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Existence and uniqueness of positive solution for nonhomogeneous sublinear elliptic equations

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ABSTRACT

In this paper, we study the existence and the uniqueness of positive solution for the sublinear elliptic equation, $-\Delta u + u = |u|^p\, sgn(u) + f$ in $\mathbb{R}^N$, $N \geqslant 3$, $0 < p < 1$, $f \in L^2(\mathbb{R}^N)$, $f > 0$ a.e. in $\mathbb{R}^N$. We show by applying a minimizing method on the Nehari manifold that this problem has a unique positive solution in $H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$. We study its continuity in the perturbation parameter f at 0.

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1. Introduction

In this paper, we study the existence and uniqueness of positive solution of the following nonhomogeneous problem

$$\begin{cases} -\Delta u + u = |u|^p\, sgn(u) + f, \\ u \in H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

$$0 < p < 1, \qquad f \in L^2(\mathbb{R}^N), \quad N \geqslant 3, \text{ satisfying } f > 0, \text{ a.e. in } \mathbb{R}^N. \tag{R_0}$$

The problem (1.1) can be considered as a perturbation of the following homogeneous problem

$$\begin{cases} -\Delta u + u = |u|^p\, sgn(u), \\ u \in H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N). \end{cases} \tag{1.2}$$

The trivial solution, namely 0, is the only solution of problem (1.2) (to be proved later).

Over the last years, many authors have studied the existence of solutions of the following problem

$$\begin{cases} -\Delta u + u = g(x,u) + f, \\ u \in H^1(\mathbb{R}^N), \quad u > 0 \end{cases} \tag{1.3}$$

where g is superlinear and subcritical, which roughly speaking means that

$$g(x,u).|u|^{-\frac{N+2}{N-2}} \to 0 \quad \text{as } |u| \to \infty \text{ (at least if } N \geqslant 3).$$

See for instance [4–7,9,15,17,20–22,27,30,31,34,35] and the references therein.

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The existence of at least two positive solutions of problem (1.3) where $g(x,u)=u^p$, $1<p<\frac{N+2}{N-2}$, is proved for a small L^2-norm and an exponential decay of f in Zhu [35] or a small $L^{\frac{p+1}{p-1}}$-norm of f in Hirano [21]. The same result is obtained by Cao, Zhou [17] and Jeanjean [22], under the assumptions below:

$$g(x,u)=a(x)u^p,\quad 1<p<\frac{N+2}{N-2},\ a(x)\geqslant 1 \text{ for all } x\in\mathbb{R}^N$$

or

$$g(x,u)\geqslant \bar{g}(u)=\lim_{|x|\to\infty} g(x,u),\quad \text{for all } x\in\mathbb{R}^N \text{ and } u>0.$$

In [5] Adachi, Tanaka considered the case

$$g(x,u)=a(x)u^p,\quad 1<p<\frac{N+2}{N-2},\ a(x)\in(0,1],\ a(x)\neq 1$$

they proved the existence of at least four positive solutions of (1.3) for sufficiently small $\|f\|_{H^{-1}(\mathbb{R}^N)}$. More general superlinear case g was considered in [4], the authors proved the existence of at least two positive solutions.

In [27], A. Malchiodi considered both subcritical and critical exponent:

$$g(x,u)=\big(1-\epsilon a(x)\big)u^p,\quad 1<p<\frac{N+2}{N-2}$$

and

$$g(x,u)=\big(1-\epsilon a(x)\big)u^p+u,\quad p=\frac{N+2}{N-2},\qquad f=\epsilon h\geqslant 0,\quad \epsilon>0 \text{ small}.$$

The author proved under more general assumptions on a and h, the existence of four classical solutions for the subcritical case and two classical solutions for critical case (see also [1–3]).

In [34], Zhou proved the existence of two positive solutions of problem (1.3) with $g(x,u)=|u|^{p-2}u$, $2<p<2^*$, f is small enough and satisfying $f(x)\leqslant \frac{C}{(1+|x|^2)^{\frac{p}{p-1}}}$ for some $C>0$.

In [15] Chen, Peng considered the following problem

$$\begin{cases} -\Delta u(x)+u(x)=\lambda\big(g(x,u)+f(x)\big), \\ u\in H^1\big(\mathbb{R}^N\big),\quad u>0 \end{cases} \tag{1.4}$$

where $\lambda>0$, g is a superlinear and $f\in L^2\cap L^{\frac{N}{2}}$. The authors proved the existence of $0<\lambda^*<\infty$ such that (1.4) has exactly two positive solutions for $\lambda\in(0,\lambda^*)$, no solution for $\lambda>\lambda^*$, a unique solution for $\lambda=\lambda^*$ under suitable conditions of g.

In [20] Ghimenti, Micheletti studied the following equation

$$-\Delta u+V(x)u=g'(u)+f(x)\quad \text{in } \mathbb{R}^N \tag{1.5}$$

under the assumptions below:

$$V\leqslant 0,\quad \lim_{|x|\to\infty} V(x)=0,\qquad g \text{ is } C^3(\mathbb{R}) \text{ with double power behaviour}$$

they proved the existence of two nonnegative solutions when $\|f\|_{\frac{2N}{N+2}}$ is sufficiently small.

It seems to us that very few results are known on perturbation of sublinear elliptic equation in $\mathbb{R}^N$. There exists a general method to solve the analogue of (1.3) in bounded domains (see P. Bolle [11], Bolle, Ghoussoub and Tehrani [12] and the references therein). In [23], Kajikiya proved the existence of infinitely many solutions of the following system

$$\begin{cases} -\Delta u=|u|^p\, sgn(u)+f(x,u), & x\in\Omega, \\ u=0, & x\in\partial\Omega \end{cases} \tag{1.6}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and $0<p<1$, under the suitable conditions of f. While in $\mathbb{R}^N$ to the author's knowledge, little is known. On this subject, in [33] Tehrani proved the existence of at least one solution of the following equation

$$-\Delta u+V(x)u=g(x,u),\quad x\in\mathbb{R}^N \tag{1.7}$$

such that g is a sublinear function and V satisfying:

$$V \in L^{\infty}(\mathbb{R}^N) \quad \text{with } v_\infty = \liminf_{|x|\to\infty} V(x) > 0, \tag{F}$$

$$\int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 + V(x)\varphi^2\right) dx < 0 \quad \text{for some } \varphi \in C_c^{\infty}(\mathbb{R}^N) \tag{FF}$$

(see also [16]). It is clear that assumption (FF) is not satisfied by our problem (1.1).

In [13] Benrhouma, Ounaies considered the following problem

$$\begin{cases} -\Delta u = u - |u|^{-2\theta}u + f & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \cap L^{2(1-\theta)}(\mathbb{R}^N) \end{cases} \tag{1.8}$$

where $f \in L^2$, $f \geqslant 0$, $f \neq 0$ and $0 < \theta < \frac{1}{2}$, they proved the existence of at least two nonnegative solutions of (1.8) for a sufficiently small $\|f\|_2$ (see also [10]).

The aim of this paper is to prove the existence and uniqueness of positive solution for nonhomogeneous problem (1.1), our approach is based on minimizing method on Nehari manifold and P.L. Lions concentration–compactness principle (see [25,36]). Our main result is the following.

Theorem 1.1. *Assume (R_0) holds. Then the problem* (1.1) *possesses a unique positive solution which converges to zero in $H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ as $\|f\|_2$ tends to zero.*

We organize this paper into four sections. In Section 2 we give some notations, preliminaries and useful results, moreover we study some properties of the Nehari manifold corresponding to problem (1.1). In Section 3, we prove the existence of positive solution of problem (1.1) which is a critical point of the associated functional in the Nehari manifold. In Section 4, we show the uniqueness of the positive solution and we prove that this solution tends to zero in $H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ as f tends to zero in L^2.

2. Notations and preliminary

We will use the following notations:

(•) $B(0,r) = \{x \in \mathbb{R}^N,\ |x| < r\}$,
(•) $\|u\|_q = (\int_{\mathbb{R}^N} |u|^q\, dx)^{\frac{1}{q}}$, $\|u\|_{H^1} = (\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\, dx)^{\frac{1}{2}}$,
(•) c_s: the constant of Sobolev, Gagliardo, Nirenberg in $\mathbb{R}^N$ such that

$$\forall\, u \in H^1, \quad \|u\|_{2^*} \leqslant c_s \|\nabla u\|_2 \quad \text{where } 2^* = \frac{2N}{N-2},$$

(•) *supp*(φ): the support of the function φ,
(•) *sgn*(u): the sign of the function u,
(•) $F'(u)$: the Fréchet derivative of F at u.

Let

$$E = H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$$

we endow E with the norm

$$\|u\| = \|\nabla u\|_2 + \|u\|_{p+1}$$

($(E, \|\ \|)$ is a Banach space). We define the functionals I^∞, I and g on E:

$$I^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\, dx,$$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\, dx - \int_{\mathbb{R}^N} f u\, dx$$

and

$$g(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx - \int_{\mathbb{R}^N} |u|^{p+1}\, dx - \int_{\mathbb{R}^N} f u\, dx.$$

The functionals I^∞, $I \in C^1$ on E and $g(u) = \langle I'(u), u\rangle$, $g \in C^1$ on E.

To get the solutions of problem (1.1) we look for critical points of the functional I. But I is not bounded neither above nor below on E so we introduce the following open subset of E

$$\alpha > 1, \quad F^\alpha = \left\{ u \in E,\ \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx > \alpha \int_{\mathbb{R}^N} |u|^{p+1}\, dx \right\}$$

and we consider the Nehari manifold (see [28])

$$N^\alpha = \left\{ u \in F^\alpha,\ \left\langle I'(u), u \right\rangle = 0 \right\},$$

α will be fixed later. Note that $g'(u) \neq 0$ for any $u \in N^\alpha$.

Let $u \in E \setminus \{0\}$, consider the function $\phi_u : [0, \infty[\to \mathbb{R}$ defined by

$$\phi_u(t) = \frac{t^2}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx - \frac{t^{p+1}}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx - t \int_{\mathbb{R}^N} f u\, dx = I(tu),$$

$$\phi_u'(t) = t \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx - t^p \int_{\mathbb{R}^N} |u|^{p+1}\, dx - \int_{\mathbb{R}^N} f u\, dx = \left\langle I'(tu), u \right\rangle,$$

$$\phi_u''(t) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx - p t^{p-1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx,$$

$$\phi_u''(t) = 0 \quad \text{if and only if} \quad t = \left[\frac{p \|u\|_{p+1}^{p+1}}{\|u\|_{H^1}^2} \right]^{\frac{1}{1-p}}.$$

Lemma 2.1. *N^α is not empty.*

Proof. $N^\alpha \neq \emptyset$ indeed, let $\varphi \in C_c^\infty(\mathbb{R}^N)$ ($\varphi \in C^\infty(\mathbb{R}^N)$ with compacted support) such that $\varphi \geqslant 0$, $\varphi \neq 0$. By assumption (R_0), there exists $x_0 \in \mathbb{R}^N$ such that $f(x_0) > 0$. Set $u(x) = \varphi(\sigma(x - x_0))$, $\sigma > 0$ and $T = (\frac{\alpha \|u\|_{p+1}^{p+1}}{\|u\|_{H^1}^2})^{\frac{1}{1-p}}$.

$\phi_u'(T) < 0$ for σ large enough, indeed:

$$\begin{aligned}
\phi_u'(T) &= T \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx - T^p \int_{\mathbb{R}^N} |u|^{p+1}\, dx - \int_{\mathbb{R}^N} f u\, dx \\
&= \left(\frac{\alpha \|u\|_{p+1}^{p+1}}{\|u\|_{H^1}^2} \right)^{\frac{1}{1-p}} \|u\|_{H^1}^2 - \left(\frac{\alpha \|u\|_{p+1}^{p+1}}{\|u\|_{H^1}^2} \right)^{\frac{p}{1-p}} \|u\|_{p+1}^{p+1} - \int_{\mathbb{R}^N} f u\, dx \\
&= \alpha^{\frac{1}{1-p}} \left(\|u\|_{p+1}^{p+1}\right)^{\frac{1}{1-p}} \left(\|u\|_{H^1}^2\right)^{\frac{-p}{1-p}} - \alpha^{\frac{p}{1-p}} \left(\|u\|_{p+1}^{p+1}\right)^{\frac{1}{1-p}} \left(\|u\|_{H^1}^2\right)^{\frac{-p}{1-p}} - \int_{\mathbb{R}^N} f u\, dx \\
&= \alpha^{\frac{p}{1-p}} (\alpha - 1) \left(\|u\|_{p+1}^{p+1}\right)^{\frac{1}{1-p}} \left(\|u\|_{H^1}^2\right)^{\frac{-p}{1-p}} - \int_{\mathbb{R}^N} f u\, dx \\
&= \frac{\alpha^{\frac{p}{1-p}} (\alpha - 1) (\|u\|_{p+1}^{p+1})^{\frac{1}{1-p}} - (\|u\|_{H^1}^2)^{\frac{p}{1-p}} \int_{\mathbb{R}^N} f u\, dx}{(\|u\|_{H^1}^2)^{\frac{p}{1-p}}}.
\end{aligned}$$

We have

$$\left(\|u\|_{p+1}^{p+1}\right)^{\frac{1}{1-p}} = \sigma^{\frac{-N}{1-p}} \|\varphi\|_{p+1}^{\frac{p+1}{1-p}},$$

$$\left(\|u\|_{H^1}^2\right)^{\frac{p}{1-p}} = \left(\sigma^{2-N} \|\nabla\varphi\|_2^2 + \sigma^{-N} \|\varphi\|_2^2\right)^{\frac{p}{1-p}} \geqslant \sigma^{\frac{(2-N)p}{1-p}} \|\nabla\varphi\|_2^{\frac{2p}{1-p}}$$

and

$$\int_{\mathbb{R}^N} f u\, dx = \int_{\mathbb{R}^N} f(x) \varphi\left(\sigma(x - x_0)\right) dx = \sigma^{-N} \int_{\mathbb{R}^N} f\left(x_0 + \frac{x}{\sigma}\right) \varphi(x)\, dx.$$

Furthermore, there exists $r > 0$ independent of σ such that

$$\int_{\mathbb{R}^N} f\left(x_0 + \frac{x}{\sigma}\right)\varphi(x)\,dx \geqslant r \quad \text{and} \quad \int_{\mathbb{R}^N} f u\,dx > r\sigma^{-N}, \quad \text{for } \sigma \text{ large enough}$$

it follows that

$$\left(\|u\|_{H^1}^2\right)^{\frac{p}{1-p}} \int_{\mathbb{R}^N} f u\,dx > r\sigma^{\frac{2p-N}{1-p}} \|\nabla\varphi\|_2^{\frac{2p}{1-p}}$$

and

$$\phi_u'(T) < \frac{\alpha^{\frac{p}{1-p}}(\alpha-1)\sigma^{\frac{-N}{1-p}}\|\varphi\|_{p+1}^{\frac{p+1}{1-p}} - r\sigma^{\frac{2p-N}{1-p}}\|\nabla\varphi\|_2^{\frac{2p}{1-p}}}{(\sigma^{2-N}\|\nabla\varphi\|_2^2 + \sigma^{-N}\|\varphi\|_2^2)^{\frac{p}{1-p}}} < 0, \quad \text{for } \sigma \text{ large enough.}$$

On the other hand,

$$\phi_u'(0) = -\int_{\mathbb{R}^N} f u\,dx < 0 \quad \text{and} \quad \lim_{t\to\infty} \phi_u'(t) = +\infty.$$

It yields that there exists a unique minimum $t_u^1 > 0$ of ϕ_u, $\phi_u'(t_u^1) > \phi_u'(T)$ then

$$t_u^1 > T \quad \text{and} \quad \left\|t_u^1 u\right\|_{H^1}^2 > \alpha\left\|t_u^1 u\right\|_{p+1}^{p+1}$$

hence $w_1 = t_u^1 u \in N^\alpha$. □

For seeking critical points of I, we need the following result.

Lemma 2.2. *Let (u_n) be a sequence in E satisfying $g(u_n) = 0$ for any $n \in \mathbb{N}$ and $(I(u_n))$ is bounded, then (u_n) is bounded in E.*

Proof. We have

$$\begin{aligned}
\frac{1}{p+1}\langle I'(u_n), u_n\rangle - I(u_n) &= \frac{1-p}{2(p+1)}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |u_n|^2\right)dx + \frac{p}{p+1}\int_{\mathbb{R}^N} f u_n\,dx \\
&\geqslant \frac{1-p}{2(p+1)}\|u_n\|_{H^1}^2 - \frac{p}{p+1}\|f\|_2\|u_n\|_2 \\
&\geqslant \frac{1-p}{4(p+1)}\|u_n\|_{H^1}^2 - c\|f\|_2^2 \qquad \text{(Young inequality)}
\end{aligned}$$

therefore (u_n) is bounded in $H^1(\mathbb{R}^N)$. Furthermore,

$$\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u_n|^2 + |u_n|^2\right)dx + I(u_n) = \frac{p}{p+1}\int_{\mathbb{R}^N}|u_n|^{p+1}\,dx$$

then (u_n) is bounded in $L^{p+1}(\mathbb{R}^N)$ and so (u_n) is bounded in E. □

At the end of this section, we study the homogeneous problem (1.2).

Lemma 2.3. *The problem* (1.2) *possesses only the trivial solution in E.*

Proof. Let u be a solution of problem (1.2), so in the distribution sense

$$-\Delta u + u = |u|^p\, sgn(u) \tag{2.1}$$

multiplying (2.1) by u and integrating by parts, we get

$$\int_{\mathbb{R}^N}\left(|\nabla u|^2 + |u|^2\right)dx = \int_{\mathbb{R}^N}|u|^{p+1}\,dx.$$

By Brezis–Kato theorem, u is a classical solution of problem (1.2) (see [32]). Moreover, $\nabla u \in L^2(\mathbb{R}^N)$, $u \in L^\infty_{loc}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \left(-\frac{1}{2}|u|^2 + \frac{1}{p+1}|u|^{p+1}\right) dx < \infty.$$

Pohožeav identity [29] gives

$$\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \int_{\mathbb{R}^N} \left(-\frac{1}{2}|u|^2 + \frac{1}{p+1}|u|^{p+1}\right) dx = -\frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx + \frac{1}{p+1} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx$$

thus

$$\left(\frac{N-2}{2N} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \left(\frac{1}{p+1} - \frac{1}{2}\right) \int_{\mathbb{R}^N} |u|^2 \, dx$$

this leads to $u = 0$. □

We study now the existence of a solution of problem (1.1).

3. Existence of a positive solution of problem (1.1)

We consider

$$\overline{N^\alpha} = \left\{ u \in E,\ g(u) = 0,\ \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx \geqslant \alpha \int_{\mathbb{R}^N} |u|^{p+1} \, dx \right\}$$

where $\alpha > 1$ will be fixed later. Set

$$m = \inf_{u \in \overline{N^\alpha}} I(u). \tag{mm}$$

Remark 3.0. $m < 0$, in fact: let w_1 be given in the proof of Lemma 2.1, then $m \leqslant I(w_1) = \phi_u(t_u^1) < \phi_u(0) = 0$.

Lemma 3.1. *The functional I is bounded below on $\overline{N^\alpha}$.*

Proof. Suppose that there exists a sequence $(u_n) \subset \overline{N^\alpha}$ such that $I(u_n) \to -\infty$

$$I(u_n) = \frac{p}{p+1} \|u_n\|_{p+1}^{p+1} - \frac{1}{2} \|u_n\|_{H^1}^2 \geqslant -\frac{1}{2} \|u_n\|_{H^1}^2$$

then $\|u_n\|_{H^1} \to +\infty$, we also have

$$0 = \frac{1}{\|u_n\|_{H^1}^2} \langle I'(u_n), u_n \rangle = 1 - \frac{\|u_n\|_{p+1}^{p+1}}{\|u_n\|_{H^1}^2} - \frac{1}{\|u_n\|_{H^1}^2} \int_{\mathbb{R}^N} f u_n \, dx \geqslant 1 - \frac{1}{\alpha} - \frac{1}{\|u_n\|_{H^1}^2} \int_{\mathbb{R}^N} f u_n \, dx.$$

Passing to the limit as $n \to \infty$, we get $1 - \frac{1}{\alpha} \leqslant 0$; that is contradiction, this leads to $m > -\infty$. □

Lemma 3.2. *Fixed $\alpha = 1 + \epsilon$, $\epsilon > 0$ small enough, then*

$$m = \inf_{u \in \overline{N^\alpha}} I(u) = \inf_{u \in N^\alpha} I(u).$$

Proof. Suppose there exists a minimizing sequence $u_n \in \overline{N^\alpha}$ of problem (mm) such that $I(u_n) \to m$, $\langle I'(u_n), u_n \rangle = 0$ and $\|u_n\|_{H^1}^2 = \alpha \|u_n\|_{p+1}^{p+1}$.

By Lemma 2.2, there exists $b > 0$ such that $\|u_n\|_{p+1}^{p+1} < b$ for any $n \in \mathbb{N}$.

Let $u \in C_c^\infty(\mathbb{R}^N)$ such that $b < \|u\|_{H^1}^2 < \|u\|_{p+1}^{p+1}$ and $\int_{\mathbb{R}^N} f(x) u(x) \, dx > 0$, take $S = \left(\frac{\alpha \|u\|_{p+1}^{p+1}}{\|u\|_{H^1}^2}\right)^{\frac{1}{1-p}} > 1$.

We have

$$\phi_u'(S) = \alpha^{\frac{p}{1-p}} (\alpha - 1) \|u\|_{p+1}^{\frac{p+1}{1-p}} \|u\|_{H^1}^{\frac{-2p}{1-p}} - \int_{\mathbb{R}^N} f u$$

put $\alpha = 1 + \epsilon$, then $\phi_u'(S) < 0$ for ϵ small enough.

On the other hand, $\lim_{t\to\infty}\phi_u'(t)=+\infty$, thus there exists $t>S>1$ such that $\phi_u'(t)=0$ and $v=tu\in N^\alpha$. Moreover

$$\|v\|_{p+1}^{p+1}=t^{p+1}\|u\|_{p+1}^{p+1}>\|u\|_{p+1}^{p+1}>b$$

therefore

$$I(v)<\left(\frac{p}{p+1}-\frac{\alpha}{2}\right)b<I(u_n).$$

We obtain

$$m=\inf_{u\in\overline{N^\alpha}}I(u)\leqslant I(v)<\left(\frac{p}{p+1}-\frac{\alpha}{2}\right)b\leqslant m$$

that is a contradiction. Hence $(u_n)\in N^\alpha$ and

$$m=\inf_{u\in\overline{N^\alpha}}I(u)=\inf_{u\in N^\alpha}I(u)$$

the proof of Lemma 3.2 is achieved. □

We apply Ekeland's variational principle [19] to problem (mm), then there exist $(u_n,\lambda_n)\subset N^\alpha\times\mathbb{R}$ such that $I(u_n)\to m$, $I'(u_n)-\lambda_n g'(u_n)\to 0$ in E'. (u_n) is called a (PS) sequence at level m of the functional I restricted to N^α.

Lemma 3.3. *Let α be fixed by Lemma 3.2, then any (PS) sequence (u_n) at level m of the functional I restricted to N^α, is a (PS) sequence of I on E.*

Proof. Let $(u_n)\in N^\alpha$ such that $I(u_n)\to m$ and $I'(u_n)-\lambda_n g'(u_n)\to 0$ in E'. By Lemma 2.2, (u_n) is bounded in E, then

$$\langle I'(u_n)-\lambda_n g'(u_n),u_n\rangle\to 0$$

and

$$\lambda_n g'(u_n).u_n\to 0\quad\text{in }\mathbb{R}$$

where

$$g'(u_n).u_n=\int_{\mathbb{R}^N}\left(|\nabla u_n|^2+|u_n|^2\right)dx-p\int_{\mathbb{R}^N}|u_n|^{p+1}\,dx,$$

$(g'(u_n).u_n)$ is bounded in $\mathbb{R}$, there exists a subsequence still denoted by $(g'(u_n).u_n)$, $g'(u_n).u_n\to l$. Suppose $l=0$, since

$$g'(u_n).u_n>(\alpha-p)\int_{\mathbb{R}^N}|u_n|^{p+1}>0$$

then $u_n\to 0$ in E and $I(u_n)\to 0=m$, that is contradiction. Hence $l\neq 0$ and $\lambda_n\to 0$ (up to a subsequence), therefore

$$I'(u_n)\to 0\quad\text{in }E'\quad\text{and}\quad(u_n)\text{ is a (PS) sequence of }I\text{ at level }m\text{ on }E.\qquad\square$$

Theorem 3.4. *There exists $u\in E$ a solution of problem (1.1), $I(u)=m$ and*

$$\int_{\mathbb{R}^N}\left(|\nabla u|^2+|u|^2\right)dx\geqslant\alpha\int_{\mathbb{R}^N}|u|^{p+1}\,dx.\tag{3.1}$$

To prove Theorem 3.4, we need the following classical lemma.

Lemma 3.5. *Let $g\in L^q(\mathbb{R}^N)$, $1\leqslant q<+\infty$, $\varphi\in C_c^\infty(\mathbb{R}^N)$ and $(y_n)\subset\mathbb{R}^N$ such that $|y_n|\to+\infty$ then*

$$\int_{\mathbb{R}^N}g(x)\varphi(x-y_n)\,dx\to 0.$$

Proof of Theorem 3.4. Let (u_n) be a (PS) sequence at level m for the functional I restricted to N^α. By Lemma 2.2, (u_n) is bounded in E.

$u_n \rightharpoonup u$ in E, $u_n \to u$ in L^q_{loc}, $\forall 1 \leqslant q < 2^*$ and $u_n \to u$ a.e. in $\mathbb{R}^N$ (up to a subsequence). Firstly, we prove that u is a solution of problem (1.1): by Lemma 3.3, (u_n) is a (PS) sequence of I on E. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$

$$\langle I'(u_n), \varphi\rangle = \int_{\mathbb{R}^N} (\nabla u_n \nabla\varphi + u_n\varphi)\,dx - \int_{\mathbb{R}^N} |u_n|^p\, sgn(u_n)\varphi\,dx - \int_{\mathbb{R}^N} f\varphi\,dx \to 0,$$

$u_n \rightharpoonup u$ in E, thus

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla\varphi + u_n\varphi)\,dx \to \int_{\mathbb{R}^N} (\nabla u \nabla\varphi + u\varphi)\,dx,$$

$u_n \to u$ in $L^{p+1}(supp(\varphi))$, then there exists a subsequence denoted by u_n, $w \in L^{p+1}$ such that $|u_n| \leqslant |w|$ and

$$\begin{cases} |u_n|^p|\varphi| \leqslant |w|^p|\varphi| \in L^1, \\ |u_n|^p\, sgn(u_n)\varphi \to |u|^p\, sgn(u)\varphi \quad \text{a.e. in } \mathbb{R}^N. \end{cases}$$

By dominated convergence theorem,

$$\int_{\mathbb{R}^N} |u_n|^p\, sgn(u_n)\varphi\,dx \to \int_{\mathbb{R}^N} |u|^p\, sgn(u)\varphi\,dx$$

therefore

$$\langle I'(u_n), \varphi\rangle \to \langle I'(u), \varphi\rangle = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

Hence u is a weak solution of the following problem

$$-\Delta u + u = |u|^p\, sgn(u) + f \quad \text{in } \mathbb{R}^N. \tag{3.2}$$

Now, we prove (3.1):

$$\int_{\mathbb{R}^N} f u_n\,dx = \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |u_n|^2\right)dx - \int_{\mathbb{R}^N} |u_n|^{p+1}\,dx > (\alpha - 1)\int_{\mathbb{R}^N} |u_n|^{p+1}$$

then

$$\int_{\mathbb{R}^N} f u\,dx \geqslant (\alpha - 1)\underline{\lim}\int_{\mathbb{R}^N} |u_n|^{p+1}\,dx \geqslant (\alpha - 1)\int_{\mathbb{R}^N} |u|^{p+1}\,dx.$$

Multiplying (3.2) by u and integrating by parts, we get

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right)dx - \int_{\mathbb{R}^N} |u|^{p+1}\,dx - \int_{\mathbb{R}^N} f u\,dx = 0$$

this gives

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right)dx \geqslant \alpha \int_{\mathbb{R}^N} |u|^{p+1}\,dx.$$

We still have to show that $I(u) = m$. We have $\|u\|_E \leqslant \underline{\lim}\|u_n\|_E$, we distinguish two cases:

($\bullet$) **(Compactness)** If $\|u\|_E = \underline{\lim}\|u_n\|_E$, then $u_n \to u$ in E.

Proof of ($\bullet$). Since $\|u_n\|_E$ is bounded, we extract a subsequence such that $\|u\|_E = \lim_{n\to\infty}\|u_n\|_E$ and

$$\overline{\lim}\|u_n\|_{p+1} = \|u\|_{p+1} + \|\nabla u\|_2 - \overline{\lim}\|\nabla u_n\|_2 \leqslant \|u\|_{p+1} + \|\nabla u\|_2 - \underline{\lim}\|\nabla u_n\|_2$$

but

$$\|\nabla u\|_2 \leqslant \underline{\lim}\|\nabla u_n\|_2 \quad \text{and} \quad \|u\|_{p+1} \leqslant \underline{\lim}\|u_n\|_{p+1}.$$

We obtain

$$\|u\|_{p+1} \leqslant \underline{\lim}\|u_n\|_{p+1} \leqslant \overline{\lim}\|u_n\|_{p+1} \leqslant \|u\|_{p+1}$$

thus

$$\begin{cases} u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \\ \|u_n\|_{p+1} \to \|u\|_{p+1}. \end{cases}$$

Applying Brezis–Lieb theorem [8], we get

$$\text{(i)} \qquad u_n \to u \quad \text{in } L^{p+1}$$

therefore

$$\|\nabla u_n\|_2 \to \|\nabla u\|_2.$$

We have

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^2\,dx = \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N} |\nabla u|^2 - 2\int_{\mathbb{R}^N} \nabla u_n \nabla u\,dx$$

by weak convergence of the sequence (u_n),

$$\int_{\mathbb{R}^N} \nabla u_n \nabla u\,dx \to \int_{\mathbb{R}^N} |\nabla u|^2\,dx$$

so

$$\text{(ii)} \qquad \|\nabla u_n - \nabla u\|_2 \to 0.$$

(i) and (ii) give the desired result, it follows that $m = I(u)$. □

(••) If $\|u\|_E < \underline{\lim}\|u_n\|_E$. We prove in the following and in three steps that this case does not occur: set $v_n(x) = u_n(x) - u(x)$, $v_n \rightharpoonup 0$ in E.

Step 1. There exists $(y_n^1) \subset \mathbb{R}^N$ such that $v_n(. + y_n^1) \rightharpoonup U_1 \neq 0$ in E.

Proof. Suppose that, $\forall (y_n) \subset \mathbb{R}^N$, $v_n(. + y_n) \rightharpoonup 0$ in E, then

$$\forall R > 0, \quad \sup_{y \in \mathbb{R}^N} \int_{B(y,R)} |v_n|^{p+1}\,dx \to 0$$

by the argument of P.L. Lions [26],

$$v_n \to 0 \quad \text{in } L^q(\mathbb{R}^N),\ \forall p+1 \leqslant q < 2^*.$$

On the other hand,

$$\langle I'(u_n), v_n\rangle = \int_{\mathbb{R}^N} (\nabla u_n \nabla v_n + u_n v_n)\,dx - \int_{\mathbb{R}^N} |u_n|^p\, sgn(u_n) v_n\,dx - \int_{\mathbb{R}^N} f v_n\,dx \to 0,$$

$u_n(x) = v_n(x) + u(x)$, then

$$\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + \nabla u \nabla v_n + |v_n|^2 + u v_n\right) dx - \int_{\mathbb{R}^N} |u_n|^p\, sgn(u_n) v_n\,dx - \int_{\mathbb{R}^N} f v_n\,dx \to 0,$$

$v_n \rightharpoonup 0$ in E, so

$$\int_{\mathbb{R}^N} (\nabla u \nabla v_n + u v_n)\,dx \to 0, \qquad \int_{\mathbb{R}^N} f v_n\,dx \to 0$$

and

$$\left|\int_{\mathbb{R}^N} |u_n|^p\, sgn(u_n) v_n\,dx\right| \leqslant \|u_n\|_{p+1}^p \|v_n\|_{p+1} \leqslant c'\|v_n\|_{p+1} \to 0.$$

This leads to

$$\|\nabla v_n\|_2 \to 0 \quad \text{and} \quad v_n \to 0 \quad \text{in } E$$

therefore $u_n \to u$ in E and $\|u_n\|_E \to \|u\|_E$ that is contradiction. Then up to a subsequence, there exists $(y_n^1) \subset \mathbb{R}^N$ such that $v_n(.+y_n^1) \rightharpoonup U_1 \neq 0$ in E. □

Step 2. $(y_n^1)_n$ is not bounded.

Proof. Suppose (y_n^1) is bounded, we extract a subsequence of (y_n^1), also denoted by (y_n^1) such that $y_n^1 \to y$.
Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, since $y_n^1 \to y$ and $v_n \rightharpoonup 0$ in E, then

$$\int_{\mathbb{R}^N} \varphi\big(x-y_n^1\big) v_n(x)\,dx \to 0,$$

$v_n(.+y_n^1) \rightharpoonup U_1$ in E, so

$$\int_{\mathbb{R}^N} \varphi\big(x-y_n^1\big) v_n(x)\,dx = \int_{\mathbb{R}^N} \varphi(x) v_n\big(x+y_n^1\big)\,dx \to \int_{\mathbb{R}^N} \varphi(x) U_1(x)\,dx.$$

It yields

$$\int_{\mathbb{R}^N} \varphi(x) U_1(x)\,dx = 0, \quad \forall \varphi \in C_c^\infty\big(\mathbb{R}^N\big).$$

Hence $U_1 = 0$ a.e. in $\mathbb{R}^N$, that is a contradiction. Thus (y_n^1) is not bounded. □

Step 3. U_1 is a solution of the homogeneous problem (1.2).

Proof. First, we prove that $u_n(.+y_n^1) \rightharpoonup U_1$ in E.
$(u(.+y_n^1))$ is bounded in E, there exists $w \in E$ such that $u(.+y_n^1) \rightharpoonup w$ in E and for any $\psi \in C_c^\infty(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} u\big(x+y_n^1\big)\psi(x)\,dx \to \int_{\mathbb{R}^N} w(x)\psi(x)\,dx,$$

$|y_n^1| \to +\infty$ then by Lemma 3.5

$$\int_{\mathbb{R}^N} u\big(x+y_n^1\big)\psi(x)\,dx \to 0, \quad \text{so} \quad \int_{\mathbb{R}^N} w(x)\psi(x)\,dx = 0, \quad \forall \psi \in C_c^\infty\big(\mathbb{R}^N\big).$$

Consequently $w = 0$ a.e. in $\mathbb{R}^N$. It holds that $u_n(x+y_n^1) \rightharpoonup U_1$ in E.
Second, let $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\big\langle I'(u_n), \varphi\big(.-y_n^1\big)\big\rangle \\
&\quad = \int_{\mathbb{R}^N} \big(\nabla u_n \nabla\varphi\big(x-y_n^1\big) + u_n(x)\varphi\big(x-y_n^1\big)\big)\,dx - \int_{\mathbb{R}^N} \big|u_n(x)\big|^p sgn(u_n)\varphi\big(x-y_n^1\big)\,dx - \int_{\mathbb{R}^N} f(x)\varphi\big(x-y_n^1\big)\,dx \\
&\quad = \int_{\mathbb{R}^N} \big(\nabla u_n\big(x+y_n^1\big)\nabla\varphi(x) + u_n\big(x+y_n^1\big)\varphi(x)\big)\,dx - \int_{\mathbb{R}^N} \big|u_n\big(x+y_n^1\big)\big|^p sgn\big(u_n\big(x+y_n^1\big)\big)\varphi(x)\,dx \\
&\qquad - \int_{\mathbb{R}^N} f(x)\varphi\big(x-y_n^1\big)\,dx.
\end{aligned}$$

Since $|y_n^1| \to +\infty$, then by Lemma 3.5

$$\int_{\mathbb{R}^N} f(x)\varphi\big(x-y_n^1\big)\,dx \to 0,$$

$u_n(.+y_n^1) \rightharpoonup U_1$ in E, so

$$\int_{\mathbb{R}^N} \nabla u_n\big(x+y_n^1\big)\nabla\varphi(x)+u_n\big(x+y_n^1\big)\varphi(x)\to\int_{\mathbb{R}^N}\nabla U_1\nabla\varphi+U_1\varphi$$

also

$$\big\langle I'(u_n),\varphi\big(.-y_n^1\big)\big\rangle\to 0.$$

Let us show that:

$$\int_{\mathbb{R}^N}\big|u_n\big(x+y_n^1\big)\big|^p\,sgn\big(u_n\big(x+y_n^1\big)\big)\varphi(x)\,dx\to\int_{\mathbb{R}^N}\big|U_1(x)\big|^p\,sgn\big(U_1(x)\big)\varphi(x)\,dx.$$

We have $u_n(.+y_n^1)\to U_1$ in $L^{p+1}(supp(\varphi))$, then there exists a subsequence denoted by $u_n(.+y_n^1)$ and $h\in L^{p+1}$ such that

$$\begin{cases}\big|u_n\big(.+y_n^1\big)\big|^p\,sgn\big(u_n\big(.+y_n^1\big)\big)\varphi\to|U_1|^p\,sgn(U_1)\varphi\quad\text{a.e. in }\mathbb{R}^N,\\ \big|u_n\big(.+y_n^1\big)\big|^p|\varphi|\leqslant|h|^p|\varphi|\in L^1\big(\mathbb{R}^N\big)\end{cases}$$

thus, by dominated convergence theorem

$$\int_{\mathbb{R}^N}\big|u_n\big(x+y_n^1\big)\big|^p\,sgn\big(u_n\big(x+y_n^1\big)\big)\varphi(x)\,dx\to\int_{\mathbb{R}^N}|U_1|^p\,sgn\big(U_1(x)\big)\varphi(x)\,dx.$$

It follows that, for any $\varphi\in C_c^\infty(\mathbb{R}^N)$

$$\big\langle I^{\infty'}(U_1),\varphi\big\rangle=\int_{\mathbb{R}^N}\nabla U_1(x)\nabla\varphi(x)+U_1(x)\varphi(x)-|U_1|^p\,sgn\big(U_1(x)\big)\varphi(x)=0$$

then, U_1 is a solution of problem (1.2). □

By Lemma 2.3, $U_1=0$ that is contradiction.

The steps 1–2–3 yield the second case (••) does not hold, so the only possible case is the compactness, this achieved the proof of Theorem 3.4. □

Now we prove the existence of positive solution of problem (1.1).

Theorem 3.6. *Assume R_0. Then the solution of problem* (1.1) *given by Theorem* 3.4 *is positive.*

Proof. We have

$$\phi'_{|u|}(0)=-\int_{\mathbb{R}^N}f|u|\,dx<0\quad\text{and}\quad\lim_{t\to\infty}\phi'_{|u|}(t)=+\infty$$

then, there exists a unique minimum $t_{|u|}>0$ of $\phi_{|u|}$.

$$\phi'_{|u|}(1)=\int_{\mathbb{R}^N}\big(|\nabla u|^2+|u|^2\big)\,dx-\int_{\mathbb{R}^N}|u|^{p+1}\,dx-\int_{\mathbb{R}^N}f|u|\,dx=\int_{\mathbb{R}^N}fu\,dx-\int_{\mathbb{R}^N}f|u|\,dx\leqslant 0,$$

$\phi'_{|u|}(1)=0$ indeed, suppose $\phi'_{|u|}(1)<0$ then $t_{|u|}>1$ and by (3.1):

$$(t_{|u|})^2\int_{\mathbb{R}^N}\big(|\nabla u|^2+|u|^2\big)\,dx>\alpha(t_{|u|})^{p+1}\int_{\mathbb{R}^N}|u|^{p+1}\,dx$$

and

$$U=t_{|u|}|u|\in N^\alpha.$$

Then

$$m\leqslant I(U)=\phi_{|u|}(t_{|u|})<\phi_{|u|}(1)=I\big(|u|\big)\leqslant I(u)=m\quad\text{that is a contradiction}$$

therefore $\phi'_{|u|}(1)=0$. Since $f>0$ a.e. in $\mathbb{R}^N$, then u is a weak nonnegative solution of the following problem

$$-\Delta u+u=u^p+f\quad\text{in }\mathbb{R}^N.\tag{3.3}$$

The right-hand side of (3.4) is nonnegative and not equivalently equal to 0, by the maximum principle u is a positive solution of problem (1.1). □

Now, we prove the uniqueness of positive solution for (1.1).

4. Uniqueness of positive solution for problem (1.1)

There are several methods for proving the uniqueness of positive solutions of semilinear elliptic equations (see [14,15,18,24] and the references therein). Here we employ the standard barrier method.

Let v_0 be a positive solution of (1.1), consider the following equation

$$-\Delta u + u = |u + v_0|^{p-1}(u + v_0) - v_0^p \quad \text{in } \mathbb{R}^N. \tag{4}$$

Lemma 4.1. *If v is a weak nonnegative solution of* (4) *in E, then $v = 0$.*

Proof. Let v be a weak nonnegative solution of (4) in E and $\xi = v + v_0$, we have

$$\int_{\mathbb{R}^N} (\nabla v \nabla w + v w)\, dx = \int_{\mathbb{R}^N} \left(|v + v_0|^p - v_0^p\right) w\, dx \leqslant \int_{\mathbb{R}^N} v^p w\, dx, \quad \forall w \in E,\ w \geqslant 0 \quad \text{a.e.}$$

and

$$\int_{\mathbb{R}^N} (\nabla \xi \nabla w + \xi w)\, dx = \int_{\mathbb{R}^N} \left(|v + v_0|^p - v_0^p\right) w\, dx + \int_{\mathbb{R}^N} \left(v_0^p + f\right) w\, dx \geqslant \int_{\mathbb{R}^N} |v + v_0|^p w\, dx, \quad \forall w \in E, w \geqslant 0 \text{ a.e.}$$

Then, v is a subsolution and ξ is a supersolution of problem (1.2). By the standard barrier method, there exists a solution h of problem (1.2) such that $v \leqslant h \leqslant v + v_0$. By Lemma 2.3, $h = 0$ and then $v = 0$.

The proof is achieved. □

Theorem 4.2. *Under condition* (R_0)*, the problem* (1.1) *has at most one positive solution.*

Proof. Assume u_1 and u_2 are two positive solutions of (1.1). Since 0 is a subsolution of problem (1.1), then by the standard barrier method there exists a solution u of problem (1.1) such that $0 \leqslant u \leqslant u_1$ and $0 \leqslant u \leqslant u_2$ a.e. in $\mathbb{R}^N$. Set $w = u_1 - u \geqslant 0$, let $\varphi \in E$

$$\int_{\mathbb{R}^N} (\nabla w \nabla \varphi + w\varphi)\, dx = \int_{\mathbb{R}^N} \left[(\nabla u_1 \nabla \varphi + u_1 \varphi) - (\nabla u \nabla \varphi + u\varphi)\right] dx = \int_{\mathbb{R}^N} \left(u_1^p - u^p\right)\varphi\, dx = \int_{\mathbb{R}^N} \left(|w + u|^p - u^p\right)\varphi\, dx$$

then w is a weak nonnegative solution of (4). By Lemma 4.1, $w = 0$ so $u_1 = u$ a.e. in $\mathbb{R}^N$. Also $u_2 = u$ a.e. then, $u_1 = u_2$ a.e. in $\mathbb{R}^N$, the proof is achieved. □

At the end of this paper, we give a continuity result. Denote by $u(f)$ the solution of (1.1) given by Theorem 3.4.

Theorem 4.3. *If $f \to 0$ in L^2 then $u(f) \to 0$ in E.*

Proof. Let $(f_n) \subset L^2$, f_n satisfy (R_0). Put $u_n = u(f_n)$,

$$\left\langle I'(u_n), u_n\right\rangle = \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + |u_n|^2\right) dx - \int_{\mathbb{R}^N} |u_n|^{p+1}\, dx - \int_{\mathbb{R}^N} f u_n\, dx = 0$$

dividing throughout by $\|u_n\|_{H^1}$, we obtain

$$0 = \|u_n\|_{H^1} - \frac{\|u_n\|_{p+1}^{p+1}}{\|u_n\|_{H^1}} - \frac{1}{\|u_n\|_{H^1}} \int_{\mathbb{R}^N} f_n u_n\, dx \geqslant \frac{\alpha - 1}{\alpha} \|u_n\|_{H^1} - \frac{1}{\|u_n\|_{H^1}} \int_{\mathbb{R}^N} f_n u_n\, dx.$$

We have

$$\left| \frac{1}{\|u_n\|_{H^1}} \int_{\mathbb{R}^N} f_n u_n\, dx \right| \leqslant \|f_n\|_2 \to 0$$

this leads to $u_n \to 0$ in H^1, $\int_{\mathbb{R}^N} f_n u_n\, dx \to 0$ and $u_n \to 0$ in E. □

Remark 4.4. Theorems 3.4 and 4.2 remain valid under condition $f \geqslant 0$, $f \neq 0$ instead of $f > 0$ a.e.

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