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On suboperators with codimension one domains ☆

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ABSTRACT

The question of the existence of a proper closed densely defined suboperator whose domain contains a given linear space is answered. Such suboperators with codimension one domains are described. The motivation for studying this topic comes from the theory of extremal extensions of (non-densely defined) positive operators.

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1. Introduction

Following Halmos [6], we call the restriction $A|_{\mathcal{L}}$ of a Hilbert space linear operator A to a linear subspace $\mathcal{L}$ of the domain $\mathcal{D}(A)$ of A a *suboperator* of A. In view of recent results in the theory of positive selfadjoint extensions (cf. [2,7,17]), the description of the set of all extremal positive selfadjoint extensions of a given positive operator T in $\mathcal{H}$ reduces to investigating the set of all linear subspaces $\mathcal{L}_1$ of $\mathcal{D}(J_T^*)$ containing $\mathcal{D}(T)$, for which the operator $J_T^*|_{\mathcal{L}_1}$ is closed and densely defined; here J_T^* is an operator which factorizes the Krein–von Neumann extension of T (cf. [1,16]; see also [13,14] for special factorizations including that of the Friedrichs extension of T). This is a particular case of a more general problem: given a closed densely defined linear operator A and a linear subspace $\mathcal{L}$ of $\mathcal{D}(A)$, find necessary and sufficient conditions for the existence of a proper closed *densely defined* suboperator of A whose domain contains $\mathcal{L}$, and if possible describe all of them. In the present paper we lay great stress on the case when the subspace $\mathcal{L}$ is not dense in $\mathcal{D}(A)$, because otherwise the first part of the problem has trivial solution. We found a description of all closed densely defined suboperators with codimension one domains.

2. Preliminaries

Throughout what follows, $\mathcal{H}$ and $\mathcal{K}$ stand for complex Hilbert spaces. By an operator from $\mathcal{H}$ to $\mathcal{K}$ we understand a linear mapping $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{K}$ defined on a linear subspace $\mathcal{D}(A)$ of $\mathcal{H}$, called the *domain* of A; $\mathcal{N}(A)$, $\mathcal{G}(A)$ and A^* stand for the *kernel*, the *graph* and the *adjoint* of A, respectively. If the set $\mathcal{D}(A)$ is dense in $\mathcal{H}$, then we say that the operator A is densely defined. The graph inner product $\langle\langle \cdot, \cdot \rangle\rangle_A$ on $\mathcal{D}(A)$ is defined by

$$\langle\langle f, g\rangle\rangle_A = \langle f, g\rangle + \langle Af, Ag\rangle, \quad f, g \in \mathcal{D}(A).$$

Set $|||f|||_A = \sqrt{\langle\langle f, f\rangle\rangle_A}$ for $f \in \mathcal{D}(A)$. Given a linear subspace $\mathcal{L}$ of $\mathcal{D}(A)$, we write

$$\mathcal{D}(A) \ominus_A \mathcal{L} = \{f \in \mathcal{D}(A) : \langle\langle f, g\rangle\rangle_A = 0 \text{ for all } g \in \mathcal{L}\}.$$

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Denote by $A|_{\mathcal{L}}$ the restriction of A to $\mathcal{L}$; if not stated otherwise, we regard $A|_{\mathcal{L}}$ as an operator from $\mathcal{H}$ to $\mathcal{K}$. If A and B are operators from $\mathcal{H}$ to $\mathcal{K}$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Ah = Bh$ for all $h \in \mathcal{D}(A)$, then we say that B *extends* A, or that A is a *suboperator* of B, and write $A \subseteq B$. An operator from $\mathcal{H}$ to $\mathcal{H}$ is called an operator *in* $\mathcal{H}$. We say that a densely defined operator A in $\mathcal{H}$ is *symmetric* (resp. *selfadjoint*) if $A \subseteq A^*$ (resp. $A = A^*$). An operator A in $\mathcal{H}$ is said to be *positive* if $\langle Ah, h\rangle \geqslant 0$ for all $h \in \mathcal{D}(A)$. As usual, $A^{1/2}$ stands for the square root of a positive selfadjoint operator A. Given a closed densely defined operator A from $\mathcal{H}$ to $\mathcal{K}$, we set $|A| = (A^*A)^{1/2}$ (cf. [3,15,19]). $\boldsymbol{B}(\mathcal{H}, \mathcal{K})$ stands for the set of all bounded linear mappings $A : \mathcal{H} \to \mathcal{K}$. We shall abbreviate $\boldsymbol{B}(\mathcal{H}, \mathcal{H})$ to $\boldsymbol{B}(\mathcal{H})$. $\boldsymbol{B}(\mathcal{H})$ is a C^*-algebra with the identity mapping I as a unit. Given vectors $e_1, \ldots, e_n$ in a vector space, we write $\langle e_1, \ldots, e_n\rangle$ for the linear span of $\{e_1, \ldots, e_n\}$.

The following lemma seems to be folklore. Though it is a particular case of Lemma A.1 in Appendix A (with $R = |A|$, $s = 1$ and $t = 2$), we provide another more elementary proof.

Lemma 2.1. *If $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{K}$ is a closed densely defined operator, then $\mathcal{D}(A) = \mathcal{D}(A^*A)$ if and only if $A \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$.*

Proof. If $\mathcal{D}(A) = \mathcal{D}(A^*A)$, then the space $\mathcal{D}(|A|) = \mathcal{D}(A)$ is invariant for the selfadjoint operator $|A|$. This implies that $|A| \in \boldsymbol{B}(\mathcal{H})$ and consequently $A \in \boldsymbol{B}(\mathcal{H}, \mathcal{K})$ (cf. [8,10–12]; see also [4, Problem 11, p. 399]). □

3. Suboperators

We begin by showing that the question of the existence of suboperators with codimension one domains is strongly related to the set $\mathcal{D}(A) \setminus \mathcal{D}(A^*A)$. This will be done in three consecutive lemmas below.

Lemma 3.1. *Let $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{K}$ be a densely defined operator and let $e \in \mathcal{D}(A)$. Then the linear functional $\xi_e : \mathcal{D}(A) \to \mathbb{C}$ defined by*

$$\xi_e(f) = \langle\langle f, e\rangle\rangle_A, \quad f \in \mathcal{D}(A), \tag{3.1}$$

is $\|\cdot\|$-continuous if and only if $e \in \mathcal{D}(A^*A)$. If the functional ξ_e is $\|\cdot\|$-continuous, then $\|\xi_e\| = \|(I + A^*A)e\|$ and $|||\xi_e|||_A = \sqrt{\langle (I + A^*A)e, e\rangle}$, where $\|\xi_e\|$ (respectively $|||\xi_e|||_A$) stands for the norm of the functional ξ_e counted with respect to the norm $\|\cdot\|$ (respectively $|||\cdot|||_A$). Moreover, if the functional ξ_e is $\|\cdot\|$-continuous and A is closed, then $|||\xi_e|||_A = \|(I + A^*A)^{1/2}e\|$.

Proof. Suppose first that the functional ξ_e is $\|\cdot\|$-continuous. Denote by $\widetilde{\xi}_e$ the unique $\|\cdot\|$-continuous extension of ξ_e to $\mathcal{H}$. Applying the Riesz representation theorem to $\widetilde{\xi}_e$, we find a unique vector $g \in \mathcal{H}$ such that $\widetilde{\xi}_e(f) = \langle f, g\rangle$ for all $f \in \mathcal{H}$. This implies that $\langle Af, Ae\rangle = \langle f, g - e\rangle$ for all $f \in \mathcal{D}(A)$. As a consequence, we get $e \in \mathcal{D}(A^*A)$ and $(I + A^*A)e = g$. Hence, we also have

$$\|\xi_e\| = \|\widetilde{\xi}_e\| = \|g\| = \big\|(I + A^*A)e\big\|$$

and

$$|||\xi_e|||_A \overset{(3.1)}{=} |||e|||_A = \sqrt{\big\langle (I + A^*A)e, e\big\rangle}.$$

Reversing the above reasoning, we see that $e \in \mathcal{D}(A^*A)$ implies the $\|\cdot\|$-continuity of ξ_e. The "moreover" part of the conclusion follows from the fact that $I + A^*A$ is a positive selfadjoint operator (cf. [3, Theorem 4.5.1]). This completes the proof. □

Lemma 3.2. *If $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{K}$ is a densely defined operator and $e \in \mathcal{D}(A) \setminus \{0\}$, then the space $\mathcal{D}(A) \ominus_A \langle e\rangle$ is dense in $\mathcal{H}$ if and only if $e \in \mathcal{D}(A) \setminus \mathcal{D}(A^*A)$.*

Proof. Let ξ_e be as in (3.1). Suppose that $\mathcal{D}(A) \ominus_A \langle e\rangle$ is dense in $\mathcal{H}$. Then clearly $\mathcal{D}(A) \ominus_A \langle e\rangle$ is dense in $(\mathcal{D}(A), \|\cdot\|)$. Since $\mathcal{D}(A) \ominus_A \langle e\rangle = \ker \xi_e$ (= the kernel of ξ_e) and $e \neq 0$, we deduce that $\ker \xi_e$ is not closed in $(\mathcal{D}(A), \|\cdot\|)$ (indeed, otherwise, by the density of $\mathcal{D}(A) \ominus_A \langle e\rangle$ in $(\mathcal{D}(A), \|\cdot\|)$, we would have $e \in \ker \xi_e$, which would contradict $e \neq 0$). Hence ξ_e is not $\|\cdot\|$-continuous. By Lemma 3.1, $e \in \mathcal{D}(A) \setminus \mathcal{D}(A^*A)$. Conversely, if $e \in \mathcal{D}(A) \setminus \mathcal{D}(A^*A)$, then again by Lemma 3.1, the functional ξ_e is not $\|\cdot\|$-continuous. Hence $\mathcal{D}(A) \ominus_A \langle e\rangle = \ker \xi_e$ is not closed in $(\mathcal{D}(A), \|\cdot\|)$. As a consequence, the space $\mathcal{D}(A) \ominus_A \langle e\rangle$ is dense in $(\mathcal{D}(A), \|\cdot\|)$ (because, due to $\xi_e \neq 0$, the codimension of $\ker \xi_e$ in $\mathcal{D}(A)$ is equal 1). Finally, since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, so also is $\mathcal{D}(A) \ominus_A \langle e\rangle$. This completes the proof. □

Lemma 3.3. *Let $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{K}$ be a closed densely defined operator. Then the following conditions are equivalent*:

(i) *the operator A is unbounded,*
(ii) *there exists a proper linear subspace $\mathcal{L}$ of $\mathcal{D}(A)$ which is dense in $\mathcal{H}$ and closed in $(\mathcal{D}(A), |||\cdot|||_A)$,*
(iii) *there exists a proper linear subspace $\mathcal{L}$ of $\mathcal{D}(A)$ which is dense in $\mathcal{H}$ and closed in $(\mathcal{D}(A), |||\cdot|||_A)$, and $\dim \mathcal{D}(A)/\mathcal{L} = 1$,*
(iv) $\mathcal{D}(A^*A) \subsetneq \mathcal{D}(A)$.

Moreover, if A is unbounded and $\mathcal{L}$ is as in (ii)*, then the operator $A|_{\mathcal{L}}$ is unbounded.*

Proof. (i) ⇒ (iv) Apply Lemma 2.1.

(iv) ⇒ (iii) Take $e \in \mathcal{D}(A) \setminus \mathcal{D}(A^*A)$. In view of Lemma 3.2, the linear space $\mathcal{L} := \mathcal{D}(A) \ominus_A \langle e \rangle$ is dense in $\mathcal{H}$. It is plain that $\mathcal{L}$ is closed in $(\mathcal{D}(A), |||\cdot|||_A)$ and $\dim \mathcal{D}(A)/\mathcal{L} = 1$ (because $\mathcal{L} = \ker \xi_e$).

(iii) ⇒ (ii) Obvious.

(ii) ⇒ (i) Suppose that, contrary to our claim, the operator A is bounded. Then plainly the norms $|||\cdot|||_A$ and $\|\cdot\|$ are equivalent on $\mathcal{D}(A)$. Hence, the linear space $\mathcal{L}$ is both closed and dense in $(\mathcal{D}(A), \|\cdot\|)$. This implies that $\mathcal{L} = \mathcal{D}(A)$, a contradiction.

To prove the "moreover" part of the conclusion, suppose that, contrary to our claim, the operator $A|_{\mathcal{L}}$ is bounded. Since, by (ii) and the closedness of A, the operator $A|_{\mathcal{L}}$ is closed and densely defined, we must have $\mathcal{L} = \mathcal{H}$, which contradicts $\mathcal{L} \subsetneq \mathcal{D}(A)$. □

A thorough inspection of the proof of Lemma 3.3 reveals that implications (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) remain true without assuming the operator A is closed. Implication (i) ⇒ (iv) is not true if A is not assumed to be closed. It is enough to consider any unbounded symmetric operator A in $\mathcal{H}$ such that $A(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$, e.g. the operator of multiplication by the independent variable x in $L^2(\mathbb{R})$ with domain $C_c^\infty(\mathbb{R})$. Then evidently $\mathcal{D}(A) = \mathcal{D}(A^*A)$.

We now formulate a sufficient condition for a closed densely defined operator to have infinitely many unbounded closed densely defined suboperators whose domains contain a fixed linear space.

Theorem 3.4. *Let $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{K}$ be a closed densely defined operator and let $\mathcal{L}$ be a linear subspace of $\mathcal{D}(A)$ such that the operator $A|_{\mathcal{L}}$ is closed. If the operator $A|_{\mathcal{D}(A) \ominus_A \mathcal{L}}$ is unbounded, then there exists a sequence $\{\mathcal{L}_k\}_{k=0}^\infty$ of linear subspaces of $\mathcal{D}(A)$ such that for every integer $n \geqslant 0$, $\mathcal{L} \subsetneq \mathcal{L}_{n+1} \subsetneq \mathcal{L}_n$, $\dim \mathcal{L}_n/\mathcal{L}_{n+1} = 1$ and the operator $A|_{\mathcal{L}_n} : \mathcal{H} \supseteq \mathcal{L}_n \to \mathcal{K}$ is unbounded closed and densely defined, where $\mathcal{L}_0 = \mathcal{D}(A)$.*

Proof. Denote by $\mathcal{H}_0$ the closure of $\mathcal{L}_0' := \mathcal{D}(A) \ominus_A \mathcal{L}$ in $\mathcal{H}$. First, we will construct a sequence $\{\mathcal{L}_k'\}_{k=1}^\infty$ of proper linear subspaces of $\mathcal{L}_0'$ such that for every integer $n \geqslant 0$, $\mathcal{L}_{n+1}' \subsetneq \mathcal{L}_n'$, $\dim \mathcal{L}_n'/\mathcal{L}_{n+1}' = 1$, the space $\mathcal{L}_n'$ is dense in $(\mathcal{H}_0, \|\cdot\|)$ and closed in $(\mathcal{D}(A), |||\cdot|||_A)$, and the operator $A|_{\mathcal{L}_n'}$ is unbounded. Suppose that the spaces $\mathcal{L}_0', \mathcal{L}_1', \ldots, \mathcal{L}_n'$ are constructed. Then the operator

$$A|_{\mathcal{L}_n'} : \mathcal{H}_0 \supseteq \mathcal{L}_n' \to \mathcal{K}$$

is unbounded closed and densely defined. Applying Lemma 3.3 to this operator, we get a proper linear subspace $\mathcal{L}_{n+1}'$ of $\mathcal{L}_n'$ which is dense in $(\mathcal{H}_0, \|\cdot\|)$ and closed in $(\mathcal{D}(A), |||\cdot|||_A)$, the codimension of $\mathcal{L}_{n+1}'$ in $\mathcal{L}_n'$ is equal to 1, and the operator $A|_{\mathcal{L}_{n+1}'}$ is unbounded. Proceeding by induction, we obtain the desired sequence of spaces.

Set $\mathcal{L}_n = \mathcal{L} \oplus_A \mathcal{L}_n'$ for $n \geqslant 0$. It is evident that $\mathcal{L} \subsetneq \mathcal{L}_{n+1} \subsetneq \mathcal{L}_n$ for all $n \geqslant 0$. Equality $\mathcal{L}_0 = \mathcal{D}(A)$ follows from the fact that $(\mathcal{D}(A), |||\cdot|||_A)$ is a Hilbert space. It is also clear that $\dim \mathcal{L}_n/\mathcal{L}_{n+1} = \dim \mathcal{L}_n'/\mathcal{L}_{n+1}' = 1$ and that $\mathcal{L}_n$ is closed in $(\mathcal{D}(A), |||\cdot|||_A)$. The latter implies that the operator $A|_{\mathcal{L}_n} : \mathcal{H} \supseteq \mathcal{L}_n \to \mathcal{K}$ is closed. Evidently, it is unbounded. It remains to show that $\mathcal{L}_n$ is dense in $(\mathcal{D}(A), \|\cdot\|)$ (because A is densely defined). For this purpose, take $g \in \mathcal{D}(A)$ and $\varepsilon > 0$. Since $\mathcal{D}(A) = \mathcal{L} \oplus_A \mathcal{L}_0'$, there exist $g_1 \in \mathcal{L}$ and $g_2 \in \mathcal{L}_0'$ such that $g = g_1 + g_2$. Because $\mathcal{L}_n'$ is dense in $(\mathcal{L}_0', \|\cdot\|)$, there exists $g_2' \in \mathcal{L}_n'$ such that $\|g_2 - g_2'\| < \varepsilon$. Taken all together, this implies that $\|g - (g_1 + g_2')\| < \varepsilon$ and $g_1 + g_2' \in \mathcal{L}_n$, which completes the proof. □

Regarding Theorem 3.4, we refer the reader to Lemma A.2 in Appendix A, which offers another tool for constructing proper closed densely defined suboperators.

The problem of the existence of a proper closed densely defined suboperator $A|_{\mathcal{L}_1}$ whose domain contains a fixed linear space $\mathcal{L}$ is solved in Theorem 3.5 below. Of course, the only nontrivial case is when the space $\mathcal{L}$ is not dense in $\mathcal{H}$.

Theorem 3.5. *Let $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{K}$ be a closed densely defined operator and let $\mathcal{L}$ be a linear subspace of $\mathcal{D}(A)$ such that the operator $A|_{\mathcal{L}}$ is closed. Then the following conditions are equivalent*:

(i) *there exists a linear subspace $\mathcal{L}_1$ of $\mathcal{D}(A)$ such that $\mathcal{L} \subseteq \mathcal{L}_1 \subsetneq \mathcal{D}(A)$ and the operator $A|_{\mathcal{L}_1} : \mathcal{H} \supseteq \mathcal{L}_1 \to \mathcal{K}$ is closed and densely defined,*

(ii) $(\mathcal{D}(A) \ominus_A \mathcal{L}) \setminus \mathcal{D}(A^*A) \neq \emptyset$,

(iii) $\mathcal{D}(A^*A) \cap (\mathcal{D}(A) \ominus_A \mathcal{L}) \subsetneq \mathcal{D}(A) \ominus_A \mathcal{L}$.

Proof. (i) ⇒ (ii) By assumption, $\mathcal{L}_1$ is a proper closed subspace of the Hilbert space $(\mathcal{D}(A), |||\cdot|||_A)$. Hence there exists a nonzero vector e in $\mathcal{D}(A) \ominus_A \mathcal{L}_1$. This yields $\mathcal{L}_1 \subseteq \mathcal{D}(A) \ominus_A \langle e \rangle$. Since $\mathcal{L}_1$ is dense in $\mathcal{H}$, so is $\mathcal{D}(A) \ominus_A \langle e \rangle$. By Lemma 3.2, $e \in \mathcal{D}(A) \setminus \mathcal{D}(A^*A)$. Because $e \in \mathcal{D}(A) \ominus_A \mathcal{L}_1 \subseteq \mathcal{D}(A) \ominus_A \mathcal{L}$, we conclude that $e \in (\mathcal{D}(A) \setminus \mathcal{D}(A^*A)) \cap (\mathcal{D}(A) \ominus_A \mathcal{L}) = (\mathcal{D}(A) \ominus_A \mathcal{L}) \setminus \mathcal{D}(A^*A)$.

(ii) ⇒ (i) Take $e \in (\mathcal{D}(A) \ominus_A \mathcal{L}) \setminus \mathcal{D}(A^*A)$. By Lemma 3.2, the space $\mathcal{L}_1 := \mathcal{D}(A) \ominus_A \langle e \rangle$ is dense in $\mathcal{H}$. Since $\mathcal{L}_1$ is a closed subspace of the Hilbert space $(\mathcal{D}(A), |||\cdot|||_A)$, the operator $A|_{\mathcal{L}_1} : \mathcal{H} \supseteq \mathcal{L}_1 \to \mathcal{K}$ is closed. Finally, because $e \in \mathcal{D}(A) \ominus_A \mathcal{L}$ and $e \neq 0$, we get $\mathcal{L} \subseteq \mathcal{L}_1 \subsetneq \mathcal{D}(A)$.

(ii) $\Leftrightarrow$ (iii) This is evident due to the following purely set-theoretical equality

$$\big(\mathcal{D}(A)\ominus_A\mathcal{L}\big)\setminus\mathcal{D}(A^*A)=\big(\mathcal{D}(A)\ominus_A\mathcal{L}\big)\setminus\big(\mathcal{D}(A^*A)\cap\big(\mathcal{D}(A)\ominus_A\mathcal{L}\big)\big). \tag{3.2}$$

Thus the proof is complete. □

A careful look at the proof of Theorem 3.5 reveals that the following description of suboperators with codimension one domains holds true.

Theorem 3.6. *Let $A:\mathcal{H}\supseteq\mathcal{D}(A)\to\mathcal{K}$ be a closed densely defined operator and let $\mathcal{L}$ be a linear subspace of $\mathcal{D}(A)$ such that the operator $A|_{\mathcal{L}}$ is closed and the set $\mathcal{X}:=(\mathcal{D}(A)\ominus_A\mathcal{L})\setminus\mathcal{D}(A^*A)$ is nonempty. Denote by $\backsimeq$ the equivalence relation on $\mathcal{X}$ which identifies vectors e and f such that $\langle e\rangle=\langle f\rangle$, and by $[e]$ the equivalence class of e with respect to $\backsimeq$. Then the mapping $[e]\mapsto\mathcal{D}(A)\ominus_A\langle e\rangle$ is a bijection between the quotient set $\mathcal{X}/\backsimeq$ and the set of all linear subspaces $\mathcal{L}_1$ of $\mathcal{D}(A)$ such that $\mathcal{L}\subseteq\mathcal{L}_1\subsetneq\mathcal{D}(A)$, $\dim\mathcal{D}(A)/\mathcal{L}_1=1$ and the operator $A|_{\mathcal{L}_1}:\mathcal{H}\supseteq\mathcal{L}_1\to\mathcal{K}$ is closed and densely defined.*

The spaces $\mathcal{D}(A)\ominus_A\mathcal{L}$ and $\mathcal{D}(A^*A)\cap(\mathcal{D}(A)\ominus_A\mathcal{L})$ which appear in Theorem 3.5 can be described as follows.

Proposition 3.7. *Let A and $\mathcal{L}$ satisfy the assumptions of Theorem* 3.5. *If $\mathcal{L}^\perp:=\{f\in\mathcal{H}\colon\langle f,g\rangle=0\ \forall g\in\mathcal{L}\}$, $P\in\boldsymbol{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ and $A|_{\mathcal{L}}$ is regarded as an operator from $\overline{\mathcal{L}}$ to $\mathcal{K}$, then*

(i) $\mathcal{D}(A)\ominus_A\mathcal{L}=\mathcal{N}(P+(A|_{\mathcal{L}})^*A)$,
(ii) $\mathcal{D}(A^*A)\cap(\mathcal{D}(A)\ominus_A\mathcal{L})=(I+A^*A)^{-1}(\mathcal{L}^\perp)$,
(iii) *the space $\mathcal{D}(A^*A)\cap(\mathcal{D}(A)\ominus_A\mathcal{L})$ is closed in $(\mathcal{D}(A^*A),|||\cdot|||_{A^*A})$,*
(iv) *the mapping*

$$(I+A^*A)^{-1}\big|_{\mathcal{L}^\perp}:\big(\mathcal{L}^\perp,\|\cdot\|\big)\longrightarrow\big(\mathcal{D}(A^*A)\cap\big(\mathcal{D}(A)\ominus_A\mathcal{L}\big),|||\cdot|||_{A^*A}\big) \tag{3.3}$$

is a linear homeomorphism; in particular, the orthogonal dimensions of $\mathcal{L}^\perp$ in $\mathcal{H}$ and $\mathcal{D}(A^*A)\cap(\mathcal{D}(A)\ominus_A\mathcal{L})$ in $(\mathcal{D}(A^*A),|||\cdot|||_{A^*A})$ are the same.

Proof. Since A is closed, so is A^*A, and consequently $(\mathcal{D}(A^*A),|||\cdot|||_{A^*A})$ is a Hilbert space. Moreover, the mapping $I+A^*A:\mathcal{D}(A^*A)\to\mathcal{H}$ is a linear bijection.

(i) Apply standard calculations.

(ii) Note first that

$$PA^*\subseteq(A|_{\mathcal{L}})^* \tag{3.4}$$

(because if $f\in\mathcal{D}(A^*)$, then $\langle(A|_{\mathcal{L}})h,f\rangle=\langle h,A^*f\rangle=\langle h,PA^*f\rangle$ for all $h\in\mathcal{L}$). Next, observe that (ii) follows from (i) and the equality

$$P(I+A^*A)f\overset{(3.4)}{=}\big(P+(A|_{\mathcal{L}})^*A\big)f,\quad f\in\mathcal{D}(A^*A).$$

(iii) Take a sequence $\{f_n\}_{n=1}^\infty\subseteq\mathcal{D}(A^*A)\cap(\mathcal{D}(A)\ominus_A\mathcal{L})$ which converges in the norm $|||\cdot|||_{A^*A}$ to $f\in\mathcal{D}(A^*A)$. Since

$$\|Ah\|^2=\langle A^*Ah,h\rangle\leqslant\|A^*Ah\|\|h\|\leqslant\frac{1}{2}\big(\|h\|^2+\|A^*Ah\|^2\big),\quad h\in\mathcal{D}(A^*A),$$

we deduce that

$$|||h|||_A\leqslant\sqrt{\frac{3}{2}}\,|||h|||_{A^*A},\quad h\in\mathcal{D}(A^*A), \tag{3.5}$$

which implies that the sequence $\{f_n\}_{n=1}^\infty$ converges in the norm $|||\cdot|||_A$ to f. As $f_n\in\mathcal{D}(A)\ominus_A\mathcal{L}$ for all $n\geqslant1$, we get $f\in\mathcal{D}(A)\ominus_A\mathcal{L}$.

(iv) In view of (ii), it is sufficient to show that the linear mapping

$$I+A^*A:\big(\mathcal{D}(A^*A),|||\cdot|||_{A^*A}\big)\longrightarrow\big(\mathcal{H},\|\cdot\|\big) \tag{3.6}$$

is a homeomorphism. Indeed, since

$$\big\|(I+A^*A)h\big\|\leqslant\|h\|+\|A^*Ah\|\leqslant\sqrt{2}\,|||h|||_{A^*A},\quad h\in\mathcal{D}(A^*A),$$

we see that the mapping (3.6) is continuous. Thus, by the inverse mapping theorem, its inverse is continuous as well. This can also be proved directly as follows. Since both the operators $(I+A^*A)^{-1}\in\boldsymbol{B}(\mathcal{H})$ and $A^*A(I+A^*A)^{-1}\in\boldsymbol{B}(\mathcal{H})$ are contractions (apply the Stone–von Neumann functional calculus to A^*A), we get

$$\big|\!\big|\!\big|(I+A^*A)^{-1}f\big|\!\big|\!\big|_{A^*A}^2\leqslant2\|f\|^2,\quad f\in\mathcal{H}.$$

This completes the proof. □

4. Comments and examples

Remark 4.1. Regarding Lemma 3.3(iv) and Theorem 3.6, note that if $A:\mathcal{H}\supseteq\mathcal{D}(A)\to\mathcal{K}$ is an unbounded closed densely defined operator, then the dimension of the quotient linear space $\mathcal{D}(A)/\mathcal{D}(A^*A)$ is at least of continuum cardinality. This can be proved as follows. Replacing A by $|A|$, we can assume without loss of generality that A is an unbounded positive selfadjoint operator in $\mathcal{H}$. Let E be the spectral measure of A. It follows from Lemma A.1 in Appendix A that there exists $f\in\mathcal{D}(A^{3/2})\setminus\mathcal{D}(A^2)\subseteq\mathcal{D}(A)\setminus\mathcal{D}(A^2)$, which means that $\int_0^\infty x^3\,\mathrm{d}\mu(x)<\infty$ and $\int_0^\infty x^4\,\mathrm{d}\mu(x)=\infty$, where $\mu(\sigma)=\langle E(\sigma)f,f\rangle$ for all Borel subsets σ of the closed half line $[0,\infty)$. This implies that for each $s\in[0,1/2]$, $\int_0^\infty x^{2+2s}\,\mathrm{d}\mu(x)<\infty$. Hence $f\in\mathcal{D}(A^s)$ and $f_s:=A^sf\in\mathcal{D}(A)$ for $s\in[0,1/2]$. Since $\int_0^\infty x^{4+2s}\,\mathrm{d}\mu(x)=\infty$ for $s\in[0,\infty)$, we see that $f_s\notin\mathcal{D}(A^2)$ for $s\in[0,1/2]$. We now show that if $g=\sum_{k=1}^n\alpha_kf_{s_k}\in\mathcal{D}(A^2)$, where $s_1,\dots,s_n$ are distinct numbers in $[0,1/2]$ and $\alpha_1,\dots,\alpha_n\in\mathbb{C}$, then $\alpha_1=\ldots=\alpha_n=0$. Indeed, otherwise, we can assume without loss of generality that $n\geqslant 2$, $s_1<\ldots<s_n$ and $\alpha_n=1$. Then there exits $R\in(0,\infty)$ such that $\delta(x):=|1+\sum_{k=1}^{n-1}\alpha_k/x^{s_n-s_k}|^2\geqslant 1/2$ for all $x\in[R,\infty)$. This leads to

$$\int_R^\infty x^4\left|\sum_{k=1}^n\alpha_kx^{s_k}\right|^2\mathrm{d}\mu(x)=\int_R^\infty x^{4+2s_n}\delta(x)\,\mathrm{d}\mu(x)\geqslant\frac{1}{2}\int_R^\infty x^{4+2s_n}\,\mathrm{d}\mu(x)=\infty,$$

and so $\int_0^\infty x^4|\sum_{k=1}^n\alpha_kx^{s_k}|^2\,\mathrm{d}\mu(x)=\infty$, which means that[1] $g\notin\mathcal{D}(A^2)$, a contradiction. Summarizing, we have proved that the family $\{f_s\}_{s\in[0,1/2]}$ is linearly independent and its linear span has trivial intersection with $\mathcal{D}(A^2)$. This completes the proof.

The example below shows that Theorem 3.4 is no longer true if we drop the assumption that the operator $A|_{\mathcal{D}(A)\ominus_A\mathcal{L}}$ is unbounded, even though A is unbounded.

Example 4.2. Consider an unbounded closed densely defined operator A_0 in a complex Hilbert space $\mathcal{H}_0$ and a bounded operator A_1 on a complex Hilbert space $\mathcal{H}_1$ (i.e. $A_1\in\boldsymbol{B}(\mathcal{H}_1)$). Then $A:=A_0\oplus A_1$ is an unbounded closed densely defined operator in $\mathcal{H}:=\mathcal{H}_0\oplus\mathcal{H}_1$. Set $\mathcal{L}=\mathcal{D}(A_0)$. Then clearly $A|_{\mathcal{L}}=A_0$ is a closed operator and $A|_{\mathcal{D}(A)\ominus_A\mathcal{L}}=A_1$ is a bounded operator. What is more, there is no linear subspace $\mathcal{L}_1$ of $\mathcal{D}(A)$ such that $\mathcal{L}\subseteq\mathcal{L}_1\subsetneq\mathcal{D}(A)$ and the operator $A|_{\mathcal{L}_1}:\mathcal{H}\supseteq\mathcal{L}_1\to\mathcal{K}$ is closed and densely defined. Indeed, this follows from Theorem 3.5 because $\mathcal{D}(A)\ominus_A\mathcal{L}=\mathcal{H}_1$ and $\mathcal{D}(A^*A)=\mathcal{D}(A_0^*A_0)\oplus\mathcal{H}_1$, which implies that $(\mathcal{D}(A)\ominus_A\mathcal{L})\setminus\mathcal{D}(A^*A)=\emptyset$. It is also worth mentioning that $\mathcal{D}(A)\ominus_A\mathcal{L}\subsetneq\mathcal{D}(A^*A)$. Moreover, if we extend A to a (necessarily unbounded closed and densely defined) operator A' in $\mathcal{H}$ in such a way that $2\leqslant\dim\mathcal{D}(A')/\mathcal{D}(A)<\infty$, and take any linear subspace $\mathcal{L}_1$ of $\mathcal{D}(A')$ such that $\mathcal{D}(A)\subseteq\mathcal{L}_1$ and $\dim\mathcal{L}_1/\mathcal{D}(A)=1$, then $A'|_{\mathcal{L}_1}:\mathcal{H}\supseteq\mathcal{L}_1\to\mathcal{K}$ is a closed densely defined proper suboperator of A' and the operator $A'|_{\mathcal{D}(A')\ominus_{A'}\mathcal{L}'}$ is bounded; here $\mathcal{L}':=\mathcal{D}(A)$.

Let us now discuss the relationship between Theorems 3.4 and 3.5. Suppose that A and $\mathcal{L}$ are as in Theorem 3.5. If the operator $A|_{\mathcal{D}(A)\ominus_A\mathcal{L}}$ is unbounded, then, by Theorem 3.4, condition (i) of Theorem 3.5 holds. However, the reverse implication is not true in general.

Example 4.3. Indeed, consider any unbounded closed densely defined operator A in a separable (necessarily infinite-dimensional) Hilbert space $\mathcal{H}$. Since the identity embedding of the Hilbert spaces $(\mathcal{D}(A),|||\cdot|||_A)$ into $\mathcal{H}$ is injective and continuous, we deduce that (cf. [5, Problem 56]) the orthogonal dimension of $(\mathcal{D}(A),|||\cdot|||_A)$ is less than or equal to $\aleph_0$. Since the operator A is unbounded, the Hilbert space $(\mathcal{D}(A),|||\cdot|||_A)$ must be infinite-dimensional. By Lemmas 3.2 and 3.3, there exists $e_1\in\mathcal{D}(A)\setminus\mathcal{D}(A^*A)$ such that the space $\mathcal{L}_1:=\mathcal{D}(A)\ominus_A\langle e_1\rangle$ is dense in $\mathcal{H}$. Without loss of generality we can assume that $|||e_1|||_A=1$. Let us complete $\{e_1\}$ to an orthonormal basis $\{e_n\}_{n=1}^\infty$ of $(\mathcal{D}(A),|||\cdot|||_A)$. Fix an integer $n\geqslant 2$ and set $\mathcal{L}=\mathcal{D}(A)\ominus_A\langle e_1,\dots,e_n\rangle$. Then the pair $(A,\mathcal{L})$ satisfies the assumptions of Theorem 3.5, $\mathcal{L}\subsetneq\mathcal{L}_1\subsetneq\mathcal{D}(A)$ and the operator $A|_{\mathcal{L}_1}:\mathcal{H}\supseteq\mathcal{L}_1\to\mathcal{H}$ is closed and densely defined. However, the operator $A|_{\mathcal{D}(A)\ominus_A\mathcal{L}}=A|_{\langle e_1,\dots,e_n\rangle}$ is bounded.

We conclude this section with some comments on Theorem 3.5 and Proposition 3.7.

Remark 4.4. Let A and $\mathcal{L}$ satisfy the assumptions of Theorem 3.5.

[1] Indeed, since

$$\langle E(\sigma)g,g\rangle=\sum_{k,l=1}^n\alpha_k\overline{\alpha}_l\langle E(\sigma)f_{s_k},f_{s_l}\rangle=\int_\sigma\left|\sum_{k=1}^n\alpha_kx^{s_k}\right|^2\mathrm{d}\mu(x),\quad\sigma\text{-Borel subset of }[0,\infty),$$

we get $\int_0^\infty x^4\langle E(\mathrm{d}x)g,g\rangle=\infty$, which is equivalent to $g\notin\mathcal{D}(A^2)$.

(a) It follows from (3.2) and Proposition 3.7 that

$$\big(\mathcal{D}(A)\ominus_A\mathcal{L}\big)\setminus\mathcal{D}(A^*A)=\mathcal{N}\big(P+(A|_{\mathcal{L}})^*A\big)\setminus(I+A^*A)^{-1}\big(\mathcal{L}^\perp\big).$$

(b) Regarding (3.6), note that the mapping

$$(I+A^*A)^{-1}:\big(\mathcal{H},\|\cdot\|\big)\longrightarrow\big(\mathcal{D}(A^*A),|||\cdot|||_A\big)$$

is continuous due to (3.5) and the continuity of the inverse of the mapping (3.6).

(c) If $\mathcal{D}(A)\ominus_A\mathcal{L}\subseteq\mathcal{D}(A^*A)$, then the norms $|||\cdot|||_A$ and $|||\cdot|||_{A^*A}$ are equivalent on $\mathcal{D}(A)\ominus_A\mathcal{L}$. Indeed, since $\mathcal{D}(A)\ominus_A\mathcal{L}$ is closed in $(\mathcal{D}(A),|||\cdot|||_A)$, we infer from (3.5) that $\mathcal{D}(A)\ominus_A\mathcal{L}$ is closed in $(\mathcal{D}(A^*A),|||\cdot|||_{A^*A})$. Thus $|||\cdot|||_A$ and $|||\cdot|||_{A^*A}$ are complete norms on $\mathcal{D}(A)\ominus_A\mathcal{L}$. When combined with (3.5) (and the inverse mapping theorem), this implies the desired equivalence. It may happen that $\mathcal{D}(A)\ominus_A\mathcal{L}$ is an infinite-dimensional subspace of $\mathcal{D}(A^*A)$ (cf. Example 4.2).

(d) If $\mathcal{L}\subseteq\mathcal{D}(A^*A)$ and A is unbounded, then condition (i) of Theorem 3.5 holds. Indeed, we have two possibilities: either $\mathcal{D}(A)\ominus_A\mathcal{L}\subseteq\mathcal{D}(A^*A)$ and so $\mathcal{D}(A)=\mathcal{L}\oplus_A(\mathcal{D}(A)\ominus_A\mathcal{L})\subseteq\mathcal{D}(A^*A)\subseteq\mathcal{D}(A)$, which, by Lemma 2.1, gives $A\in\boldsymbol{B}(\mathcal{H},\mathcal{K})$, a contradiction; or $(\mathcal{D}(A)\ominus_A\mathcal{L})\setminus\mathcal{D}(A^*A)\neq\emptyset$, which enables us to apply Theorem 3.5. It may happen that $\mathcal{L}$ is an infinite-dimensional subspace of $\mathcal{D}(A^*A)$ (see Example 4.2 with the new space $\mathcal{L}=\mathcal{D}(A_1)$).

Appendix A

The following lemma seems to be well-known, as it is strongly related to the theory of scales of Hilbert spaces. Since we were unable to find an appropriate reference, we decided to include its proof for the sake of completeness.

Lemma A.1. *Let R be a positive selfadjoint operator in $\mathcal{H}$ and let s,t be real numbers such that $0<s<t$. Then $\mathcal{D}(R^t)\subseteq\mathcal{D}(R^s)$. Moreover, the following two conditions are equivalent*:

(i) $\mathcal{D}(R^s)=\mathcal{D}(R^t)$,
(ii) $R\in\boldsymbol{B}(\mathcal{H})$.

Proof. The inclusion $\mathcal{D}(R^t)\subseteq\mathcal{D}(R^s)$ is a direct consequence of the inequality

$$\int_0^\infty x^{2s}\big\langle E(\mathrm{d}x)f,f\big\rangle\leqslant\int_0^\infty\big(1+x^{2t}\big)\big\langle E(\mathrm{d}x)f,f\big\rangle<\infty,\quad f\in\mathcal{D}\big(R^t\big),$$

where E stands for the spectral measure of R.

(i) $\Rightarrow$ (ii) Suppose that $\mathcal{D}(R^s)=\mathcal{D}(R^t)$. Since the operators R^s and R^t are closed, the normed spaces $(\mathcal{D}(R^s),|||\cdot|||_{R^s})$ and $(\mathcal{D}(R^s),|||\cdot|||_{R^t})$ are complete. It is an elementary check that the identity operator $(\mathcal{D}(R^s),|||\cdot|||_{R^s})\to(\mathcal{D}(R^s),|||\cdot|||_{R^t})$ is a closed operator. Hence, by the closed graph theorem, there exists a constant $c>0$ such that $\|R^tf\|^2\leqslant c(\|f\|^2+\|R^sf\|^2)$ for all $f\in\mathcal{D}(R^s)$. Thus we have

$$\int_0^\infty\big(c\big(1+x^{2s}\big)-x^{2t}\big)\big\langle E(\mathrm{d}x)f,f\big\rangle\geqslant0,\quad f\in\mathcal{D}\big(R^s\big).$$

Since $E(\sigma)\mathcal{D}(R^s)\subseteq\mathcal{D}(R^s)$ for all Borel subsets of the closed half line $[0,\infty)$, we get

$$\int_\sigma\big(c\big(1+x^{2s}\big)-x^{2t}\big)\big\langle E(\mathrm{d}x)f,f\big\rangle\geqslant0,\quad\sigma\text{-Borel subset of }[0,\infty),\ f\in\mathcal{D}\big(R^s\big).$$

This implies that $\langle E(Z_c)f,f\rangle=0$ for all $f\in\mathcal{D}(R^s)$, where

$$Z_c:=\big\{x\in\mathbb{R}\colon x\geqslant0,\ c\big(1+x^{2s}\big)-x^{2t}<0\big\}.$$

As $\mathcal{D}(R^s)$ is dense in $\mathcal{H}$, we obtain $E(Z_c)=0$. This and the fact that Z_c is an open subset of $[0,\infty)$ lead to $Z_c\subseteq[0,\infty)\setminus\operatorname{supp}E$, where $\operatorname{supp}E$ is the closed support of E. Since $\operatorname{supp}E$ coincides with the spectrum $\operatorname{Sp}(R)$ of R, we see that $\operatorname{Sp}(R)\subseteq[0,\infty)\setminus Z_c$, or equivalently $x^{2t}\leqslant c(1+x^{2s})$ for all $x\in\operatorname{Sp}(R)$. This and $t>s$ implies that $\operatorname{Sp}(R)$ is a bounded subset of $\mathbb{R}$. As a consequence, the selfadjoint operator R is bounded.

The reverse implication (ii) $\Rightarrow$ (i) is obvious. □

The following result is found in [18]. For the convenience of the reader we sketch its proof. It shows that the problem of looking for proper closed suboperators of a given closed densely defined operator A is equivalent to the problem of searching for proper closed extensions of A^*.

Lemma A.2. *Let $A : \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{K}$ and $B : \mathcal{H} \supseteq \mathcal{D}(B) \to \mathcal{K}$ be closed densely defined operators such that $A \subseteq B$. Then $B^* \subseteq A^*$ and*

$$\dim_{\mathcal{D}(B)} \mathcal{D}(B) \ominus_B \mathcal{D}(A) = \dim_{\mathcal{D}(A^*)} \mathcal{D}(A^*) \ominus_{A^*} \mathcal{D}(B^*), \tag{A.1}$$

where $\dim_{\mathcal{D}(B)}$ (*respectively*: $\dim_{\mathcal{D}(A^*)}$) *stands for the orthogonal dimension considered in the Hilbert space* $(\mathcal{D}(B), \langle\langle \cdot, \cdot \rangle\rangle_B)$ (*respectively*: $(\mathcal{D}(A^*), \langle\langle \cdot, \cdot \rangle\rangle_{A^*})$). *Moreover,* $\mathcal{D}(B) \ominus_B \mathcal{D}(A) = \mathcal{N}(I + A^*B)$, $\mathcal{D}(A^*) \ominus_{A^*} \mathcal{D}(B^*) = \mathcal{N}(I + BA^*)$ *and*

$$\dim_{\mathcal{D}(B)} \mathcal{N}(I + A^*B) = \dim_{\mathcal{D}(A^*)} \mathcal{N}(I + BA^*).$$

Proof. Let $W \in \boldsymbol{B}(\mathcal{K} \oplus \mathcal{H}, \mathcal{H} \oplus \mathcal{K})$ be the unitary operator given by $W(f \oplus g) = (-g) \oplus f$ for $f \in \mathcal{K}$ and $g \in \mathcal{H}$. Then, by [9, Lemma 3.6.1] (see also [3, Theorem 3.3.1]), we have

$$\mathcal{G}(B) \ominus \mathcal{G}(A) = \mathcal{G}(A)^\perp \ominus \mathcal{G}(B)^\perp = W\big(\mathcal{G}(A^*)\big) \ominus W\big(\mathcal{G}(B^*)\big) = W\big(\mathcal{G}(A^*) \ominus \mathcal{G}(B^*)\big),$$

which means that the Hilbert spaces $\mathcal{G}(B) \ominus \mathcal{G}(A)$ and $\mathcal{G}(A^*) \ominus \mathcal{G}(B^*)$ are unitarily equivalent. However, $\mathcal{G}(B) \ominus \mathcal{G}(A)$ is unitarily equivalent to $(\mathcal{D}(B) \ominus_B \mathcal{D}(A), \langle\langle \cdot, \cdot \rangle\rangle_B)$ and $\mathcal{G}(A^*) \ominus \mathcal{G}(B^*)$ is unitarily equivalent to $(\mathcal{D}(A^*) \ominus_{A^*} \mathcal{D}(B^*), \langle\langle \cdot, \cdot \rangle\rangle_{A^*})$. This leads to (A.1). The rest of the conclusion can be proved in a standard way. □

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