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# On the set of limit points of the partial sums of series rearranged by a given divergent permutation

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ABSTRACT

We give a new characterization of divergent permutations. We prove that for any divergent permutation $p$, any closed interval $I$ of $\mathbb{R}^*$ (the 2-point compactification of $\mathbb{R}$) and any real number $s \in I$, there exists a series $\sum a_n$ of real terms convergent to $s$ such that $I = \sigma a_{p(n)}$ (where $\sigma a_{p(n)}$ denotes the set of limit points of the partial sums of the series $\sum a_{p(n)}$). We determine permutations $p$ of $\mathbb{N}$ for which there exists a conditionally convergent series $\sum a_n$ such that $\sum a_{p(n)} = +\infty$. If the permutation $p$ of $\mathbb{N}$ possesses the last property then we prove that for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^*$ there exists a series $\sum a_n$ convergent to $\alpha$ and such that $\sigma a_{p(n)} = [\beta, +\infty]$. We show that for any countable family $P$ of divergent permutations there exist conditionally convergent series $\sum a_n$ and $\sum b_n$ such that any series of the form $\sum a_{p(n)}$ with $p \in P$ is convergent to the sum of $\sum a_n$, while $\sigma b_{p(n)} = \mathbb{R}^*$ for every $p \in P$.



## 0. Introduction

The paper is in a sense historic. It was written in 1993 and accepted in 1995 for publication in the Journal of Mathematical Analysis and Applications but finally not published there – which was my private decision. The reason for this was my discovery, between 1994–1995 of the findings of [4], where, as it turned out, two essential results of my work were presented (Theorem 3 of [4] $\equiv$ Theorem 3.1 of the present paper and Theorem 5 of [4] which is "almost" my Theorem 5.2).

Moreover, Nash-Williams and White, in paper [9] from 1999 resolved the fundamental problem of the description of the set of limit points of the partial sums of the rearranged series convergence $\sum_{n=1}^{\infty} a_{p(n)}$ if, a real series $\sum_{n=1}^{\infty} a_n = 0$ and a permutation $p$ of $\mathbb{N}$ are given (see also the final remarks in this paper). The same authors generalized these results to the series with elements from $\mathbb{R}^n$ (see [10,11]). Surely, the theorems concerning the series in $\mathbb{R}^n$, $n \geqslant 2$ are much more profound than those drawn in the case of real series. It should also be emphasized that the investigations in this field have a longer history:

- A.S. Kronrod paper [7] (written under the auspices of D.I. Menszov) was one of the first papers in this field – the paper that is hardly known; but is directly connected with the results derived in the paper;
- Jasek's papers (see [5,6] and the references in these papers);
- The research conducted under the supervision of Prof. Z. Zahorski (a well-known specialist on the theory of real function – the so-called "Zahorski classes" $M_k$ are commonly recognized) – see papers [17,19–24].

In view of the above, why have I decided to resuscitate and publish the "old" paper supplemented by some changes and additions? The answer is to be found in the presented techniques of proving, and, more precisely, in using:

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– A combinatoric characterization of divergent permutations (see also Theorem 5 in paper [28], where it was properly substantiated);
– The introduction of two new notions: families $\mathbb{U}(p)$ and $\mathbb{V}(p)$ of positive integers assigned to divergent permutation $p$ (see Section 4), facilitating a very easy and clear manner of constructing the series with the required properties.

Accordingly, it seems that the presented proofs are neater (more compact and short presented) than similar proofs from [4,9,10].

## 1. Basic notation and terminology

The sets of reals, positive integers, nonnegative integers, even and odd positive integers will be denoted by $\mathbb{R}$, $\mathbb{N}$, $\mathbb{N}_0$, $2\mathbb{N}$ and $2\mathbb{N}-1$, respectively.

**Definition 1.1.** We say that two nonempty, finite and disjoint subsets $\mathbb{X}$ and $\mathbb{Y}$ of $\mathbb{N}$ are spliced when $\operatorname{card}\mathbb{X} = \operatorname{card}\mathbb{Y}$ and the following condition is satisfied:

$$\text{either} \quad x_i < y_i < x_{i+1} < y_n \quad \text{for every } i = 1, 2, \ldots, n-1,$$

$$\text{or} \quad y_i < x_i < y_{i+1} < x_n \quad \text{for every } i = 1, 2, \ldots, n-1,$$

where $\{x_i\colon\ i = 1, 2, \ldots, n\}$ and $\{y_i\colon\ i = 1, 2, \ldots, n\}$ are the sets $\mathbb{X}$ and $\mathbb{Y}$ listed in increasing order. In other words, two nonempty, finite and disjoint sets $\mathbb{X}, \mathbb{Y} \subset \mathbb{N}$ are spliced iff they have the same cardinality and an increasing sequence in which the elements of $\mathbb{X}$ and $\mathbb{Y}$ alternate can be created from the set of all elements of $\mathbb{X}$ and $\mathbb{Y}$.

**Definition 1.2.** Let $p$ be a permutation of $\mathbb{N}$ and let $k \in \mathbb{N}$. We say that a $2k$-set $\mathbb{X} \subset \mathbb{N}$ is spliced by $p$ if the sets:

$$\bigl\{p(x_n)\colon\ n = 1, 2, \ldots, k\bigr\} \quad \text{and} \quad \bigl\{p(x_n)\colon\ n = k+1, k+2, \ldots, 2k\bigr\}$$

are spliced. Here $\{x_n\colon\ 1, 2, \ldots, 2k\}$ denotes the increasing sequence of all elements of $\mathbb{X}$.

**Definition 1.3.** A subset $I$ of $\mathbb{N}$ is said to be an interval if either $I = \emptyset$ or it can be expressed in the form $I = \{m, m+1, \ldots, m+n-1\}$ for some $m, n \in \mathbb{N}$. We will use the following symbols: $[m,n]$, $[m,n)$, $(m,n]$ and $(m,n)$ with $m, n \in \mathbb{N}$, $m < n$, to denote the sets: $\{m, m+1, \ldots, n\}$, $\{m, m+1, \ldots, n-1\}$, $\{m+1, m+2, \ldots, n\}$ and $\{m+1, m+2, \ldots, n-1\}$, respectively. Also the set $\{x \in \mathbb{R}^*\colon\ a \leqslant x\}$ will be denoted by $[a, +\infty]$.

**Definition 1.4.** We say that a set $A$ of positive integers is a union of $n$ mutually separated intervals (abbrev.: $n$ **MSI**) if there exists a family $\mathfrak{A} \subset 2^{\mathbb{N}}$ of nonempty intervals with $\bigcup \mathfrak{A} = A$ and $\operatorname{card}\mathfrak{A} = n$ satisfying the condition: $\operatorname{dist}(I, J) \geqslant 2$ for any two different members $I$ and $J$ of $\mathfrak{A}$.

For brevity, we write $K < L$ for two nonempty subsets $K$ and $L$ of $\mathbb{N}$ when $k < l$ for any $k \in K$ and $l \in L$. In the sequel we will write $k < L (L < k)$ instead of $\{k\} < L$ ($L < \{k\}$, resp.)

In this paper we will often identify a given sequence with its set of values. Moreover, we will consider only series $\sum a_n$ consisting of real terms. We will use the notation $\sigma a_n$ for the set of points of accumulation (the derived set) of the partial sums of a series $\sum a_n$. The set $\sigma a_n$ will be treated as a subset of the extended reals:

$$\mathbb{R}^* := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\} = \text{the 2-point compactification of } \mathbb{R}.$$

When $\sigma a_n$ is a one element set $\{s\}$ we will write $s = \sum a_n$, the usual notation.

We note that $\sigma a_n$ is closed (because derived sets are always closed) and that if $a_n \to 0$ it is also convex, hence a closed interval of $\mathbb{R}^*$. In particular, $\sigma a_n$ is always a closed interval of $\mathbb{R}^*$ when $\{a_n\}$ is a rearrangement of a convergent series.

## 2. Divergent permutations

**Definition 2.1.** A permutation $p$ of $\mathbb{N}$ is called a divergent permutation if there exists a conditionally convergent series $\sum a_n$ such that the series $\sum a_{p(n)}$ is divergent. The family of all divergent permutations is denoted by $\mathfrak{D}$.

It is well known (see [1,7,12,13,16,28]) that a permutation $p$ of $\mathbb{N}$ is divergent if and only if for every $k \in \mathbb{N}$ there exists a nonempty interval $I \subset \mathbb{N}$ such that the set $p(I)$ is a union of at least $k$ **MSI** (see also [2,3,8,14,15,18,25,27,29] for supplementary material).

The following theorem gives a new characterization of divergent permutations.

**Theorem 2.1.** *A permutation $p$ of $\mathbb{N}$ is divergent if and only if for every $k \in \mathbb{N}$ there exists a $2k$-set $\mathbb{X} \subset \mathbb{N}$ spliced by $p$.*

**Proof.** Let us fix $k \in \mathbb{N}$. If $p$ is a divergent permutation then there exists an interval $I \subset \mathbb{N}$ such that the set $p(I)$ is a union of at least $(2k+1)$ **MSI**. Then the set $J$ defined by

$$J := \big[\min p(I), \max p(I)\big] \setminus p(I)$$

is a union of at least $2k$ **MSI** and, additionally, $p^{-1}(J) \cap I = \emptyset$. Hence, an increasing sequence $\{x_n\colon n = 1, 2, \ldots, 2k\}$ of positive integers can be chosen in such a way that either

$$\begin{aligned} &\{x_n\colon n = 1, 2, \ldots, k\} \subset I \quad \text{and} \\ &\{x_n\colon n = k+1, k+2, \ldots, 2k\} \subset p^{-1}(J), \end{aligned} \tag{1}$$

or

$$\begin{aligned} &\{x_n\colon n = 1, 2, \ldots, k\} \subset p^{-1}(J) \quad \text{and} \\ &\{x_n\colon n = k+1, k+2, \ldots, 2k\} \subset I \end{aligned} \tag{2}$$

and furthermore the set $\mathbb{X} := \{x_n\colon n = 1, 2, \ldots, 2k\}$ is spliced by $p$.

On the other hand, if $\{x_n\colon n = 1, 2, \ldots, 2k\}$ is an increasing sequence spliced by $p$ then each of the sets

$$p\big([x_1, x_k]\big) \quad \text{and} \quad p\big([x_{k+1}, x_{2k}]\big)$$

is a union of at least $k$ **MSI**. □

**Corollary 2.2.** *Let $p \in \mathfrak{D}$. Then for any $r, s \in \mathbb{N}$ there exists a $2r$-set $\mathbb{X} \subset \mathbb{N}$ spliced by $p$ such that $\mathbb{X} \cup p(\mathbb{X}) > s$.*

**Proof.** Take $t \in \mathbb{N}$ with $t \geqslant p([1, s])$. By Theorem 2.1 there exists an increasing sequence of positive integers $\{x_n\colon n = 1, 2, \ldots, 2(r+t)\}$ spliced by $p$. Let $\mathbb{Y} = \{y_n\colon n = 1, 2, \ldots, 2(r+t)\}$ be the increasing sequence formed from the elements of the set $\{p(x_n)\colon n = 1, 2, \ldots, 2(r+t)\}$ and let $\mathbb{Y}^* = \{y_{r_i}\colon i = 1, 2, \ldots, t\}$ be the subsequence of $\mathbb{Y}$ formed from the elements of the set $\{p(x_n)\colon n = 1, 2, \ldots, t\}$. Without loss of generality suppose that $r_1 > 1$. Then it is sufficient to take

$$\mathbb{X} = p^{-1}\big(\mathbb{Y} \setminus (\mathbb{Y}^* \cup \mathbb{Y}^{**})\big),$$

where $\mathbb{Y}^{**} := \{y_{r_i - 1}\colon i = 1, 2, \ldots, t\}$. □

## 3. Characterization of the sets of limit points

**Theorem 3.1.** *Let $p \in \mathfrak{D}$. Then for any nonempty, closed interval $I \subset \mathbb{R}^*$ and for any $s \in (I \cap \mathbb{R})$, there exists a sequence $\{a_n\} \subset \mathbb{R}$ with $\sum a_n = s$ and $\sigma a_{p(n)} = I$.*

**Proof.** Let us fix a divergent permutation $p$. Then Corollary 2.2 guarantees the existence of a countable family of increasing sequences of positive integers $\mathbb{X}_r = \{x_n^{(r)}\colon n = 1, 2, \ldots, 2r\}$, $r \in \mathbb{N}$, spliced by $p$ such that for each $r$

$$p(1) < \mathbb{X}_r \cup p(\mathbb{X}_r) < \mathbb{X}_{r+1} \cup p(\mathbb{X}_{r+1}). \tag{3}$$

Let

$$\begin{aligned} &\mathbb{X}_r^{(1)} := \big\{p\big(x_n^{(r)}\big)\colon n = 1, 2, \ldots, r\big\} \quad \text{and} \\ &\mathbb{X}_r^{(2)} := \big\{p\big(x_n^{(r)}\big)\colon n = r+1, r+2, \ldots, 2r\big\}. \end{aligned} \tag{4}$$

We may assume that either

$$\min \mathbb{X}_r^{(1)} < \min \mathbb{X}_r^{(2)} \quad \text{for every } r \in \mathbb{N} \tag{5}$$

or

$$\min \mathbb{X}_r^{(2)} < \min \mathbb{X}_r^{(1)} \quad \text{for every } r \in \mathbb{N}. \tag{6}$$

If $I = \mathbb{R}^*$, $s \in \mathbb{R}$, then the terms of the desired series $\sum a_n$ we define by

$$a_n = \begin{cases} (-1)^r / r^{\frac{1}{2}} & \text{for } n \in \mathbb{X}_r^{(1)},\ r \in \mathbb{N}, \\ (-1)^{r+1} / r^{\frac{1}{2}} & \text{for } n \in \mathbb{X}_r^{(2)},\ r \in \mathbb{N}. \end{cases}$$

Moreover, we put $a_{p(1)} = s$ and $a_n = 0$ for all remaining indices $n \in \mathbb{N}$.

When $I \neq \mathbb{R}^*$ the definition of the elements $a_n$ depends on which of the conditions (5) or (6) holds. First let us assume that condition (5) is satisfied. There are two cases:

(i) $I=\{x\in\mathbb{R}:\ a\leqslant x\leqslant b\}$, $a,b\in\mathbb{R}$, $a\leqslant b$ and $s\in I$. We set

$$a_n=\begin{cases}(b-s)/r & \text{for } n\in\mathbb{X}_r^{(1)},\ r\in(2\mathbb{N}-1),\\ (s-b)/r & \text{for } n\in\mathbb{X}_r^{(2)},\ r\in(2\mathbb{N}-1),\end{cases}$$

and

$$a_n=\begin{cases}(a-s)/r & \text{for } n\in\mathbb{X}_r^{(1)},\ r\in 2\mathbb{N},\\ (s-a)/r & \text{for } n\in\mathbb{X}_r^{(2)},\ r\in 2\mathbb{N}.\end{cases}$$

(ii) $I=[a,+\infty]$, $a,s\in\mathbb{R}$, $a\leqslant s$. We set

$$a_n=\begin{cases}1/r^{\frac{1}{2}} & \text{for } n\in\mathbb{X}_r^{(1)},\ r\in(2\mathbb{N}-1),\\ -1/r^{\frac{1}{2}} & \text{for } n\in\mathbb{X}_r^{(2)},\ r\in(2\mathbb{N}-1),\end{cases}$$

and

$$a_n=\begin{cases}(a-s)/r & \text{for } n\in\mathbb{X}_r^{(1)},\ r\in 2\mathbb{N},\\ (s-a)/r & \text{for } n\in\mathbb{X}_r^{(2)},\ r\in 2\mathbb{N}.\end{cases}$$

Furthermore, we put $a_{p(1)}=s$ and $a_n=0$ for all indices

$$n\in\Bigl(\mathbb{N}\setminus\bigcup_{r\in\mathbb{N}}\bigl(\mathbb{X}_r^{(1)}\cup\mathbb{X}_r^{(2)}\bigr)\Bigr)$$

such that $n\neq p(1)$.

We leave it to the reader to verify that the series $\sum a_n$ is convergent to $s$ and that $\sigma a_{p(n)}=I$ in each case considered above.

If condition (6) holds then the definition of the elements $a_n$ requires only one change: we should replace the sets $\mathbb{X}_r^{(1)}$ and $\mathbb{X}_r^{(2)}$ in the definition of the elements $a_n$ by the sets $\mathbb{X}_r^{(2)}$ and $\mathbb{X}_r^{(1)}$, respectively, for every $r\in\mathbb{N}$.

In order to make the definitions of the series $\sum a_n$ as clear as possible we present the detailed form of such series in the case (i) when condition (5) holds. From conditions (3) and (4) we deduce that

$$\begin{aligned}\sum a_n={}&\underbrace{\mathbf{0}_1+s}_{0\text{-}block}+\underbrace{\mathbf{0}_2+(b-s)+\mathbf{0}_3+(s-b)}_{1\text{-}block}+\underbrace{\mathbf{0}_4+(a-s)+\mathbf{0}_5+(s-a)}_{2\text{-}block}\\ &+\underbrace{\mathbf{0}_6+(b-s)/2+\mathbf{0}_7+(s-b)/2+\mathbf{0}_8+(b-s)/2+\mathbf{0}_9+(s-b)/2}_{3\text{-}block}\\ &+\underbrace{\mathbf{0}_{10}+(a-s)/2+\mathbf{0}_{11}+(s-a)/2+\mathbf{0}_{12}+(a-s)/2+\mathbf{0}_{13}+(s-a)/2}_{4\text{-}block}+\cdots\\ &+\underbrace{\mathbf{0}_{k(r)}+x_r+\mathbf{0}_{k(r)+1}-x_r+\mathbf{0}_{k(r)+2}+x_r+\cdots}_{r\text{-}block}\\ &+\underbrace{\mathbf{0}_{k(r)+i}+(-1)^i x_r+\cdots+\mathbf{0}_{k(r)+2r-1}-x_r}_{r\text{-}block}+\cdots\end{aligned}$$

where $\mathbf{0}_r$ is a finite sum of zeros, $k(r):=0.5(r^2+3)$ and $x_r:=(b-s)/r$ when $r$ is odd, and $k(r):=0.5(r^2+4)$ and $x_r:=(a-s)/r$ when $r$ is even. However the series $\sum a_{p(n)}$ has the following form:

$$\begin{aligned}\sum a_{p(n)}={}&\underbrace{\mathbb{O}_1+s}_{0\text{-}block}+\underbrace{\mathbb{O}_2+(b-s)+\mathbb{O}_3+(s-b)}_{1\text{-}block}+\underbrace{\mathbb{O}_4+(a-s)+\mathbb{O}_5+(s-a)}_{2\text{-}block}\\ &+\underbrace{\mathbb{O}_6+(b-s)/2+\mathbb{O}_7+(b-s)/2+\mathbb{O}_8+(s-b)/2+\mathbb{O}_9+(s-b)/2}_{3\text{-}block}\\ &+\underbrace{\mathbb{O}_{10}+(a-s)/2+\mathbb{O}_{11}+(a-s)/2+\mathbb{O}_{12}+(s-a)/2+\mathbb{O}_{13}+(s-a)/2}_{4\text{-}block}+\cdots\\ &+\underbrace{\mathbb{O}_{k(r)}+x_r+\mathbb{O}_{k(r)+1}+x_r+\cdots+\mathbb{O}_{k(r)+r-1}+x_r}_{r\text{-}block}\\ &+\underbrace{\mathbb{O}_{k(r)+r}-x_r+\mathbb{O}_{k(r)+r+1}-x_r+\cdots+\mathbb{O}_{k(r)+2r-1}-x_r}_{r\text{-}block}+\cdots\end{aligned}$$

where the symbol $\mathbb{O}_r$ denotes a finite sum of zeros for every $r \in \mathbb{N}$ and where $k(r)$ and $x_r$ are defined as above. We note that the rearranged series $\sum a_{p(n)}$ is built from "blocks". First, the partial sums of this series whose upper index belongs to the given $r$-block ($r$ is assumed to be odd) run through the set $\{x \in \mathbb{R}\colon s \leqslant x \leqslant b\}$ from $s$ to $b$ with the step equal to either $(b-s)/r$ or 0 (if $r$ is even then the partial sums of $\sum a_{p(n)}$ run through the set $\{x \in \mathbb{R}\colon a \leqslant x \leqslant s\}$ from $s$ to $a$ with the step equal to either $(a-s)/r$ or 0). Next, they run from $b$ to $s$ with the step equal to either $(s-b)/r$ or 0 (run from $a$ to $s$ with the step equal to either $(s-a)/r$ or 0, resp.). Hence $\sigma a_{p(n)}$ is a dense subset of the set $\{x \in \mathbb{R}\colon a \leqslant x \leqslant b\}$ which is the desired result. □

**Remark.** The series $\sum a_n$ can be chosen to be conditionally convergent also in the case $I = \{s\}$, $s \in \mathbb{R}$. It is sufficient to define

$$a_n = \begin{cases} r^{-2} & \text{for } n \in \mathbb{X}_r^{(1)},\ r \in \mathbb{N}, \\ -r^{-2} & \text{for } n \in \mathbb{X}_r^{(2)},\ r \in \mathbb{N}. \end{cases}$$

## 4. Divergence to infinity

For each $p \in \mathfrak{D}$ and for each $n \in \mathbb{N}$ there exist a positive integer $t(p,n)$ and a family $\{I_i^{(p,n)}\colon i = 1, 2, \ldots, t(p,n)\} \subset 2^{\mathbb{N}}$ of nonempty intervals such that

$$\bigcup_{i=1}^{t(p,n)} I_i^{(p,n)} = p\big([1,n]\big) \quad \text{and} \quad \operatorname{dist}\big(I_i^{(p,n)}, I_j^{(p,n)}\big) \geqslant 2 \quad \text{for } i \neq j.$$

This enables us to express the following results.

**Theorem 4.1.** *Let $p$ be a permutation of $\mathbb{N}$. Then $p \in \mathfrak{D}$ if and only if $\limsup_{n\to\infty} t(p,n) = \infty$.*

**Proof.** By the definition of $t(p,n)$,

$$p\big([1,n]\big) \text{ is a union of } k \textbf{ MSI} \quad \Leftrightarrow \quad t(p,n) = k.$$

Since $p([n,m]) = p([1,m]) \setminus p([1,n])$, for $n < m$, we see that the set $p([n,m])$ is a union of at most $(t(p,n) + t(p,m))$ **MSI**. Hence $p \in \mathfrak{D}$ if and only if the sequence $\{t(p,n)\colon n \in \mathbb{N}\}$ is unbounded. □

**Theorem 4.2.** *Let $p \in \mathfrak{D}$. If $\liminf_{n\to\infty} t(p,n) < \infty$ then the sum $\sum a_n$ is a limit point of the partial sums of the series $\sum a_{p(n)}$ for each conditionally convergent series $\sum a_n$.*

**Proof.** Assume $\alpha = \liminf_{n\to\infty} t(p,n) < \infty$. Then there exists an infinite subset $A$ of $\mathbb{N}$ such that $t(p,n) = \alpha$ for every $n \in A$. We will denote by $k(n)$ the positive integer which is defined to be $\max\{k \in \mathbb{N}\colon [1,k] \subset p([1,n])\}$ for sufficiently large $n \in \mathbb{N}$. Let $\sum a_n$ be a conditionally convergent series. A trivial verification shows that

$$\left|\sum_{i=1}^{n} a_{p(i)} - \sum a_i\right| \leqslant \max\left\{\left|\sum_{i=u}^{v} a_i\right|\colon u, v \in \mathbb{N},\ k(n) < u \leqslant v\right\}$$

for large enough $n \in A$. Since $k(n) \to \infty$ as $n \to \infty$, this clearly implies that the sum $\sum a_n$ is a limit point of the series $\sum a_{p(n)}$. □

**Remark.** Theorem 4.2 tells us that, if Theorem 3.1 is to be valid for all divergent permutations, the condition $s \in I \cap \mathbb{R}$ cannot be weakened.

With each divergent permutation $p$ we can associate two basic sets:

$$\mathbb{U}(p) := \big\{u \in \mathbb{N}\colon t(p,u) - t(p,u-1) = 1\big\} \tag{7}$$

and

$$\mathbb{V}(p) := \big\{v \in \mathbb{N}\colon t(p,v) - t(p,v-1) = -1\big\} \tag{8}$$

where $t(p,0) := 0$. Obviously, both sets $\mathbb{U}(p)$ and $\mathbb{V}(p)$ are infinite. The increasing sequences of all elements of the sets $p(\mathbb{U}(p))$ and $p(\mathbb{V}(p))$ will be denoted by $\{u_n(p)\}$ and by $\{v_n(p)\}$, respectively.

**Lemma 4.3.** *For every divergent permutation $p$ and for every positive integer $n$ the following relations are satisfied:*

(i) $u_n(p) < v_n(p) < u_{n+1}(p)$,
(ii) $p^{-1}(u_n(p)) < p^{-1}(v_n(p))$,
(iii) $\mathrm{card}(p^{-1}(I_i^{(p,n)}) \cap \mathbb{U}(p)) - \mathrm{card}(p^{-1}(I_i^{(p,n)}) \cap \mathbb{V}(p)) = 1$ *for every index* $i = 1, 2, \ldots, t(p,n)$,
(iv) $\mathrm{card}([1,n] \cap \mathbb{U}(p)) - \mathrm{card}([1,n] \cap \mathbb{V}(p)) = t(p,n)$.

**Proof.** The relations (i), (ii) and (iv) can be deduced immediately from the definitions of the sets $\mathbb{U}(p)$ and $\mathbb{V}(p)$. The equality (iii) can be proved by induction on the cardinality of the interval $I_i^{(p,n)}$. The relation (iv) also follows from (iii). □

**Theorem 4.4.** *Let $\{p_i\}$ be a sequence of divergent permutations such that $\lim_{n\to\infty} t(p_i,n) = \infty$ for every $i \in \mathbb{N}$. Moreover, suppose that*

$$p_i\big(\mathbb{U}(p_i)\big) = p_j\big(\mathbb{U}(p_j)\big) \quad \textit{and} \quad p_i\big(\mathbb{V}(p_i)\big) = p_j\big(\mathbb{V}(p_j)\big) \tag{9}$$

*for any $i, j \in \mathbb{N}$. Then for every $\alpha \in \mathbb{R}$ there exist conditionally convergent series $\sum a_n$ and $\sum b_n$, such that*

(i) $\sum a_n = \sum b_n = \alpha$,
(ii) $\sum_{n=1}^{\infty} a_{p_i(n)} = +\infty$ *for every* $i \in \mathbb{N}$,

*and*

(iii) *the set of limit points of the partial sums of a series $\sum_{n=1}^{\infty} b_{p_i(n)}$ is equal to $[\alpha, +\infty]$ for every $i \in \mathbb{N}$.*

**Proof.** We will use the symbols (i), (ii) and (iv) to denote the conclusions (i), (ii) and (iv) of Lemma 4.3, respectively.

To simplify the notation, we will write $u_n$ and $v_n$ instead of $u_n(p_1)$ and $v_n(p_1)$, respectively, for each $n \in \mathbb{N}$.

First we will construct the terms of the series $\sum a_n$. For this purpose we choose by induction an increasing sequence $\{w_n\}$ of members of the set $\mathbb{U}(p_1)$ with $w_n > p_n^{-1}(u_1)$, such that, for any $n \in \mathbb{N}$,

$$t(p_i, w) > n^2 \tag{10}$$

for $i = 1, 2, \ldots, n$ and for any $w \geqslant w_n$. Put

$$a(u_i) = n^{-1} \quad \text{and} \quad a(v_i) = -n^{-1}$$

when $p_1^{-1}(u_i) \in [w_n, w_{n+1})$ and put $a(u_1) = \alpha$ and $a(n) = 0$ for the remaining $n \in \mathbb{N}$.

By (i) the series $\sum a(n)$ is convergent to $\alpha$. On the other hand, applying the relations (ii), (iv) and (10) we get

$$\sum_{s=1}^{w} a\big(p_i(s)\big) \geqslant \alpha + \big(t(p_i, w) - 1\big)/n \geqslant \alpha + n$$

for $w \in [w_n, w_{n+1})$, and $i = 1, 2, \ldots, n$. This yields $\sum_{n=1}^{\infty} a(p_i(n)) = +\infty$ for every positive integer $i$.

For the construction of the series $\sum b_n$ we need to choose increasing sequences $\{x_n\}$ and $\{y_{i,n}\colon n \in \mathbb{N}\}$, $i \in \mathbb{N}$, of positive integers satisfying, for every $i, n \in \mathbb{N}$ with $i \leqslant n$, the following conditions:

$$p_1^{-1}(u_1) < X_{i,n} < X_{i,n+1} \quad \text{and} \quad y_{i,n} \in X_{i,n}, \tag{11}$$

$$t(p_i, y_{i,n}) > n^2 \tag{12}$$

and

$$p_i\big([1, y_{i,n}]\big) \leqslant u_{x_{2n}}. \tag{13}$$

Here $X_{i,n} := p_i^{-1}([u_{x_{2n-1}}, v_{x_{2n}}])$, for $i \leqslant n$. Then we set

$$b(u_1) = \alpha, \qquad b(u_i) = n^{-1} \quad \text{and} \quad b(v_i) = -n^{-1}$$

for $i \in \{x_{2n-1}, x_{2n-1} + 1, \ldots, x_{2n}\}$. Moreover, we set $b(n) = 0$ for all remaining indices $n \in \mathbb{N}$. We conclude from (i) that the series $\sum b(n)$ is convergent. Using (ii), (iv), (13) and (12) we obtain

$$\sum_{s=1}^{y_{i,n}} b\big(p_i(s)\big) \geqslant \alpha + \big(t(p_i, y_{i,n}) - 1\big)/n \geqslant \alpha + n \tag{14}$$

for $i \leqslant n$. Simultaneously, by (11), we have

$$\sum_{s=1}^{\max X_{i,n}} b\big(p_i(s)\big) = \alpha \tag{15}$$

for $i \leqslant n$. From (14), (15) and from the convergence to zero of the sequence $\{b(n)\}$ it follows that the set of limit points of the partial sums of any series $\sum_{n=1}^{\infty} b(p_i(n))$, $i \in \mathbb{N}$, includes the set $\{x \in \mathbb{R}\colon\ \alpha \leqslant x\}$. However, by (ii) almost all partial sums of a series $\sum_{n=1}^{\infty} b(p_i(n))$, $i \in \mathbb{N}$, are $\geqslant \alpha$. Hence, all these series have the same set of limit points of the partial sums, namely the set $[\alpha, +\infty]$. The proof is finished. □

**Remark.** The assumptions of Theorem 4.4 are strong, yet they make it possible to derive a very intriguing thesis. In [26] (see also Ex. 3.3 from [24]) there is an example of two divergent permutations $p$ and $q$ such that:

$$\lim_{n\to\infty} t(p,n) = \lim_{n\to\infty} t(q,n) = \infty,$$
$$p\big(U(p)\big) = q\big(U(q)\big) \quad \text{and} \quad p\big(V(p)\big) = q\big(V(q)\big),$$

for which there exist two conditionally convergent series $\sum a_n$ and $\sum b_n$ such that

$$\sum a_n = \sum b_n = \sum a_{q(n)} = \sum b_{p(n)} = 0,$$

and

$$\sum a_{p(n)} = \sum b_{q(n)} = +\infty.$$

These example could be extended to countable infinite family of divergent permutations.

Together with Theorems 4.1 and 4.2 the above theorem (we need only the simple case when all permutations $p_i$ are the same) implies the following result.

**Theorem 4.5.** *Let $p \in \mathfrak{D}$. We have $\sum a_{p(n)} = +\infty$ for some conditionally convergent series $\sum a_n$ if and only if $\lim_{n\to\infty} t(p,n) = \infty$.*

Summarizing the main results of Sections 3 and 4 we get the following theorem.

**Theorem 4.6.** *Let $p \in \mathfrak{D}$ and let $I$ be a closed interval of $\mathbb{R}^*$. Then for every $s \in (I \cap \mathbb{R})$ there exists a conditionally convergent series $\sum a_n$ such that $s = \sum a_n \in I = \sigma a_{p(n)}$. For a conditionally convergent series $\sum a_n$ the set $\sigma a_n$ is necessarily a closed interval of $\mathbb{R}^*$. If $\liminf_{n\to\infty} t(p,n) < \infty$ and the series $\sum a_n$ is convergent then $\sum a_n \in \sigma a_{p(n)}$. If $\lim_{n\to\infty} t(p,n) = \infty$ then there exists a convergent series $\sum a_n$ such that $\sigma a_{p(n)} = \{+\infty\}$.*

## 5. Miscellaneous results

**Theorem 5.1.** *For each sequence $\{p_n\}$ of divergent permutations there exist two conditionally convergent series $\sum a_k$ and $\sum b_k$ of real terms with the properties*:

(i) *the series $\sum_{k=1}^{\infty} a_{p_n(k)}$ is convergent to zero for every $n \in \mathbb{N}$ and*
(ii) *the set of limit points of the partial sums of the series $\sum_{k=1}^{\infty} b_{p_n(k)}$ is equal to $\mathbb{R}^*$ for every $n \in \mathbb{N}$.*

**Proof.** Take $\{p_n\}$ to be a sequence of divergent permutations and write $p_0$ for the identity function of $\mathbb{N}$. First we will construct the series $\sum a_k$. Put

$$s_1 = 1, \qquad t_1 = 2 \quad \text{and} \quad v_1 = \max p_1^{-1}\big([1,2]\big).$$

Suppose the elements $s_1$, $t_1$ and $v_1$ have been defined for every $i = 1, 2, \ldots, n-1$. Then we choose the positive integers $s_n$ and $t_n$ in such a way that:

$$v_{n-1} < s_n < t_n \tag{16}$$

and

$$v_{n-1} < \bigcup_{i=1}^{n} p_i^{-1}\big(\{s_n, t_n\}\big). \tag{17}$$

Next we set

$$v_n = \max \bigcup_{i=1}^{n} p_i^{-1}\big([1, t_n]\big). \tag{18}$$

From (16)–(18) we get

$$v_{n-1} < s_n < t_n \leqslant v_n \tag{19}$$

for $n \in \mathbb{N}$, and

$$p_i\big((v_{n-1}, v_n]\big) \cap \{s_k, t_k\colon k \in \mathbb{N}\} = \{s_n, t_n\} \tag{20}$$

for $i \leqslant n$. Now define

$$a_n = \begin{cases} k^{-1} & \text{for } n = s_k,\ k \in \mathbb{N}, \\ -k^{-1} & \text{for } n = t_k,\ k \in \mathbb{N}, \\ 0 & \text{for all remaining } n \in \mathbb{N}. \end{cases}$$

The inequalities (19) imply that the series $\sum a_n$ is conditionally convergent. On the other hand, the relation (20) yields the convergence of the series $\sum_{k=1}^{\infty} a_{p_n(k)}$ for every $n \in \mathbb{N}$. Moreover, we see that

$$\sum_{k=1}^{\infty} a_{p_n(k)} = \sum a_k = 0$$

for each index $n \in \mathbb{N}$.

Now we construct the series $\sum b_k$. By Theorem 3.1 with each permutation $p_i$, $i \in \mathbb{N}$, can be associated a conditionally convergent series $\sum_{n=1}^{\infty} a_n^{(i)}$ such that the set of limit points of the partial sums of the series $\sum_{n=1}^{\infty} a_{p_i(n)}^{(i)}$ is equal to $\mathbb{R}^*$. Proceeding by induction, we can select positive integers

$$k(i,n), \qquad s(i,n), \quad \text{and} \quad t(i,n)$$

for $i, n \in \mathbb{N}$, $i \leqslant n$, satisfying the following conditions:

$$k(i,n) < s(i,n) \leqslant t(i,n) < \begin{cases} k(1,n+1), & i = n, \\ k(i+1,n), & i < n, \end{cases} \tag{21}$$

$$p_i\big(\big[s(i,n), t(i,n)\big]\big) \subset \big[k(i,n), k(i+1,n)\big) \quad \text{whenever } i < n, \quad \text{and}$$

$$p_n\big(\big[s(n,n), t(n,n)\big]\big) \subset \big[k(n,n), k(1,n+1)\big), \tag{22}$$

$$(-1)^n \sum_{r=s(i,n)}^{t(i,n)} a_{p_i(r)}^{(i)} \geqslant n, \tag{23}$$

$$\left| \sum_{r=u}^{v} a_r^{(i)} \right| \leqslant n^{-3} \quad \text{for any } u, v \in \mathbb{N} \text{ with } v \geqslant u \geqslant k(i,n). \tag{24}$$

We define the desired series $\sum b_r$ by, for each $i \in \mathbb{N}$, setting $b_r = a_r^{(i)}$ for every $r$ which belongs to the set

$$\big[k(i,i), k(1,i+1)\big) \cup \bigcup_{n=i+1}^{\infty} \big[k(i,n), k(i+1,n)\big).$$

Furthermore, we set $b_r = 0$ for all indices $r \in [1, k(1,1))$. By (21) this definition is consistent. By (24) we have

$$\left| \sum_{r=k(1,n)}^{k(1,n+1)-1} b_r \right| \leqslant n^{-2}$$

for every $n \in \mathbb{N}$. Hence $\sum b_r$ is convergent. On the other hand, by (22) we obtain

$$\sum_{r=s(i,n)}^{t(i,n)} b_{p_i(r)} = \sum_{r=s(i,n)}^{t(i,n)} a_{p_i(r)}^{(i)} \tag{25}$$

for any $i, n \in \mathbb{N}$, $i \leqslant n$, which by (23) implies that both $-\infty$ and $+\infty$ are limit points of the partial sums of each series $\sum_{r=1}^{\infty} b_{p_i(r)}$, $i \in \mathbb{N}$. Thus the set of limit points of the partial sums of each of these series is equal to $\mathbb{R}^*$. $\square$

**Theorem 5.2.** *Let $p \in \mathfrak{D}$ and let $\lim_{n\to\infty} t(p,n) = \infty$. Then for every $\alpha \in \mathbb{R}$ and for every $\beta \in \mathbb{R}^*$ there exists a conditionally convergent series $\sum a_n$ such that*

$$\sum a_n = \alpha \quad \textit{and} \quad \sigma a_{p(n)} = [\beta, +\infty].$$

**Proof.** Let $\beta \in \mathbb{R}_+$. For abbreviation, we will write $\mathbb{U}$ and $\mathbb{V}$ instead of $\mathbb{U}(p)$ and $\mathbb{V}(p)$, respectively. The increasing sequences of all elements of the sets $p(\mathbb{U})$ and $p(\mathbb{V})$ will be denoted by $\{u_n\}$ and $\{v_n\}$, respectively. The function $w : p(\mathbb{U}) \setminus \{u_1\} \to \mathbb{V}$ is defined by the relation $w(u_{n+1}) = p^{-1}(v_n)$ for every positive integer $n$.

Two auxiliary sequences: $\{r_n\colon\ n \in \mathbb{N}_0\}$ of positive integers and $\{y_n\colon\ n \in \mathbb{N}_0\}$ of positive reals will be then created as follows. First choose $r_0 \in \mathbb{U}$, $r_0 > p^{-1}(u_1)$ and set $y_0 = 0$,

$$\mu_n = \operatorname{card}\big([r_0, n] \cap \mathbb{U}\big) \quad \text{and} \quad \nu_n = \operatorname{card}\big([r_0, n] \cap \mathbb{V}\big)$$

for every $n \in \mathbb{N}$, $n \geqslant r_0$. Next assume that for some $k \in \mathbb{N}_0$ all elements $r_i$ and $y_i$, $0 \leqslant i \leqslant k$, have been defined and $y_i \leqslant 1/i$ for every $1 \leqslant i \leqslant k$. Moreover, suppose that the sequence $\{r_i\colon\ 0 \leqslant i \leqslant k\}$ is increasing,

$$r_i > \big\{w(u)\colon\ u \in p\big([r_0, r_{i-1}] \cap \mathbb{U}\big)\big\} \tag{26}$$

and

$$\mu_n - \nu_n \geqslant \beta(i+1), \tag{27}$$

$$y_i(\mu_n - \nu_n) > \max\{i, \beta + 1\}, \tag{28}$$

for every $i, n \in \mathbb{N}$, $i \leqslant k$ and $n \geqslant r_i$. Define

$$f(x, n) = y_k(\mu_{r_k} - \gamma_n) + x(\mu_n - \mu_{r_k} - \nu_n + \gamma_n)$$

and

$$\gamma_n = \operatorname{card}\big([1, n] \cap \big\{w(u)\colon\ u \in p\big([r_0, r_k] \cap \mathbb{U}\big)\big\}\big)$$

for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$. We note that by (28)

$$f(y_k, n) = y_k(\mu_n - \nu_n) > 1 + \beta \tag{29}$$

for every $n \in \mathbb{N}$, $n \geqslant r_k$, $f(0, n) \geqslant 0$ for every $n \in \mathbb{N}$ and $f(0, n) = 0$ for all sufficiently large $n \in \mathbb{N}$. Moreover, since $(\mu_n - \nu_n) \to \infty$ as $n \to \infty$, it follows that $x$ is arbitrarily small whenever $f(x, n) = \beta$ and $n$ is sufficiently large. Choose $s \in \mathbb{N}$, $s > r_k$ such that $\gamma_s = \mu_{r_k}$, $f(1/(k+1), s) \geqslant \beta$ and

$$\mu_n - \nu_n \geqslant \mu_s - \nu_s \quad \text{for every } n \in \mathbb{N},\ n \geqslant s. \tag{30}$$

Put

$$y_{k+1} = \max\big\{x \in \mathbb{R}_+\colon\ f(x, n) = \beta \text{ for some } r_k \leqslant n \leqslant s\big\}.$$

Clearly by (27) we have $y_{k+1} \leqslant 1/(k+1)$ and, by (29), (30), $f(y_{k+1}, n) \geqslant \beta$ for every $n \in \mathbb{N}$, $n \geqslant r_k$. We finish a step of induction by choosing an index $r_{k+1} > s$ such that

$$\mu_n - \nu_n > \beta(k+1)$$

and

$$y_{k+1}(\mu_n - \nu_n) > \max\{k+1, \beta + 1\}$$

for every $n \in \mathbb{N}$, $n \geqslant r_{k+1}$.

Set

$$a_n(\beta) = \begin{cases} y_{k+1}, & n \in p((r_k, r_{k+1}] \cap \mathbb{U}),\ k \in \mathbb{N}_0, \\ -y_{k+1}, & n \in \{p(w(u))\colon\ u \in p((r_k, r_{k+1}] \cap \mathbb{U})\},\ k \in \mathbb{N}_0, \end{cases}$$

and $a_n(\beta) = 0$ for the remaining indices $n \in \mathbb{N}$. Since $y_k \to 0$ as $k \to \infty$ we get $\sum a_n(\beta) = 0$. On the other hand, by (26), (28) and (29), for every positive integer $k$ we have

$$\sum_{i=1}^{n} a_{p(i)}(\beta) = f(y_{k+1}, n) \geqslant \begin{cases} \beta & \text{for every } n \in (r_k, r_{k+1}], \\ y_{k+1}(\mu_{r_{k+1}} - \nu_{r_{k+1}}) \geqslant k+1 & \text{for } n = r_{k+1}, \end{cases}$$

and, by the definition of $y_{k+1}$, there exists an index $n \in (r_k, r_{k+1}]$ such that $f(y_{k+1}, n) = \beta$. Hence $\sigma a_{p(n)}(\beta) = [\beta, +\infty]$.

Now let $\alpha, \gamma \in \mathbb{R}$, $\alpha < \gamma$. Consider the series

$$a_k = \begin{cases} \alpha & \text{for } k = u_1, \\ a_k(\gamma - \alpha) & \text{for } k \in \mathbb{N},\ k \neq u_1. \end{cases}$$

Then $\sum a_n = \alpha$ and $\sigma a_{p(n)} = [\gamma, +\infty]$. The cases $\gamma \leqslant \alpha$ and $\gamma = +\infty$ obey Theorem 3.1 and Theorem 4.4, respectively. This finishes the proof. $\square$

**Theorem 5.3.** *For each increasing sequence $\{x_n\}$ of positive integers and for each conditionally convergent series $\sum a_n$ there exists a divergent permutation $p$ satisfying the following conditions*:

(i) $\lim_{n\to\infty} t(p,n)=\infty$,
(ii) $p(\mathbb{U}(p))=\{x_{2n-1}\colon n\in\mathbb{N}\}$ *and* $p(\mathbb{V}(p))=\{x_{2n}\colon n\in\mathbb{N}\}$

*and*

(iii) *the series* $\sum a_{p(n)}$ *is convergent and* $\sum a_{p(n)}=\sum a_n$.

**Proof.** Fix positive integers $k_n\geqslant 2$, $n\in\mathbb{N}$. Let a subsequence $\mathbb{Y}=\{y_n\}$ of the sequence $\{x_{2n-1}\}$ be chosen so that $y_1\geqslant x_3$ and

$$\sum |a_{y_n}|<\infty. \tag{31}$$

Denote by $\{I_n\}\subset 2^{\mathbb{N}}$ the sequence of intervals given by the relations:

$$\bigcup_{n\in\mathbb{N}} I_n=\{i\in\mathbb{N}\colon i>x_1\},\quad I_n<I_{n+1},$$
$$\operatorname{card}(I_{2n})=k_n \quad\text{and}\quad \operatorname{card}(I_{2n-1})=x_{2n+1}-x_{2n-1}-\chi_{\mathbb{Y}}(x_{2n+1}),$$

for every $n\in\mathbb{N}$. Here $\chi_{\mathbb{Y}}:\mathbb{N}\to\{0,1\}$ denotes the characteristic function of $\mathbb{Y}$.

Now we are ready to define the desired permutation $p$. We will define the restrictions of $p$ to the intervals $I_n$, $n\in\mathbb{N}$, as follows. First we set $p(1)=x_1$ and $p(i)=x_1-i+1$ for $i=2,3,\dots,x_1$. Then for every positive integer $n$, the restriction $p|I_{2n}$ is defined to be the increasing mapping of the interval $I_{2n}$ onto the set $\{y_i\colon i=1+h_{n-1},2+h_{n-1},\dots,h_n\}$ where $h_0:=0$ and $h_n:=1+k_1+k_2+\cdots+k_n$.

If $x_{2n+1}\notin\mathbb{Y}$ then the restriction $p|I_{2n-1}$ is defined by requiring $p$ to be the increasing mapping of the interval $(\min I_{2n-1},\min I_{2n-1}+x_{2n}-x_{2n-1}-1]$ onto the interval $(x_{2n-1},x_{2n})$ and the decreasing mapping of the interval $[\min I_{2n-1}+x_{2n}-x_{2n-1},\max I_{2n-1})$ onto the interval $(x_{2n},x_{2n+1})$. Moreover, we set $p(\min I_{2n-1})=x_{2n+1}$ and $p(\max I_{2n-1})=x_{2n}$.

On the other hand, if $x_{2n+1}\in\mathbb{Y}$ then we define $p|I_{2n-1}$ to be the increasing mapping of the interval $[\min I_{2n-1},\min I_{2n-1}+x_{2n}-x_{2n-1}-1)$ onto the interval $(x_{2n-1},x_{2n})$ and the decreasing mapping of the interval $[\min I_{2n-1}+x_{2n}-x_{2n-1}-1,\max I_{2n-1})$ onto the interval $(x_{2n},x_{2n+1})$. Furthermore, we set $p(\max I_{2n-1})=x_{2n}$.

We remark that $t(p,i+1)=1+t(p,i)$ whenever both indices $i$ and $i+1$ belong to the same interval $I_{2n}$ for some $n\in\mathbb{N}$. Moreover, we have $t(p,i)\geqslant t(p,\max I_{2n})-1$ for any $i\in I_{2n+1}$ and $n\in\mathbb{N}$. From the inequality $k_n\geqslant 2$, $n\in\mathbb{N}$, it may be concluded that

$$t(p,\max I_{2n})\geqslant n\quad\text{for } n\in\mathbb{N}.$$

Hence we deduce that $\lim_{n\to\infty} t(p,n)=\infty$.

It is easy to show that

$$\left|\sum_{i=1}^{s} a_{p(i)}-\sum_{i=1}^{x_{2n+1}} a_i\right|\leqslant \sum_{y_i>x_{2n+1}} |a_{y_i}|$$

whenever $s\in I_{2n}$ and

$$\left|\sum_{i=1}^{s} a_{p(i)}-\sum_{i=1}^{x_{2n-1}} a_i\right|\leqslant \sum_{y_i>x_{2n-1}} |a_{y_i}|+2\max\left\{\left|\sum_{i=u}^{v} a_i\right|\colon u,v\in\mathbb{N},\ x_{2n-1}<u\leqslant v\right\}$$

when $s\in I_{2n-1}$. These two estimates, together with the assumption (31), show that the series $\sum a_{p(n)}$ is convergent to the sum of the series $\sum a_n$. The relations

$$p\big(\mathbb{U}(p)\big)=\{x_{2n-1}\colon n\in\mathbb{N}\}\quad\text{and}\quad p\big(\mathbb{V}(p)\big)=\{x_{2n}\colon n\in\mathbb{N}\}$$

follow directly from the definition of $p$, and the proof is complete. □

## 6. Final remarks

The form of the set of the limit points of the rearranged series $\sum_{n=1}^{\infty} a_{p(n)}$ if real series $\sum a_n=0$ and permutation $p$ of $\mathbb{N}$ are given, is determined by the so-called: "width $w(p)$ of $p$" (this concept was introducing by Nash-Williams and White in [9]); we note that $w(p)\in\mathbb{N}\cup\{0,\infty\}$. We set $w(p):=0$ if $p$ is a convergent permutation.

**Definition 1.** We use symbol $t(A;p)$, where $A\subset\mathbb{N}$ and $p$ is a permutation of $\mathbb{N}$, to denote the number of **MSI** which form the partition of the set $p(A)$.

**Definition 2.** Let $p$ be a divergent permutation of $\mathbb{N}$. If there exist $q \in \mathbb{N}$ and an increasing sequence $\{N_k\}_{k=1}^{\infty}$ of the finite subsets of $\mathbb{N}$ such that $N_k$ is a union of $q$ **MSI** and sequence $\{t(N_k; p)\}_{k=1}^{\infty}$ is bounded, then $w(p)$ is equal to the smallest $q \in \mathbb{N}$ with this property. In other cases, we set $w(p) = \infty$.

The following facts are fundamental to this theory (see [9,10]):

1° $w(p) \geqslant 2$ iff $\lim_{n\to\infty} t([1,n], p) = \lim_{n\to\infty} t(p,n) = \infty$;

2° if $w(p) = q \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ is a real series with the sum 0, then $\sigma a_{p(n)}$ is the closed interval of $\mathbb{R}^*$ that must contain set $\{(q-1)x, qx\}$ for some $x \in \mathbb{R}^*$. In the sequel, if $w(p) = 1$, then $\sigma a_{p(n)} = [x, y]$ for some $x, y \in \mathbb{R}^*$ with $x \leqslant 0 \leqslant y$. On the other hand, if $2 \leqslant w(p) = q < \infty$ then

$$\sigma a_{p(n)} \in \left\{ [q-1, q], \left[ -\frac{1}{q}, \frac{1}{1-q} \right], \text{every interval } [a, +\infty] \text{ and } [-\infty, a] \text{ with } a \in \mathbb{R}^*, \ldots \right\},$$

but $[q, q+\varepsilon] \neq \sigma a_{p(n)}$ for any $\varepsilon \in \mathbb{R}$, $0 \leqslant \varepsilon < \frac{q}{q-1}$;

3° $w(p) = \infty$ iff there exists a real convergent series $\sum a_n$ such that $p$-rearranged series $\sum a_{p(n)}$ is also convergent but to a different sum.

In my PhD dissertation [26] (see also [24, Th. 3.2]) it is proven that if $p$ is a divergent permutation and there exists a conditionally convergent series $\sum a_n$ such that series $\sum a_{p(n)}$ is also convergent and $\sum a_n \neq \sum a_{p(n)}$, then for every $\alpha \in \mathbb{R}$, and every nonempty closed interval $I \subset \mathbb{R}^*$ there exists a conditionally convergent series $\sum b_n$ such as:

$$\sum b_n = \alpha \quad \text{and} \quad \sigma b_{p(n)} = I.$$

This fact from Theorem 4.6 can be easily deduced.

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## References

[1] R.P. Agnew, Permutations preserving convergence of series, Proc. Amer. Math. Soc. 6 (1955) 563–564.
[2] F. Garibay, P. Greenberg, L. Resendis, J.J. Rivaud, The geometry of sum preserving permutations, Pacific J. Math. 135 (1988) 313–322.
[3] U.C. Guha, On Levi's theorem on rearrangement of convergent series, Indian J. Math. 9 (1967) 91–93.
[4] M.C. Hu, J.K. Wang, On rearrangement of series, Bull. Inst. Math. Acad. Sin. (N.S.) 7 (1979) 363–376.
[5] B. Jasek, Über umordnung von reihen, Colloq. Math. 7 (1960) 257–259.
[6] B. Jasek, Complex series and connected sets, Dissertationes Math. (Rozprawy Mat.) 52 (1966) 47.
[7] A.S. Kronrod, On rearrangements of real and complex series, Mat. Sb. 18 (1946) 237–277.
[8] F.W. Levi, Rearrangement of convergent series, Duke Math. J. 13 (1946) 579–585.
[9] C.St.J.A. Nash-Williams, D.J. White, An application of network flows to rearrangement of series, J. Lond. Math. Soc. (2) 59 (1999) 637–646.
[10] C.St.J.A. Nash-Williams, D.J. White, Rearrangement of vector series. I, Math. Proc. Cambridge Philos. Soc. 130 (2001) 89–109.
[11] C.St.J.A. Nash-Williams, D.J. White, Rearrangement of vector series. II, Math. Proc. Cambridge Philos. Soc. 130 (2001) 111–134.
[12] P.A.B. Pleasants, Rearrangements that preserve convergence, J. Lond. Math. Soc. (2) 15 (1977) 134–142.
[13] P.A.B. Pleasants, Addendum: Rearrangements that preserve convergence, J. Lond. Math. Soc. (2) 18 (1978) 576.
[14] M.A. Sarigöl, Permutation preserving convergence and divergence of series, Bull. Inst. Math. Acad. Sin. (N.S.) 16 (1988) 221–227.
[15] M.A. Sarigöl, A short proof of Levi's theorem on rearrangements of convergent series, Doga-Tr. J. Math. 16 (1992) 201–205.
[16] P. Schaefer, Sum-preserving rearrangements of infinite series, Amer. Math. Monthly 88 (1981) 33–40.
[17] A. Sowa, Niektóre własności zbioru punktów osiągalnych w przestrzeni $\mathbb{R}^3$ za pomocą mnożników $+1$ i $-1$, Zesz. Nauk. Politech. Śl. Mat.-Fiz. 52 (1990) 129–136 (in Polish).
[18] G.S. Stoller, The convergence-preserving rearrangements of real infinite series, Pacific J. Math. 73 (1977) 227–231.
[19] J. Timmler, On presenting series in $E^n$ spaces, Zesz. Nauk. Politech. Śl. Mat.-Fiz. 30 (1979) 289–293.
[20] J. Timmler, The proof for certain theorem given by Dvoretzky and Hanani, Zesz. Nauk. Politech. Śl. Mat.-Fiz. 30 (1979) 295–300.
[21] J. Timmler, Uzbieżnianie szeregów wektorowych w przestrzeniach $\mathbb{R}^n$ mnożnikami $+1$ lub $-1$, PhD thesis, Gliwice, 1980 (in Polish).
[22] J. Timmler, On a property of vector series in $\mathbb{R}^3$ space, Zesz. Nauk. Politech. Śl. Mat.-Fiz. 48 (1986) 203–216.
[23] R. Wituła, A new proof of a theorem on convergence in $\mathbb{R}^n$, Zesz. Nauk. Politech. Śl. Mat.-Fiz. 52 (1990) 137–142.
[24] R. Wituła, The Riemann theorem and divergent permutations, Colloq. Math. 69 (1995) 275–287.
[25] R. Wituła, Convergence-preserving functions, Nieuw Arch. Wiskd. 13 (1995) 31–35.
[26] R. Wituła, On convergent and divergent permutations, PhD thesis, Katowice, 1997 (in Polish).
[27] R. Wituła, M. Przybyła, The strongly and weakly divergent permutations, Demonstratio Math. 39 (2006) 107–116.
[28] R. Wituła, D. Słota, R. Seweryn, On Erdös' theorem for monotonic subsequences, Demonstratio Math. 40 (2007) 239–259.
[29] R. Wituła, D. Słota, The convergence classes of divergent permutations, Demonstratio Math. 43 (2010).