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Column continuous transition functions ☆

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Abstract

A column continuous transition function is by definition a standard transition function $P(t)$ whose every column is continuous for $t \geqslant 0$ in the norm topology of bounded sequence space l_∞. We will prove that it has a stable q-matrix and that there exists a one-to-one relationship between column continuous transition functions and increasing integrated semigroups on l_∞. Using the theory of integrated semigroups, we give some necessary and sufficient conditions under which the minimal q-function is column continuous, in terms of its generator (of the Markov semigroup) as well as its q-matrix. Furthermore, we will construct all column continuous Q-functions for a conservative, single-exit and column bounded q-matrix Q. As applications, we find that many interesting continuous-time Markov chains (CTMCs), say Feller–Reuter–Riley processes, monotone processes, birth–death processes and branching processes, etc., have column continuity.

Keywords: Continuous-time Markov chains; Transition functions; q-Matrices; Integrated semigroups; Generators

1. Introduction

We shall in this paper study a generalization of Feller–Reuter–Riley transition functions (FRR), which play a key role in continuous-time Markov chains (CTMCs). See [1,3,4,11,12,15,18].

We consider CTMCs on a linear ordered set, that is, we take the state space $E = Z^+ = \{0, 1, 2, \ldots\}$, and assume that all transition functions are standard, as in Anderson [1]. For more notations and preliminaries, we refer to Anderson [1] and Chen [5].

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Definition 1.1. A transition function $P(t) = (P_{ij}(t);\ i, j \in E)$ is called to be column continuous if

$$P_{ij}(t) \to \delta_{ij} \quad \text{uniformly in } i \in E \text{ as } t \to 0 \text{ for every } j \in E. \tag{1.1}$$

Obviously, $P(t)$ is column continuous if and only if $P(t)e_j \to e_j$ in the norm topology of l_∞ as $t \downarrow 0$ for every $j \in E$, where e_j is the jth column unit vector. In other words, every column of $P(t)$ is continuous in l_∞ for $t \geqslant 0$. This is the reason why we call it *column* continuity.

Although a transition function is always *row* continuous (see [1, Proposition 1.1.3]), that is,

$$P_{ij}(t) \to \delta_{ij} \quad \text{uniformly in } j \in E \text{ as } t \to 0 \text{ for every } i \in E, \tag{1.2}$$

it is not necessarily column continuous, the example (no column continuity) is easy given by our discussions. However, many interesting and useful transition functions are column continuous. Reuter and Riley [15] proved that FRR transition function is column continuous. More examples (contain monotone processes, birth–death processes, branching processes) are given in Section 6.

Recall that a transition function $P(t)$ deduces a positive contraction continuous semigroup on l_1. We denote Ω be its generator. Another similar notation is the so-called q-matrix $Q = (q_{ij},\ i, j \in E)$, $Q = P'(0)$ componentwise, and satisfying

$$0 \leqslant q_{ij} < +\infty \quad \forall\, i, j \in E,\ i \neq j; \tag{1.3}$$

$$\sum_{j \neq i} q_{ij} \leqslant -q_{ii} \equiv q_i \leqslant +\infty \quad \forall\, i \in E. \tag{1.4}$$

If $q_i < +\infty$ for every $i \in E$, Q is called to be stable. It is well known that there always exists a (Feller) minimal Q-function for a given stable q-matrix Q, where a Q-function means a transition function $P(t)$ such that $P'(0) = Q$ componentwise. For the details, we refer to Anderson [1].

An interesting property of a column continuous transition function $P(t)$ is that it has a stable q-matrix (Theorem 2.2). More importantly, we will consider the following two questions:

Question 1. What are the necessary and sufficient conditions for a transition function $P(t)$ being column continuous, in terms of its generator Ω?

Question 2. For a given (stable) q-matrix Q, what are the necessary and sufficient conditions for the minimal Q-function to be column continuous? How is it for the non-minimal case?

To study the above two questions, we will use the theory of integrated semigroups, which introduced by Arendt [2], and extensively developed by many authors (see [6–11,16], etc.). It has many applications to the partial differential equations. However, it is the recent event that the theory of integrated semigroups has been used to deal with the CTMCs (see [11]). In this paper, we will show that there exists a one-to-one relationship between column continuous transition functions and increasing contraction integrated semigroups (Section 3). By using this fact, we give a complete answer to Question 1 (Theorem 3.1), and give a partial answer to Question 2 (Theorem 4.2). For the non-minimal case, we concentrate our attention upon a special class of q-matrices: conservative, single-exit and column bounded. For such a q-matrix Q, we will construct *all* column continuous Q-functions (Theorem 5.1).

2. Basic properties of column continuous transition functions

We first give some equivalent conditions of column continuity.

Proposition 2.1. *Let $P(t) = (P_{ij}(t);\ i, j \in E)$ be a transition function. Then the followings are equivalent*:

(i) *$P(t)$ is column continuous*;
(ii) $\sup_{i \neq j} P_{ij}(t) \to 0$ *as $t \downarrow 0$ for every $j \in E$*;
(iii) *the mapping $t \mapsto P(t)e_j$ from $[0, \infty)$ into l_∞ is continuous, for every column unit vector e_j, $j \in E$*;
(iv) *the mapping $t \mapsto P(t)x$ is continuous from $[0, \infty)$ into l_∞ for every $x \in c_0$.*

Proof. (i) $\Leftrightarrow$ (ii) is obvious, since $P(t)$ is standard, $P_{ij}(t) \to \delta_{ij}$ as $t \downarrow 0$ for every $i, j \in E$. (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) is clear. (i) $\Rightarrow$ (iii) is elementary and standard. We have only to prove (iii) $\Rightarrow$ (iv). Let $Z = \{x \in l_\infty;\ t \mapsto P(t)x$ is continuous in $l_\infty\}$. It is easy to prove that Z is a closed subspace of l_∞. Thus $c_0 = \overline{\mathrm{Span}_{j \in E}\{e_j\}} \subset Z$, which proved (iv). $\square$

We then give an interesting result of column continuity.

Theorem 2.2. *Every column continuous transition function $P(t)$ has a stable q-matrix Q.*

Proof. We have to prove that

$$\lim_{h \to 0} \frac{1 - P_{jj}(h)}{h} = q_j < +\infty \quad \text{for every } j \in E. \tag{2.1}$$

Let $j \in E$ be fixed, and ε be given such that $0 < \varepsilon < \frac{1}{4}$. Since $P(t)$ is column continuous, there exists a $\delta > 0$ such that

$$\left|P_{ij}(t) - \delta_{ij}\right| < \varepsilon \quad \text{for } 0 \leqslant t < \delta \text{ and all } i \in E.$$

Thus, if $0 \leqslant t, s < \delta$, we have

$$\begin{aligned}
1 - P_{jj}(t+s) &= 1 - P_{jj}(t)P_{jj}(s) - \sum_{i \neq j} P_{ji}(s)P_{ij}(t) \\
&\geqslant 1 - P_{jj}(t)P_{jj}(s) - \varepsilon\big(1 - P_{jj}(s)\big) \\
&= 1 - P_{jj}(t) + \big(P_{jj}(t) - \varepsilon\big)\big(1 - P_{jj}(s)\big) \\
&= 1 - P_{jj}(t) + \big(1 - \big(1 - P_{jj}(t)\big) - \varepsilon\big)\big(1 - P_{jj}(s)\big) \\
&\geqslant 1 - P_{jj}(t) + (1 - 2\varepsilon)\big(1 - P_{jj}(s)\big).
\end{aligned}$$

Writing $f(\mu) = 1 - P_{jj}(\mu)$, this becomes

$$f(t+s) \geqslant f(t) + (1 - 2\varepsilon)f(s) \tag{2.2}$$

for $0 \leqslant t, s < \delta$. Iterating (2.2), we find that

$$f(t+ns) \geqslant f(t) + n(1 - 2\varepsilon)f(s) \tag{2.3}$$

provided $t, s \geqslant 0$ and $t + ns < \delta$.

Now for any $0 < h < \delta$, let $\delta = t(h) + n(h)h$ such that $n(h)$ is an integer and $0 \leqslant t(h) < h$. Then (2.3) implies that

$$\frac{f(h)}{h} \leqslant \frac{1}{n(h)h(1 - 2\varepsilon)} f(\delta).$$

Letting $h \to 0$, and using the fact that $n(h)h \to \delta$, we obtain that

$$q_j = \lim_{h\to 0} \frac{1 - P_{jj}(h)}{h} = \lim_{h\to 0} \frac{f(h)}{h} \leqslant \frac{f(\delta)}{\delta(1-2\varepsilon)} < +\infty.$$

Finally we state that the column continuity can also be characterized through the resolvent function. Its proof is not elementary, we will give the proof by using the theory of integrated semigroups in Section 3. □

Recall that a function $R(\lambda) = (r_{ij}(\lambda);\ i, j \in E)$, $\lambda > 0$, is called to be a *resolvent function* if

(i) $r_{ij}(\lambda) \geqslant 0$ and $\lambda \sum_{k\in E} r_{ik}(\lambda) \leqslant 1$ for all $i, j \in E$ and $\lambda > 0$;
(ii) $r_{ij}(\lambda) - r_{ij}(\mu) + (\lambda - \mu) \sum_{k\in E} r_{ik}(\lambda) r_{kj}(\mu) = 0$ for all $i, j \in E$ and $\lambda, \mu > 0$;
(iii) $\lim_{\lambda\to\infty} \lambda r_{ii}(\lambda) = 1$ for all $i \in E$.

It is also well known that a transition function $P(t) = (P_{ij}(t))$ can be uniquely determined by a resolvent function $R(\lambda) = (r_{ij}(\lambda))$ through the Laplace transform

$$r_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} P_{ij}(t)\,dt, \quad \lambda > 0,\ i, j \in E.$$

Definition 2.3. A resolvent function $R(\lambda) = (r_{ij}(\lambda))$ is column continuous if

$$\lambda r_{ij}(\lambda) \to \delta_{ij} \quad \text{uniformly in } i \text{ as } \lambda \to \infty \text{ for every } j \in E, \tag{2.4}$$

or equivalently if $\lambda R(\lambda) e_j \to e_j$ as $\lambda \to \infty$ in l_∞ for every $j \in E$.

Theorem 2.4. *Let $P(t)$ be a transition function and $R(\lambda)$ be the corresponding resolvent function. Then $P(t)$ is column continuous if and only if $R(\lambda)$ is.*

3. Characterizations of generators of column continuous semigroups

Let $P(t)$ be a transition function. It is well known that $P(t)$ is a positive continuous contraction semigroup on l_1, where the operator $P(t)$ is defined by $y \mapsto yP(t)$, $y \in l_1$ being regarded as a row vector. Note that the dual operator $P^*(t)$ on l_∞ of $P(t)$ can be written as $x \mapsto P(t)x$, $x \in l_\infty$ being a column vector. The following result states that column continuity can be characterized through its generator Ω, which answers Question 1.

Theorem 3.1. *Let $P(t)$ be a transition function, and Ω be the generator of semigroup $P(t)$ on l_1. Then $P(t)$ is column continuous if and only if*

$$e_j \in \overline{D(\Omega^*)} \quad \textit{for every } j \in E, \tag{3.1}$$

where e_j is the jth column unit vector, Ω^* is the dual operator of Ω and $\overline{D(\Omega^*)}$ is the closure of the domain of Ω^*.

To prove the above theorem, we will use the theory of integrated semigroups (see [2,6–11,16], etc.).

Definition 3.2. [2] A strongly continuous family $\{T(t);\ t \geqslant 0\}$ of bounded linear operators on a Banach space X is called to be an *integrated semigroup* if $T(0) = 0$ and

$$T(t)T(s)x = \int_0^t \bigl(T(\tau+s) - T(\tau)\bigr)x\,d\tau \tag{3.2}$$

for all $t, s \geqslant 0$ and $x \in X$. Moreover, $T(t)$ is *non-degenerate* if $T(t)x = 0$ for all $t \geqslant 0$ implies $x = 0$. The generator A of $T(t)$ is defined by

$$D(A) = \left\{ x \in X,\ \text{there exists } y \in X \text{ such that } T(t)x = tx + \int_0^t T(r)y\,dr,\ t \geqslant 0 \right\}$$

with

$$Ax = y \quad \text{for } x \in D(A).$$

Recall that an operator A generates an exponentially bounded and non-degenerate integrated semigroup $T(t)$ if and only if $(\omega, +\infty) \subset \rho(A)$ for some $\omega \in R$ and $\lambda \mapsto (\lambda - A)^{-1}/\lambda$ is a Laplace transformation. In this case,

$$(\lambda - A)^{-1} = \int_0^\infty \lambda e^{-\lambda t} T(t)\,dt \quad \text{for } \lambda > \omega. \tag{3.3}$$

The following theorem states that there exists a one-to-one relationship between column continuous transition functions and increasing contraction integrated semigroups on l_∞.

Theorem 3.3. *Let $P(t) = (P_{ij}(t))$ be a column continuous transition function and write*

$$T_{ij}(t) = \int_0^t P_{ij}(s)\,ds \quad \textit{for } i, j \in E,\ t \geqslant 0. \tag{3.4}$$

Then $T(t) = (T_{ij}(t))$ is an integrated semigroup on l_∞, and satisfies

(i) *$T(t)$ is non-degenerate*;
(ii) *$\|T(t)\| \leqslant t$ for all $t \geqslant 0$*;
(iii) *$T(t)$ is increasing*;
(iv) *$e_j \in \overline{D(\Omega_\infty)}$ for every $j \in E$, where Ω_∞ is the generator of $T(t)$.*

Conversely, if $T(t) = (T_{ij}(t))$ is an integrated semigroup satisfying (i)–(iv), then there exists a unique column continuous transition function $P(t)$ such that (3.4) holds. Moreover, we have

(v) $\Omega_\infty = \Omega^*$, Ω is the generator of semigroup $P(t)$ on l_1.

Proof. In [11], I have proved that if $P(t)$ is a transition function, then $T(t)$ defined as in (3.4) is an integrated semigroup. We will prove (i)–(v) provided $P(t)$ is column continuous.

(i) Let $x=(x_j)\in l_\infty$ such that $T(t)x=0$ for all $t\geqslant 0$. Then

$$\sum_j \frac{1}{t}\int_0^t P_{ij}(s)\,ds\,x_j=0 \quad \text{for } i\in E \text{ and } t\geqslant 0. \tag{3.5}$$

Letting $t\to 0$, by using bounded convergence theorem (here, the bounded controlling function is taken as follows: Fixed $i\in E$ and let $c=\min_{0\leqslant s\leqslant 1}P_{ii}(s)$, then $c>0$. Since $P_{ij}(1)\geqslant P_{ii}(1-s)P_{ij}(s)$, it follows that $P_{ij}(s)\leqslant P_{ij}(1)/P_{ii}(1-s)\leqslant P_{ij}(1)/c$ for $0\leqslant s\leqslant 1$. Therefore $|\frac{1}{t}\int_0^t p_{ij}(s)\,ds\,x_j|\leqslant P_{ij}(1)|x_j|/c$ for every $j\in E$ and $0\leqslant t\leqslant 1$. Now the right-hand side is just the needed bounded function, because $\sum_j P_{ij}(1)|x_j|\leqslant \|x\|<+\infty$), we get

$$\begin{aligned}\lim_{t\to 0}\sum_j \frac{1}{t}\int_0^t P_{ij}(s)\,ds\,x_j &= \sum_j \lim_{t\to 0}\frac{1}{t}\int_0^t P_{ij}(s)\,ds\,x_j\\ &=\sum_j P_{ij}(0)x_j=\sum_j \delta_{ij}x_j=x_i\end{aligned}$$

for every $i\in E$. Thus (3.5) implies that $x_i=0$ for all $i\in E$. That is, $x=0$. Thus $T(t)$ is non-degenerate.

(ii) and (iii) are elementary, since $P(t)$ is contractive and positive.

(iv) For every fixed $j\in E$, by Proposition 2.1, $P(t)e_j$ is continuous on l_∞ for $t\geqslant 0$. Thus $\int_0^t P(s)e_j\,ds$ is integrable on l_∞. Therefore, for every row unit vector $e_i\in l_1$, we have

$$\left\langle e_i,\int_0^t P(s)e_j\,ds\right\rangle=\int_0^t \langle e_i,P(s)e_j\rangle\,ds=\int_0^t P_{ij}(s)\,ds=\langle e_i,T(t)e_j\rangle.$$

Since $\mathrm{Span}\{e_i\}$ is dense in l_1, above equality implies that $T(t)e_j=\int_0^t P(s)e_j\,ds\in l_\infty$ for $t\geqslant 0$. Since $T(t)e_j\in\overline{D(\Omega_\infty)}$ (see [2, Corollary 3.4]), it follows that $e_j=\lim_{t\to 0}\frac{1}{t}\int_0^t P(s)e_j\,ds=\lim_{t\to 0}\frac{1}{t}T(t)e_j\in\overline{D(\Omega_\infty)}$, which proved (iv).

(v) Since Ω generates the continuous contraction semigroup $P(t)$ on l_1, it follows from [2, Corollary 4.4] and [11, Corollary 3.7] that Ω^* generates an integrated semigroup $S(t)$ on l_∞. We show that $S(t)=T(t)$ for all $t\geqslant 0$. If this is true, then their generator Ω^* and Ω_∞ agree. By Yosida [17], $(\lambda-\Omega^*)^{-1}=((\lambda-\Omega)^{-1})^*$. Thus if $x\in l_\infty$, by using Laplace transform, we have

$$\left\langle\int_0^\infty e^{-\lambda t}e_iP(t)\,dt,x\right\rangle=\left\langle e_i,\lambda\int_0^\infty e^{-\lambda t}S(t)x\,dt\right\rangle$$

which implies, by integration by parts, that

$$\lambda\int_0^\infty e^{-\lambda t}\langle e_i,S(t)x\rangle\,dt=\lambda\int_0^\infty e^{-\lambda t}\int_0^t\langle e_iP(s),x\rangle\,ds\,dt.$$

Thus, by the uniqueness of Laplace transform,

$$\langle e_i,S(t)x\rangle=\int_0^t\langle e_iP(s),x\rangle\,ds=\left\langle\int_0^t e_iP(s)\,ds,x\right\rangle=\langle e_i,T(t)x\rangle$$

for $i\in E$, which implies that $S(t)x=T(t)x$. Thus we have proved the desired conclusion.

Now we prove the converse question. Let $T(t) = (T_{ij}(t))$ be an integrated semigroup such that (i)–(iv) hold. We first show that $t \mapsto T(t)e_j$ from $[0, \infty)$ into l_∞ is continuously differentiable for every $j \in E$. Since $\|T(t)\| \leqslant t$ for all $t \geqslant 0$, $T(t)$ is exponentially bounded with any bound $\omega > 0$, and it follows from [2, Theorem 3.1] that $(0, +\infty) \subset \rho(\Omega_\infty)$ and

$$R(\lambda, \Omega_\infty)x = (\lambda - \Omega_\infty)^{-1}x = \int_0^\infty \lambda e^{-\lambda t} T(t)x\, dt \quad \text{for } \lambda > 0,\ x \in X,$$

which implies, since $\|T(t)\| \leqslant t$, that

$$\big\|R(\lambda, \Omega_\infty)\big\| \leqslant \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

Differentiating $R(\lambda, \Omega_\infty)$ n-times in λ and using repeatedly above inequality, we get

$$\sup\big\{\big\|\lambda^{n+1} R^{(n)}(\lambda, \Omega_\infty)/n!\big\|\colon \lambda > 0,\ n = 0, 1, 2, \ldots\big\} \leqslant 1.$$

Thus, by the integrated version of Widder's theorem [2, Theorem 1.1], there exists a function $S : [0, \infty) \mapsto B(l_\infty)$ satisfying

$$S(0) = 0 \quad \text{and} \quad \big\|S(t + h) - S(t)\big\| \leqslant h \quad (h \geqslant 0,\ t \geqslant 0),$$

such that

$$R(\lambda, \Omega_\infty)x = \int_0^\infty \lambda e^{-\lambda t} S(t)x\, dt \quad \text{for } \lambda > 0,\ x \in l_\infty.$$

By the uniqueness of Laplace transform, $S(t) = T(t)$, and thus we have

$$\big\|T(t + h) - T(t)\big\| \leqslant h \quad (h \geqslant 0,\ t \geqslant 0). \tag{3.6}$$

Let $Z = \{x \in l_\infty\colon\ T(t)x$ is continuously differentiable function of $t \geqslant 0\}$. By using (3.6), it is easy to show that Z is a closed subspace of l_∞, which implies that $\overline{D(\Omega_\infty)} \subset Z$, since $D(\Omega_\infty) \subset Z$ (see [2]). By assumption (iv), it follows that $e_j \in Z$ for every $j \in E$. That is, we have proved that $T(t)e_j$ is continuously differentiable for $t \geqslant 0$ in l_∞.

We then define

$$P_{ij}(t) = \left\langle e_i, \frac{d}{dt} T(t)e_j \right\rangle = T'_{ij}(t), \quad i, j \in E,\ t > 0, \tag{3.7}$$

and show that $P(t) = (P_{ij}(t))$ is a column continuous transition function. Indeed, since $T(t)$ is increasing, it follows that $P_{ij}(t) \geqslant 0$ and $P_{ij}(0) = \langle e_i, \frac{d}{dt} T(t)e_j|_{t=0}\rangle = \delta_{ij} + \Omega_\infty T(t)e_j|_{t=0} = \delta_{ij}$. (3.6), together with Fatou–Lebesgue lemma, implies that

$$\begin{aligned}
\sum_{j \in E} P_{ij}(t) &= \sum_{j \in E} \lim_{h \downarrow 0} \frac{T_{ij}(t + h) - T_{ij}(t)}{h} \\
&\leqslant \liminf_{h \to 0} \frac{1}{h} \sum_{j \in E} \big|T_{ij}(t + h) - T_{ij}(t)\big| \\
&\leqslant \liminf_{h \to 0} \frac{1}{h} \big\|T(t + h) - T(t)\big\| \leqslant 1.
\end{aligned}$$

Obviously, $P_{ij}(t)$ is a continuous function of $t \geqslant 0$, and $P(t)e_j = \frac{d}{dt}T(t)e_j$ is continuous in l_∞, as we have proved, that is, $P(t)$ is column continuous. To complete the proof, we have only to show that $P(t)$ has the semigroup property:

$$P_{ij}(t+s) = \sum_{k\in E} P_{ik}(t)P_{kj}(s) \quad \text{for all } s, t \geqslant 0,\ i, j \in E. \tag{3.8}$$

To this end, we let, for fixed $i, j \in E$, and $t, s \geqslant 0$,

$$f(t,s) = \langle e_i, T(t)T(s)e_j\rangle = \left\langle e_i, \int_0^t \big(T(\tau+s)e_j - T(\tau)e_j\big)\,d\tau\right\rangle. \tag{3.9}$$

Differentiating (3.9) with respect to t and with respect to s again, we get

$$\frac{\partial}{\partial t} f(t,s) = \langle e_i, T(t+s)e_j - T(t)e_j\rangle; \tag{3.10}$$

$$\frac{\partial^2}{\partial s\partial t} f(t,s) = \left\langle e_i, \frac{\partial}{\partial s}T(t+s)e_j\right\rangle = P_{ij}(t+s). \tag{3.11}$$

On the other hand,

$$f(t,s) = \langle e_i, T(s)T(t)e_j\rangle = \sum_{k\in E} T_{ik}(s)T_{kj}(t). \tag{3.12}$$

Differentiating (3.12) in t, and using the Lebesgue bounded differentiable theorem (since $T_{ik}(s)T'_{kj}(t) = T_{ik}(s)P_{kj}(t) \leqslant T_{ik}(s)$, and $\sum_k T_{ik}(s) \leqslant \|T(s)\| < +\infty$), we get

$$\frac{\partial}{\partial t} f(t,s) = \sum_{k\in E} T_{ik}(s)T'_{kj}(t). \tag{3.13}$$

Thus, it follows from Fatou–Lebesgue lemma, that

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t} f(t,s) &= \lim_{h\to 0^+} \sum_{k\in E} \frac{T_{ik}(s+h) - T_{ik}(s)}{h} T'_{kj}(t)\\ &\geqslant \sum_{k\in E} \lim_{h\to 0^+} \frac{T_{ik}(s+h) - T_{ik}(s)}{h} T'_{kj}(t)\\ &= \sum_{k\in E} T'_{ik}(s)T'_{kj}(t) = \sum_{k\in E} P_{ik}(s)P_{kj}(t).\end{aligned}$$

This together with (3.11), implies that

$$P_{ij}(t+s) \geqslant \sum_{k\in E} P_{ik}(s)P_{kj}(t). \tag{3.14}$$

To prove (3.14) is an equality, we will use the Lebesgue bounded differentiable theorem on (3.13). To find the bounded controlling function, we let $i \in E$ and $\tau > 0$ be fixed. It follows form (3.14) that $P_{ii}(\tau) \geqslant P_{ii}(\frac{\tau}{n})^n$ for $n = 1, 2, \ldots,$ which implies that $P_{ii}(s) > 0$ for every $s \geqslant 0$. If $c = \min_{0\leqslant s\leqslant\tau} P_{ii}(s)$, then $c > 0$. By (3.14) again, $P_{ik}(\tau) \geqslant P_{ii}(\tau - s)P_{ik}(s) \geqslant cP_{ik}(s)$ for every $k \in E$ and $0 < s < \tau$, that is

$$\sup_{0\leqslant s\leqslant\tau} P_{ik}(s) \leqslant \frac{P_{ik}(\tau)}{c} \quad \text{for every } k \in E. \tag{3.15}$$

Differentiating (3.13) with respect to s for $0 \leqslant s \leqslant \tau$, by the bounded differentiable theorem (where the bounded function is $\sum_k P_{ik}(\tau)T'_{kj}(t) = \sum_k P_{ik}(\tau)P_{kj}(t) \leqslant P_{ij}(\tau + t) < +\infty$, since$|T'_{ik}(s)T'_{kj}(t)| \leqslant \frac{P_{ik}(\tau)}{c} T'_{kj}(t)$ for $0 \leqslant s \leqslant \tau$), we obtain

$$\frac{\partial^2}{\partial s \partial t} f(t,s) = \sum_{k \in E} T'_{ik}(s)T'_{kj}(t) = \sum_{k \in E} P_{ik}(s)P_{kj}(t) \tag{3.16}$$

for $0 \leqslant s \leqslant \tau$ and $t \geqslant 0$. Since τ is arbitrary, (3.16) holds for all $s \geqslant 0$ and $t \geqslant 0$. This together with (3.11) implies (3.8), which completes the proof of Theorem 3.3. □

Remark. (1) $T_{ij}(t)$ in Theorem 3.3 has clearly probabilistic explanation: $T_{ij}(t)$ is the mean time spent in j before time t if the chain starts in i. Indeed, if we denote $X(t)$ being the corresponding CTMC, then

$$\begin{aligned} T_{ij}(t) &= \int_0^t P_{ij}(s)\,ds = \int_0^t E\big[I_{X(s)=j} \mid X(0) = i\big]\,ds \\ &= E\left[\int_0^t I_{X(s)=j}\,ds \mid X(0) = i\right] \\ &= E[\text{time spent in } j \text{ before time } t \mid \text{start in } i]. \end{aligned}$$

(2) $T(t)$ is not an integrated c_0-semigroup except that the generator Ω_∞ is bounded. This is easy to prove by using William's theorem (see [15] or [1]).

Finally we give the proof of Theorems 3.1 and 2.4 by using Theorem 3.3.

Proof of Theorem 3.1. Necessity follows directly from Theorem 3.3(iv) and (v).

Sufficiency. Let $Y = \overline{D(\Omega^*)} \subset l_\infty$. By the theory of dual semigroup (see [13, Theorem 1.10.4]), it follows that $P(t)Y \subset Y$, and the restriction $P(t)|_Y$ of $P(t)$ on Y is a strongly continuous semigroup on Y, and its generator $\Omega^+ = \Omega^*|_Y$, the part of Ω^* in Y. Now if $e_j \in Y$, then $P(t)e_j$ is continuous in Y and thus in l_∞ for $t \geqslant 0$, which implies that $P(t)$ is column continuous. □

Proof of Theorem 2.4. Let $R(\lambda)$ be the resolvent function, $R(\lambda, \Omega)$ and $R(\lambda, \Omega^*)$ be the resolvent operator of Ω and Ω^*, respectively. It is obvious that $R(\lambda) = R(\lambda, \Omega)$ (as two bounded operators on l_1) and $R(\lambda) = R(\lambda, \Omega^*)$ (as two bounded operators on l_∞), where the operator $R(\lambda)$ is defined by $y \mapsto yR(\lambda)$ for $y \in l_1$ and by $x \mapsto R(\lambda)x$ for $x \in l_\infty$, respectively. For $x \in D(\Omega^*)$, we have

$$\begin{aligned} \big\|\lambda R(\lambda)x - x\big\|_\infty &= \big\|\lambda R\big(\lambda : \Omega^*\big)x - x\big\|_\infty \\ &= \big\|R\big(\lambda, \Omega^*\big)\Omega^* x\big\|_\infty \leqslant \frac{1}{\lambda}\big\|\Omega^* x\big\|_\infty \to 0 \end{aligned} \tag{3.17}$$

as $\lambda \to \infty$. Since $\|\lambda R(\lambda)\| \leqslant 1$, it follows that (3.17) holds for all $x \in \overline{D(\Omega^*)}$. Now if $P(t)$ is column continuous, then, by Theorem 3.1, every $e_j \in \overline{D(\Omega^*)}$ for $j \in E$. Then, by (3.17), $\lambda R(\lambda)e_j \to e_j$ as $\lambda \to \infty$, that is, $R(\lambda)$ is column continuous. Conversely, suppose $\lambda R(\lambda)e_j \to e_j$ as $\lambda \to \infty$ in l_∞ for every $j \in E$. Since $R(\lambda)e_j = R(\lambda, \Omega^*)e_j \in D(\Omega^*)$, it follows that thus $e_j \in \overline{D(\Omega^*)}$. Therefore, by Theorem 3.1, $P(t)$ is column continuous. □

4. Characterizations of q-matrices for column continuities

In this section, we will consider Question 2 announced as in introduction. We first introduce some definitions.

Definition 4.1. A q-matrix $Q = (q_{ij})$ is called to be *column bounded* if every column of Q is bounded, that is

$$\sup_i |q_{ij}| < +\infty \quad \text{for every } j \in E; \tag{4.1}$$

Q is called to be *column almost-bounded* if for every $j \in E$,

$$\liminf_{i_n \to \infty} \frac{q_{i_n j}}{q_{i_n}} = 0 \tag{4.2}$$

for every subsequence $\{i_n\}$ of $\{i\}$ satisfying $q_{i_n} \uparrow \infty$ as $n \to \infty$.

We then give our main result in this section.

Theorem 4.2. *Let $Q = (q_{ij})$ be a given (stable) q-matrix. If Q is column bounded, then all Q-functions which satisfy forward equations are column continuous, in particular, the minimal one is. Conversely, if there exists a Q-function $P(t)$ such that $P(t)$ is column continuous, then the minimal Q-function $F(t)$ is column continuous and Q is column almost-bounded.*

To prove Theorem 4.2, we need some notations and lemmas. Note that a q-matrix Q defines two operators Q_1 and Q_∞ with the maximum domain, on Banach space l_1 and l_∞, respectively,

$$yQ_1 = yQ, \qquad y \in D(Q_1) = \{y \in l_1 \mid yQ \text{ is well defined and } yQ \in l_1\};$$
$$Q_\infty x = Qx, \qquad x \in D(Q_\infty) = \{x \in l_\infty \mid Qx \in l_\infty\},$$

where y is row vector and x is column vector.

Lemma 4.3. *Given a q-matrix Q, let $P(t)$ be a column continuous Q-function and satisfy the backward equations, then*

$$e_j \in \overline{D(Q_\infty)} \quad \textit{for every } j \in E. \tag{4.3}$$

Proof. Let Q_0 be an operator on l_1 defined by

$$yQ_0 = yQ \quad \forall y \in D(Q_0) = \operatorname{span}_{i \in E}\{e_i\}. \tag{4.4}$$

Then Q_0 is densely defined on l_1. It is easy to prove that the dual operator $Q_0^* = Q_\infty$ (see [11, 14]). Let Ω be the generator of the semigroup $P(t)$ on l_1. Since $P(t)$ satisfies the backward equations, it follows from [1, Proposition 1.4.5] that $Q_0 \subset \Omega$. Thus $Q_\infty = Q_0^* \supset \Omega^*$. Since $P(t)$ is also column continuous, it follows from Theorem 3.1 that $e_j \in \overline{D(\Omega^*)} \subset \overline{D(Q_\infty)}$ for every $j \in E$, which proved (4.3). □

Lemma 4.4. *If Q is zero-exit, then the minimal Q-function is column continuous if and only if Q satisfies* (4.3).

Proof. It follows from [11, Lemma 5.6] that $\overline{Q_0} = \Omega$ if and only if Q is zero-exit, where Q_0 is defined as in (4.4), $\overline{Q_0}$ is the closure of the closable operator Q_0 and Ω is the generator of the minimal semigroup. Thus if Q is zero-exit then $\Omega^* = \overline{Q_0}^* = Q_0^* = Q_\infty$. Now assume Q also satisfies (4.3), then

$$e_j \in \overline{D(Q_\infty)} = \overline{D(\Omega^*)} \quad \text{for every } j \in E,$$

which implies by Theorem 3.1 that the minimal Q-function $F(t)$ is column continuous. □

Lemma 4.5. *If Q satisfies*

$$e_j \in \overline{D(Q_1^*)} \quad \textit{for every } j \in E, \tag{4.5}$$

then all Q-functions which satisfy the forward equations are column continuous.

Proof. Let $P(t)$ be a Q-function which satisfies the forward equations with the generator Ω. Then, by [1, Proposition 1.4.6] or [14], $\Omega \subset Q_1$, and therefore $Q_1^* \subset \Omega^*$. Thus if (4.5) holds then $e_j \in \overline{D(Q_1^*)} \subset \overline{D(\Omega^*)}$. Therefore by Theorem 3.1 $P(t)$ is column continuous. □

Lemma 4.6. *If the minimal Q-function $F(t)$ is column continuous, then*

$$\lim_{\lambda\to\infty}\left(\sup_{i\neq j}\frac{q_{ij}}{\lambda+q_i}\right) = 0 \quad \textit{for every } j \in E. \tag{4.6}$$

Proof. Because the minimal Q-resolvent $\phi(\lambda)$ satisfies the backward equations, we have (see [1])

$$(\lambda + q_i)\phi_{ij}(\lambda) = \sum_{k\neq i} q_{ik}\phi_{kj}(\lambda) \geqslant q_{ij}\phi_{jj}(\lambda) \quad \text{for } i \neq j,$$

which implies that

$$\frac{q_{ij}}{\lambda+q_i} \leqslant \frac{\phi_{ij}(\lambda)}{\phi_{jj}(\lambda)} = \frac{\lambda\phi_{ij}(\lambda)}{\lambda\phi_{jj}(\lambda)}, \quad i \neq j.$$

Thus

$$\sup_{i\neq j}\frac{q_{ij}}{\lambda+q_i} \leqslant \left[\sup_{i\neq j}\lambda\phi_{ij}(\lambda)\right]\cdot\frac{1}{\lambda\phi_{jj}(\lambda)}.$$

Let $\lambda \to \infty$ in above inequality yields (4.6), because we see from Theorem 2.4 that $\sup_{i\neq j}\lambda\phi_{ij}(\lambda) \to 0$ and $\lambda\phi_{jj}(\lambda) \to 1$ as $\lambda \to \infty$. □

Lemma 4.7. *If there exists a Q-function $P(t)$ such that $P(t)$ is column continuous, then the minimal Q-function $F(t)$ is column continuous.*

Proof. Since for every $j \in E$,

$$\sup_{i\neq j} f_{ij}(h) \leqslant \sup_{i\neq j} P_{ij}(h) \to 0$$

as $h \to 0$. The desired conclusion follows from Proposition 2.1. □

Proof of Theorem 4.2. Let Q be column bounded. By Lemma 4.5, we have only to show that $e_j \in \overline{D(Q_1^*)}$ for every $j \in E$. Indeed, let $j \in E$ be fixed. For $x \in D(Q_1) \subset l_1$, we have

$$\langle xQ_1, e_j\rangle = \langle xQ, e_j\rangle = \left\langle \sum_{i\in E} x_i q_{ij}, e_j \right\rangle = \sum_{i\in E} x_i q_{ij}.$$

Let $z = (q_{0j}, q_{1j}, q_{2j}, \ldots)^T$. Since Q is column bounded, it follows that $z \in l_\infty$ and $\sum_i x_i q_{ij} = \langle x, z\rangle$. Therefore

$$\langle xQ_1, e_j\rangle = \langle x, z\rangle \quad \text{for every } x \in D(Q_1),$$

which implies that $e_j \in D(Q_1^*)$ and $Q_1^* e_j = z \in l_\infty$ for every $j \in E$. Thus the first argument in Theorem 4.2 follows from Lemma 4.5.

Conversely, if there exists a column continuous Q-function, then, by Lemma 4.7, the minimal Q-function $F(t)$ is column continuous. By Lemma 4.6, Q must satisfy (4.6). We have to show that Q is column almost-bounded. If not, then there exist a state $j \in E$ and a subsequence $\{i_n\}$ such that

$$\limsup_{i_n\to\infty} \frac{q_{i_n j}}{q_{i_n}} = c_j > 0 \quad \text{and} \quad q_{i_n} \uparrow \infty \quad (n \to \infty). \tag{4.7}$$

Let $\lambda > 0$ be fixed. Then by (4.7) there exist an $N = N(\lambda)$ such that

$$\frac{q_{i_n j}}{q_{i_n}} \geqslant \frac{1}{2} c_j \quad \text{and} \quad q_{i_n} \geqslant \lambda$$

for $n \geqslant N$, which implies that

$$\sup_{i\neq j} \frac{q_{ij}}{\lambda + q_i} \geqslant \sup_{n\geqslant N} \frac{q_{i_n j}}{q_{i_n} + q_{i_n}} \geqslant \frac{1}{4} c_j.$$

Noting that $\lambda > 0$ is arbitrary, we have proved that

$$\lim_{\lambda\to\infty} \left(\sup_{i\neq j} \frac{q_{ij}}{\lambda + q_i} \right) \geqslant \frac{1}{4} c_j > 0,$$

which contradicts (4.6). Thus Q must be column almost-bounded. □

5. Constructions of the non-minimal column continuous Q-functions

We consider only a special class of q-matrices: $Q = (q_{ij})$ is conservative, single-exit and column bounded (this class contains the important birth–death process and branching process). For such q-matrices, the constructions of all Q-functions are clear. Let $\Phi(\lambda) = (\phi_{ij}(\lambda))$ be the minimal Q-resolvent, $z(\lambda) = 1 - \lambda\Phi(\lambda)1$. Then, by the construction theorem (see [1, Theorem 4.2.6]), all the Q-resolvent $\Psi(\lambda) = (\psi_{ij}(\lambda))$, $\lambda > 0$, is of the following form:

$$\psi_{ij}(\lambda) = \phi_{ij}(\lambda) + z_i(\lambda) y_j(\lambda), \tag{5.1}$$

where

$$y(\lambda) = \frac{\eta(\lambda)}{c + \lambda\eta(\lambda)1}. \tag{5.2}$$

Here,

$$\eta(\lambda) = \bar\eta(\lambda) + b\phi(\lambda), \qquad \bar\eta(\lambda) = \eta + (\lambda_0 - \lambda)\eta\phi(\lambda) \tag{5.3}$$

and $c \geqslant 0, \lambda_0 > 0, \eta \in l_1^+(\lambda_0), b \geqslant 0$ such that $b\Phi(\lambda) \in l_1$ for some (and thus for all) $\lambda > 0$. The following result constructs all column continuous Q-function for above q-matrix Q.

Theorem 5.1. *Let Q be a conservative, single-exit and column-bounded q-matrix. Then the Q-resolvent $\Psi(\lambda)$ is column continuous if and only if either $b = 0$ or $b \neq 0$ such that $\lim_{\lambda\to\infty} \lambda\eta(\lambda)1 = +\infty$.*

Proof. *Sufficiency*. Note that $\Psi(\lambda)$ satisfies the forward equations if and only if $b = 0$ (see [1, Theorem 4.2.6]). Thus if $b = 0$ then $\Psi(\lambda)$ is column continuous by Theorem 4.2. Assume now $b \neq 0$ such that $\lim_{\lambda\to\infty} \lambda\eta(\lambda)1 = +\infty$. Then, for every fixed $j \in E$,

$$\lim_{\lambda\to\infty} \lambda y_j(\lambda) = \lim_{\lambda\to\infty} \frac{\lambda\eta_j(\lambda)}{c + \lambda\eta(\lambda)1} = \lim_{\lambda\to\infty} \frac{b_j}{c + \lambda\eta(\lambda)1} = 0,$$

here, we use the fact that $\lim_{\lambda\to\infty} \lambda\eta_j(\lambda) = b_j$ (see [1, Proposition 4.1.12]). Since $|z_i(\lambda)| \leqslant 1$ for all $i \in E$, it follows from Theorems 4.2 and 2.4 that

$$\sup_{i\neq j} \lambda\psi_{ij}(\lambda) \leqslant \sup_{i\neq j} \lambda\phi_{ij}(\lambda) + \sup_{i\neq j} \lambda z_i(\lambda) y_j(\lambda) \leqslant \sup_{i\neq j} \lambda\phi_{ij}(\lambda) + \lambda y_j(\lambda) \to 0$$

as $\lambda \to \infty$. That is, $\Psi(\lambda)$ is column continuous, and thus, by Theorem 2.4, the corresponding Q-function $P(t)$ is column continuous.

Necessity. Let $\Psi(\lambda)$ be column continuous. Since $\Phi(\lambda)$ is column continuous by Theorem 4.2, it follows from (5.1) that

$$\sup_i z_i(\lambda)\big(\lambda y_j(\lambda)\big) \leqslant \sup_i \big|\lambda\psi_{ij}(\lambda) - \delta_{ij}\big| + \sup_i \big|\lambda\phi_{ij}(\lambda) - \delta_{ij}\big| \to 0 \tag{5.4}$$

as $\lambda \to \infty$ for every $j \in E$. We claim that

$$\lim_{\lambda\to\infty} \lambda y_j(\lambda) = 0 \quad \text{for every } j \in E. \tag{5.5}$$

Indeed, if not, then there exists a state $j_0 \in E$ such that $\lim_{\lambda\to\infty} \lambda y_{j_0}(\lambda) > 0$, here we use the fact that the limit $\lim_{\lambda\to\infty} \lambda y_j(\lambda)$ always exists (see [1]). Thus it follows from (5.4) that

$$\lim_{\lambda\to\infty} \Big[\sup_i z_i(\lambda)\Big] = \lim_{\lambda\to\infty} \Big[\sup_i z_i(\lambda)\big(\lambda y_{j_0}(\lambda)\big)\Big] \cdot \frac{1}{\lambda y_{j_0}(\lambda)} = 0,$$

which means that, for $0 < \varepsilon < 1/2$ and large $\lambda > 0$,

$$\sup_i z_i(\lambda) = \sup_i \left(1 - \lambda \sum_k \phi_{ik}(\lambda)\right) = 1 - \inf_i \lambda \sum_k \phi_{ik}(\lambda) < \varepsilon$$

that is

$$\lambda\Phi(\lambda)1 \geqslant (1 - \varepsilon) \cdot 1 \quad \text{for large } \lambda > 0,$$

which implies by [1, Lemma 4.3.1] that Q is zero-exit. This contradicts to the assumption that Q is single-exit and thus (5.5) holds. Since now $\lambda\eta_j(\lambda) \to b_j$ as $\lambda \to \infty$ for every $j \in E$, it follows from (5.5) and (5.2) that

$$\lim_{\lambda\to\infty} \lambda y_j(\lambda) = \frac{b_j}{c + \lim_{\lambda\to\infty} \lambda\eta(\lambda)1} = 0 \quad \text{for every } j \in E. \tag{5.6}$$

Here $\lambda\eta(\lambda)1$ increases as $\lambda \to \infty$ (see [1]). Thus, if $b \neq 0$, then there exists a state $j_0 \in E$ such that $b_{j_0} > 0$. It follows from (5.6) that $\lim_{\lambda\to\infty} \lambda\eta(\lambda)1 = +\infty$, which proved the needed conclusion. $\square$

6. Examples

Example 6.1. All *monotone transition functions* $P(t)$ ($P(t)$ is not necessarily the minimal one) are column continuous. To prove this fact, we let Q be its q-matrix (Q must be stable), and let $\widetilde{P}(t)$ be its dual function with q-matrix $\widetilde{Q}$. Then we have (see [4,12,18])

$$q_{ij} = \sum_{k=0}^{i} (\tilde{q}_{jk} - \tilde{q}_{j+1,k}) \quad (\forall i, j \in E),$$

which implies that

$$|q_{ij}| \leqslant \sum_{k=0}^{\infty} |\tilde{q}_{jk}| + \sum_{k=0}^{\infty} |\tilde{q}_{j+1k}| \leqslant 2(\tilde{q}_j + \tilde{q}_{j+1}) \quad \forall i \in E.$$

Thus Q is column bounded. Since by [4, Theorem 2.3] all monotone transition functions satisfy the forward equations, it follows from Theorem 4.2 that $P(t)$ is column continuous.

Example 6.2. All *Feller–Reuter–Riley transition functions* (and thus all *dual functions* (see [12, 18])) are column continuous. This has been proved by Reuter and Riley [15].

Example 6.3. *Birth–death matrices*, *branching matrices*. For the two classes of Q-matrices, the minimal Q-function is column continuous. Applying Theorem 5.1, we have in fact constructed all column continuous birth–death Q-functions, as well as column continuous branching Q-functions.

Example 6.4. Let Q be conservative, single-exit and column bounded q-matrix. Then there exist indeed infinitely many Q-functions which are not column continuous.

Take $\eta = 0$ and $0 \neq b \in l_1$ in Theorem 5.1. Then $\eta(\lambda) = b\phi(\lambda) \in l_1$ for every $\lambda > 0$. Thus

$$\lambda\eta(\lambda)1 = \|\lambda\eta(\lambda)\|_1 = \|\lambda b\Phi(\lambda)\|_1 \leqslant \|\lambda\Phi(\lambda)\|_1 \|b\|_1 \leqslant \|b\|_1 < +\infty.$$

Thus, $\Psi(\lambda)$, defined as in (5.1), is not column continuous by Theorem 5.1.

Example 6.5. There exists a column almost-bounded q-matrix Q, which is not column bounded, such that the minimal Q function $F(t)$ is column continuous. Let

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 2 & 0 & -2^2 & 0 & \cdots \\ 3 & 0 & 0 & -3^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Then it is easy to see that Q is column almost-bounded (not column bounded) and Q is zero-exit. We will show that Q satisfied (4.3), that is $e_j \in \overline{D(Q_\infty)}$ for all $j \in E$. Indeed, obviously $e_j \in D(Q_\infty)$ for $j \neq 0$. Take

$$x = \left(1, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \in l_\infty.$$

Then $Qx = 0$. Thus $x \in D(Q_\infty)$. Then

$$x^{(m)} = x - \sum_{n=1}^{m} \frac{1}{n} e_n = \left(1, 0, \ldots, 0, \frac{1}{m+1}, \frac{1}{m+2}, \ldots\right) \in D(Q_\infty).$$

But $\|x^{(m)} - e_0\| = \frac{1}{m+1} \to 0$ (as $m \to \infty$), which implies that $e_0 = \lim_{m\to\infty} x^{(m)} \in \overline{D(Q_\infty)}$. Therefore the minimal Q-function $F(t)$, by Lemma 4.4, is column continuous as desired.

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