon.$$

This implies that the sequence $\{x^k\}$ converges to x^∞. □

5. Numerical experiments

In this section we shall utilize a special case of the proposed method to solve some nonlinear complementarity problems. The main task of the hybrid LQP method is the prediction part. More clearly, how to choose an appropriate ξ^k is crucial to make the new method practical and efficient. For example, let

$$\xi^k=\beta_k\left(F\left(\tilde{x}^k\right)-F\left(x^k\right)\right). \tag{42}$$

Substitute (42) into (11) and then the equations are reduced to

$$\beta_k F\left(x^k\right)+\tilde{x}^k-(1-\mu)x^k-\mu X_k^2\left(\tilde{x}^k\right)^{-1}=0, \tag{43}$$

whose positive solution can be component-wise computed by

$$\tilde{x}_j^k=\frac{1}{2}\left(s_j^k+\sqrt{\left(s_j^k\right)^2+4\mu\left(x_j^k\right)^2}\right),\quad j=1,\dots,n, \tag{44}$$

where

$$s^k=(1-\mu)x^k-\beta_k F\left(x^k\right). \tag{45}$$

The similar strategy (42) can be seen in Xu and Bnouhachem [20]. In particular, in case of (42), σ in (13) can be 1 and the generated sequence $\{x^k\}$ lie in R_+^n not only in R_{++}^n. The details of analysis and convergence proof are entirely as same as those in [20]. Now we let *Algorithm* 1 refer to the algorithm given by (44), (12) and (13) when $\sigma\in(0,1)$. *Algorithm* 2 refers to the algorithm given by (44), (12) and (13) when $\sigma=1$. The following is a brief analysis on the progresses of both algorithms at the $(k+1)$th iteration.

Remark 5.1. As is well noticed, it follows from Algorithm 2 that

$$\Phi_k\big(\gamma\alpha_k^*\big) \geqslant \frac{\gamma(2-\gamma)(1-\eta\sqrt{1-\mu^2})}{2(1+\mu)}\big\|x^k-\tilde{x}^k\big\|^2 := \Upsilon_k^2 \tag{46}$$

while it follows from Algorithm 1 that

$$\Phi_k\big(\gamma\alpha_k^*\big) \geqslant \sigma\frac{\gamma(2-\gamma)(1-\eta\sqrt{1-\mu^2})}{2(1+\mu)}\big\|x^k-\tilde{x}^k\big\|^2 := \Upsilon_k^1, \tag{47}$$

where $\sigma \in (0,1)$. If both algorithms start from the same point x^k, we have $\Upsilon_k^1 \leqslant \Upsilon_k^2$. This is why we say Algorithm 2 may be more effective than Algorithm 1.

In order to verify the theoretical assertions, we test some NCPs that $F(x) = D(x) + Mx + q$, where $D(x)$ and $Mx + q$ are the nonlinear and linear parts of $F(x)$, respectively.

We form the linear part of the test problems similarly as in [10]. The matrix $M = A^\top A + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, +5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500, +500)$. The components of $D(x)$, the nonlinear part of $F(x)$, are $D_j(x) = d_j * \arctan(x_j)$ where d_j is a random variable in $(0, 1)$.

The implemental details of the hybrid inexact LQP method

Step 0. Let $\beta_0 = 1$, $\eta(:= 0.95) < 1$, $\mu = 0.01$, $\gamma = 1.9$ and $x^0 \in R_+^n$.
For $k = 0, 1, \ldots,$ do:

Step 1. $s := (1-\mu)x^k - \beta_k F\big(x^k\big)$, $\quad \tilde{x}_i^k := \Big(s_i + \sqrt{(s_i)^2 + 4\mu\big(x_i^k\big)^2}\Big)\Big/2$,

$\xi := \beta_k\big(F\big(\tilde{x}^k\big) - F\big(x^k\big)\big)$, $\quad r := \|\xi\|\Big/\Big(\sqrt{1-\mu^2}\big\|x^k - \tilde{x}^k\big\|\Big)$.

while $(r > \eta)$

$\beta_k := \beta_k * 0.8\big/r$,

$s := (1-\mu)x^k - \beta_k F\big(x^k\big)$, $\quad \tilde{x}_i^k := \Big(s_i + \sqrt{(s_i)^2 + 4\mu\big(x_i^k\big)^2}\Big)\Big/2$,

$\xi := \beta_k\big(F\big(\tilde{x}^k\big) - F\big(x^k\big)\big)$, $\quad r := \|\xi\|\Big/\Big(\sqrt{1-\mu^2}\big\|x^k - \tilde{x}^k\big\|\Big)$.

end while

Step 2. Update x^{k+1} by correction step (35)

$$\beta_{k+1} := \begin{cases} \beta_k * 0.7/r & \text{if } r \leqslant 0.4, \\ \beta_k & \text{otherwise.} \end{cases}$$

All the codes are written in Matlab 6.1 and run on a desk computer with CPU Intel P4-1.6G and SDRAM 256M. We test the problem with $n = 100$ up to $n = 1000$. The iterations begin with $x^0 = (0.3, \ldots, 0.3)^\top$. It is well known that solving NCP is equivalent to finding a zero point of

$$\mathrm{e}(x) := x - P_{R_+^n}\big[x - F(x)\big] = \min\big(x, F(x)\big).$$

Therefore, the iteration stops once

$$\big\|\min\big\{x, F(x)\big\}\big\|_\infty \leqslant 10^{-8}.$$

To obtain more stable results, we run 10 times for each test case. The average numbers of iterations and the computational time are given in Table 1.

Table 1
Numerical results for NCP when $x^0 = (0.3, \dots, 0.3)^\top$

Dimension of the problem	Algorithm 1 ($\sigma = 0.8$)		Algorithm 1 ($\sigma = 0.99$)		Algorithm 2 ($\sigma = 1$)	
	No. it.	CPU (sec)	No. it.	CPU (sec)	No. it.	CPU (sec)
$n = 100$	411	0.192	353	0.175	330	0.134
$n = 200$	574	1.46	512	1.16	488	1.03
$n = 400$	564	9.11	436	6.80	424	6.35
$n = 600$	492	20.19	406	15.85	379	15.18
$n = 800$	511	38.04	413	29.54	399	28.35
$n = 1000$	593	68.15	491	53.37	468	50.89

As is shown in Table 1, Algorithm 2 needs fewer iterations and less CPU time than Algorithm 1; when we choose Algorithm 1, it is expected that σ is close to 1. Then, the numerical results agree with the theoretical analysis in Remark 5.1.

The computation load for both the predictor and the corrector is quite tiny, thus both algorithms are very efficient. In addition, the iterative number is insensitive to the size of NCP. Preliminarily speaking, the new hybrid inexact LQP method is easy to be implemented and effective for large-scale NCP.

6. Conclusion

We have proposed a hybrid inexact Logarithmic–Quadratic Proximal method for NCP. The intermediate point is computed by solving the LQP system approximately under relaxed inexact criterion; and the new iterate is produced by making use of the improved extragradient method. Preliminary numerical results show that the new method is attractive for large-scale NCPs. How to extend the techniques developed in this paper to general variational inequalities is worthy of further investigations.

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Further reading

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