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A new characterization of hyperbolic cylinder in anti-de Sitter space $H_1^{n+1}(-1)$

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Abstract

By investigating n-dimensional complete maximal spacelike hypersurfaces with two distinct principal curvatures in an $(n+1)$-dimensional anti-de Sitter space $H_1^{n+1}(-1)$, we give a new characterization of hyperbolic cylinder $H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$ in $H_1^{n+1}(-1)$.

Keywords: Hyperbolic cylinder; Maximal spacelike hypersurface; Anti-de Sitter space

1. Introduction

Let $M_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian manifold of constant curvature c, we also call it a *Lorentzian space form*. When $c > 0$, $M_1^{n+1}(c) = S_1^{n+1}(c)$ (i.e., $(n+1)$-dimensional de Sitter space); when $c = 0$, $M_1^{n+1}(c) = L^{n+1}$ (i.e., $(n+1)$-dimensional Lorentz–Minkowski space); when $c < 0$, $M_1^{n+1}(c) = H_1^{n+1}(c)$ (i.e., $(n+1)$-dimensional anti-de Sitter space). A hypersurface M of $M_1^{n+1}(c)$ is said to be spacelike if the induced metric on M from that of the ambient space is positive definite.

E. Calabi [1] first studied the Bernstein problem for a maximal spacelike entire graph in L^{n+1} and proved that it has to be hyperplane, when $n \leqslant 4$. S.Y. Cheng and S.T. Yau [2] proved that the conclusion remains true for all n. As a generalization of the Bernstein type problem, Cheng

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and Yau [2] and T. Ishihara [3] proved that a complete maximal spacelike submanifold M of $M_1^{n+1}(c)$ $(c \geqslant 0)$ is totally geodesic. T. Ishihara [3] also proved the following well-known result:

Theorem 1.1. [3] *Let M be an n-dimensional $(n \geqslant 2)$ complete maximal spacelike hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$, then the norm square of the second fundamental form of M satisfies*

$$S \leqslant n, \tag{1.1}$$

and $S = n$ if and only if $M^n = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$ $(1 \leqslant m \leqslant n-1)$.

In this paper, we prove the following result, which gives a new characterization of hyperbolic cylinder in anti-de Sitter space $H_1^{n+1}(-1)$.

Theorem 1.2. *Let M be an n-dimensional $(n \geqslant 3)$ complete maximal spacelike hypersurface with two distinct principal curvatures λ and μ in anti-de Sitter space $H_1^{n+1}(-1)$. If $\inf(\lambda-\mu)^2 > 0$, then $M = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$ $(1 \leqslant m \leqslant n-1)$.*

2. Preliminaries

Let M be an n-dimensional complete spacelike hypersurface of anti-de Sitter space $H_1^{n+1}(-1)$. For any $p \in M$, we choose a local orthonormal frame $e_1, \dots, e_n, e_{n+1}$ in $H_1^{n+1}(-1)$ around p such that $e_1, \dots, e_n$ are tangent to M. Take the corresponding dual coframe $\omega_1, \dots, \omega_n, \omega_{n+1}$ with the matrix of connection one forms being ω_{ij}. The metric of $H_1^{n+1}(-1)$ is given by $\overline{ds^2} = \sum_i \omega_i^2 - \omega_{n+1}^2$. We make the convention on the range of indices that $1 \leqslant i, j, k \leqslant n$.

A well-known argument [2] shows that the forms ω_{in+1} may be expressed as $\omega_{in+1} = \sum_j h_{ij}\omega_j$, $h_{ij} = h_{ji}$. The second fundamental form is $B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$. The mean curvature of M is given by $H = \frac{1}{n}\sum_i h_{ii}$. If $H = 0$, then M is said to be *maximal*. The Gauss equations are

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.1}$$

$$R_{ij} = -(n-1)\delta_{ij} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{2.2}$$

$$n(n-1)(R+1) = -n^2H^2 + S, \tag{2.3}$$

where R is the normalized scalar curvature of M and the norm square of the second fundamental form is

$$S = \sum_{i,j}(h_{ij})^2. \tag{2.4}$$

The Codazzi equations are

$$h_{ijk} = h_{ikj}, \tag{2.5}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.6}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{lj} + \sum_l h_{ijl}\omega_{lk}. \tag{2.7}$$

By exterior differentiation of (2.6), we have the following Ricci identities:

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.8}$$

We may choose a frame field $\{e_1, \dots, e_{n+1}\}$ such that

$$\omega_{in+1} = \lambda_i\omega_i, \quad \text{that is} \quad h_{ij} = \lambda_i\delta_{ij}, \quad i = 1, 2, \dots, n, \tag{2.9}$$

where λ_i are principal curvatures. If we assume that M is a maximal spacelike hypersurface with two distinct principal curvatures, then we may put

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda, \quad \lambda_{m+1} = \cdots = \lambda_n = \mu, \quad \lambda \neq \mu,$$

and we obtain

$$m\lambda + (n-m)\mu = 0. \tag{2.10}$$

Example. Hyperbolic cylinder:

$$M_{m,n-m} = H^m\left(-\frac{n}{m}\right) \times H^{n-m}\left(-\frac{n}{n-m}\right) \quad (1 \leqslant m \leqslant n-1).$$

We know (see [3]) that $M_{m,n-m}$ is a complete maximal spacelike hypersurface in $H_1^{n+1}(-1)$ with two distinct principal curvatures λ and μ, where

$$\lambda_1 = \cdots = \lambda_m = \lambda = \sqrt{\frac{n-m}{m}}, \qquad \lambda_{m+1} = \cdots = \lambda_n = \mu = -\sqrt{\frac{m}{n-m}}.$$

By direct computation, we get that the square norm of the second fundamental form of $M_{m,n-m}$ satisfy $S = n$.

Now we have to consider two cases.

Case 1. $2 \leqslant m \leqslant n-2$. In this case, we make the convention on range of indices that

$$1 \leqslant a, b, c \leqslant m, \quad m+1 \leqslant \alpha, \beta, \gamma \leqslant n, \quad 1 \leqslant i, j, k \leqslant n.$$

Proposition 2.1. *Let M be an n-dimensional maximal spacelike hypersurface with two distinct principal curvature in anti-de Sitter $H_1^{n+1}(-1)$. If the multiplicities of these two distinct principal curvatures are greater than* 1, *then $h_{ijk} = 0$ and $M = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$ $(2 \leqslant m \leqslant n-2)$.*

Proof. Choosing $i = a$, $j = b$ in (2.6) and noting $h_{ab} = \lambda_a\delta_{ab}$, $\lambda_a = \lambda_b$, we have

$$\sum_{k=1}^n h_{abk}\omega_k = dh_{ab} = d\lambda_a \cdot \delta_{ab}. \tag{2.11}$$

Because $m \geqslant 2$, we can choose $a \neq b$, then $h_{abk} = 0$, in particular we obtain

$$h_{aba} = 0, \quad 1 \leqslant a \neq b \leqslant m. \tag{2.12}$$

Choosing $a = b$ in (2.11), we get $\sum_{k=1}^{n} h_{aak}\omega_k = d\lambda = \sum_k \lambda_{,k}\omega_k$, then it follows that

$$h_{aac} = \lambda_{,c}, \quad 1 \leqslant a, c \leqslant m. \tag{2.13}$$

From (2.5), we have $h_{aab} = h_{aba}$, then (2.12) and (2.13) imply

$$\lambda_{,b} = 0, \quad 1 \leqslant b \leqslant m. \tag{2.14}$$

Since $n - m \geqslant 2$, we also can get that

$$\mu_{,\alpha} = 0, \quad m + 1 \leqslant \alpha \leqslant n. \tag{2.15}$$

Combining (2.10), (2.14) with (2.15), we see that $\lambda =$ constant and $\mu =$ constant, also $h_{ijk} = 0$. From Ishihara's works [3], we conclude $M = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$ $(2 \leqslant m \leqslant n-2)$. We complete the proof of Proposition 2.1. □

Case 2. $m = n - 1$.

From (2.10), we assume that

$$\lambda_1 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu,$$

then it follows that

$$\lambda - \mu = n\lambda, \quad \lambda\mu = -(n-1)\lambda^2. \tag{2.16}$$

Because $n \geqslant 3$, we have $m = n - 1 \geqslant 2$. By similar discussion as Case 1, we obtain

$$\lambda_{,1} = \cdots = \lambda_{,n-1} = 0. \tag{2.17}$$

Combining (2.16) with (2.17), we get

$$\mu_{,1} = \cdots = \mu_{,n-1} = 0. \tag{2.18}$$

Noting $h_{ij} = \lambda_i \delta_{ij}$ and (2.6), we have

$$\sum_k h_{ijk}\omega_k = \delta_{ij} d\lambda_i + (\lambda_i - \lambda_j)\omega_{ij}. \tag{2.19}$$

From (2.19), (2.17) and (2.18), we obtain

$$h_{ijk} = 0, \quad \text{if } i \neq j,\ \lambda_i = \lambda_j, \tag{2.20}$$

$$h_{aab} = 0, \qquad h_{aan} = \lambda_{,n}, \tag{2.21}$$

$$h_{nna} = 0, \qquad h_{nnn} = \mu_{,n}. \tag{2.22}$$

We introduce the following generalized maximum principle (see Omori [5] and Yau [10]) in order to prove our Theorem 1.2.

Lemma 2.1. (Omori [5], Yau [10]) *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2 function which bounded from below on M. Then there is a sequence of points p_k in M such that*

$$\lim_{k\to\infty} f(p_k) = \inf(f), \qquad \lim_{k\to\infty} \left|\nabla f(p_k)\right| = 0, \qquad \lim_{k\to\infty} \inf \Delta f(p_k) \geqslant 0.$$

3. Proof of Theorem 1.2

At first, we prove the following key lemma.

Lemma 3.1. *Let M be an n-dimensional $(n \geqslant 3)$ maximal spacelike hypersurface with two distinct principal curvatures in $H_1^{n+1}(-1)$, then we have*

$$|\nabla S|^2 = \sum_{k=1}^{n} (S_{,k})^2 = \frac{4nS}{n+2} \sum_{i,j,k} h_{ijk}^2. \tag{3.1}$$

Proof. We have to consider two cases.

Case 1. $m \geqslant 2$, $n-m \geqslant 2$, that is, the multiplicity of two distinct principal curvature are great than 1. From Proposition 2.1, we know that $\lambda = \text{constant}$, $\mu = \text{constant}$ and $h_{ijk} = 0$. Thus $S = \text{constant}$ and $h_{ijk} = 0$, (3.1) holds in this case.

Case 2. $m = n-1$, that is, $\lambda_1 = \cdots = \lambda_{n-1} = \lambda$, $\lambda_n = \mu$, then we have

$$S = n(n-1)\lambda^2, \tag{3.2}$$

$$S_{,i} = 2n(n-1)\lambda\lambda_{,i}. \tag{3.3}$$

By use of (2.14), (3.2) and (3.3), we obtain

$$\sum_{k=1}^{n} (S_{,k})^2 = (S_{,n})^2 = 4n(n-1)S(\lambda_{,n})^2. \tag{3.4}$$

On the other hand, by use of (2.20)–(2.22), we know

$$\begin{aligned}
\sum_{i,j,k} h_{ijk}^2 &= \sum_{a,b,c=1}^{n-1} h_{abc}^2 + 3\sum_{a,b=1}^{n-1} h_{abn}^2 + 3\sum_{a=1}^{n-1} h_{ann}^2 + h_{nnn}^2 \\
&= 3\sum_{a=1}^{n-1} h_{naa}^2 + h_{nnn}^2 = 3(n-1)(\lambda_{,n})^2 + (\mu_{,n})^2 \\
&= (n-1)(n+2)(\lambda_{,n})^2.
\end{aligned} \tag{3.5}$$

Combining (3.4) with (3.5), we complete the proof of Lemma 3.1. □

Lemma 3.2. *Let M be an n-dimensional $(n \geqslant 3)$ complete maximal hypersurface of anti-de Sitter space $H_1^{n+1}(-1)$ with two distinct principal curvatures one of which is simple (i.e., $\lambda_1 = \cdots = \lambda_{n-1} = \lambda \neq \lambda_n = \mu$). If $\inf(\lambda-\mu)^2 > 0$, then*

$$S \geqslant n, \tag{3.6}$$

where S is the norm square of the second fundamental form of M.

Proof. Making use of the (2.8), (2.5), (2.4), we can compute the Laplacian ΔS of S as follows (also see [3,4]):

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ij}h_{ijkk}$$

$$
\begin{aligned}
&= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ij} h_{ikjk} \\
&= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ij} \Big(h_{ikkj} + \sum_m h_{km} R_{mijk} + \sum_m h_{mj} R_{mkjk} \Big) \\
&= \sum_{i,j,k} h_{ijk}^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (\lambda_i - \lambda_j)^2. \qquad (3.7)
\end{aligned}
$$

By use of Gauss equation (2.1), we obtain $R_{anan} = -1 - \lambda\mu = -1 + (n-1)\lambda^2$ for $1 \leqslant a \leqslant n-1$. Then we have by use of (3.2)

$$
\begin{aligned}
\frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + (n-1)(\lambda - \mu)^2 \big[(n-1)\lambda^2 - 1\big] \\
&= \sum_{i,j,k} h_{ijk}^2 + (n-1)n^2\lambda^2 \big[(n-1)\lambda^2 - 1\big] \\
&= \sum_{i,j,k} h_{ijk}^2 + S(S-n). \qquad (3.8)
\end{aligned}
$$

Since we assume $\inf(\lambda - \mu)^2 = b^2 > 0$, from $(n-1)\lambda + \mu = 0$ and (3.2), we have

$$
S \geqslant \frac{n-1}{n} b^2 > 0. \qquad (3.9)
$$

Combining (3.9), Lemma 3.1 with (3.8) we have

$$
\frac{1}{2}\Delta S = \frac{n+2}{4nS} |\nabla S|^2 + S(S-n). \qquad (3.10)
$$

Noting $R_{ii} = -(n-1) + \lambda_i^2 \geqslant -(n-1)$ and $S \geqslant \frac{n-1}{n} b^2 > 0$, we know that Omori and Yau's generalized maximum principle (Lemma 2.1) can be applied to the function S on M. Then there is a sequence of points p_k in M such that

$$
\lim_{k\to\infty} S(p_k) = \inf S, \qquad \lim_{k\to\infty} \big|\nabla S(p_k)\big| = 0, \qquad \lim_{k\to\infty} \inf \Delta S(p_k) \geqslant 0.
$$

Approaching limit of the both side of equality (3.10), we obtain

$$
0 \leqslant \inf S \cdot (\inf S - n). \qquad (3.11)
$$

From (3.9) we have $\inf S > 0$, then we obtain

$$
\inf S \geqslant n. \qquad (3.12)
$$

We complete the proof of Lemma 3.2. □

Proof of Theorem 1.2. We assume that M has two distinct principal curvature λ (multiplicity m) and μ (multiplicity $n-m$).

Case 1. $2 \leqslant m \leqslant n-2$. By Proposition 2.1 we know $M = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$, $2 \leqslant m \leqslant n-2$.

Case 2. $m = n-1$. From Theorem 1.1 of T. Ishihara [3], we know that $S \leqslant n$. From Lemma 3.2, we get $S \geqslant n$. Hence, $S = n$ on M. Since $S = n(n-1)\lambda^2$, we have $\lambda^2 = \frac{1}{n-1}$, $\mu^2 = n-1$. Then M is isometric to $H^1(-n) \times H^{n-1}(-\frac{n}{n-1})$. We complete the proof of Theorem 1.2. □

Remark 3.1. In [6–9], the authors studied n-dimensional complete hypersurfaces with two distinct principal curvatures in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$.

Remark 3.2. The referee proposed the following:

Conjecture. *The only complete spacelike hypersurfaces in* $M_1^{n+1}(c)$ $(c \leqslant 0)$ *with constant mean curvature and two distinct principal curvatures* λ *and* μ *satisfying* $\inf(\lambda-\mu)^2 > 0$ *are the hyperbolic cylinders.*

This conjecture is interesting, but our method in this paper is not effective to the conjecture.

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