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Multiple periodic solutions of the second order Hamiltonian systems with superlinear terms ☆

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ABSTRACT

We study the existence of 2π-periodic solutions of the second order Hamiltonian systems $-\ddot{x}-A(t)x=\lambda x+V_x'(t,x)$ with superlinear terms and with saddle structure near the origin. Some multiplicity results are obtained by using bifurcation method, homological linking and Morse theory.

1. Introduction

In this paper we are concerned with the existence of 2π-periodic solutions of the second order Hamiltonian systems of the form

$$-\ddot{x}-A(t)x=\lambda x+V_x'(t,x) \tag{HS$_\lambda$}$$

where $\lambda\in\mathbb{R}$ is a parameter, $A(t)$ is a continuous symmetric matrix function in $\mathbb{R}^N$ and 2π-periodic in t, and the potential V satisfies the following conditions

(V_1) $V\in C^2(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ is 2π-periodic in t.
(V_2) $V(t,0)=0$, $V_x'(t,0)=0$, $V_x''(t,0)=0$.
(V_3) There exist $\bar{r}>0$ and $\theta>2$ such that

$$0<\theta V(t,x)\leqslant\big(V_x'(t,x),x\big),\quad \text{for } |x|\geqslant\bar{r},\ t\in[0,2\pi].$$

(V_4) $V_x''(t,x)>0$ for $|x|>0$ small and $t\in[0,2\pi]$.
(V_5) $V_x''(t,x)<0$ for $|x|>0$ small and $t\in[0,2\pi]$.

Here and in the sequel, $|\cdot|$ and $(\cdot,\cdot)$ denote the norm and the inner product in $\mathbb{R}^N$, Bx denotes the matrix product in $\mathbb{R}^N$ for an $N\times N$ matrix B and $x\in\mathbb{R}^N$. For two symmetric matrices B and C in $\mathbb{R}^N$, $B>C$ means that $B-C$ is positive definite. We always use 0 to denote the origin in various spaces.

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Due to $V'_x(t,0)=0$, $x\equiv 0$ is a trivial solution of $(\mathrm{HS})_\lambda$ for any parameter $\lambda\in\mathbb{R}$. Our interest is the multiplicity of nontrivial 2π-periodic solutions of $(\mathrm{HS})_\lambda$ for certain range of the parameter. Let $\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots$ be the distinct eigenvalues of the linear Hamiltonian systems

$$\begin{cases} -\ddot{x}-A(t)x=\lambda x, \\ x(0)=x(2\pi), \quad \dot{x}(0)=\dot{x}(2\pi). \end{cases} \tag{LHS}$$

We make a convention $\lambda_0=-\infty$ for the use of convenience below. Denote $S^1=\mathbb{R}/(2\pi\mathbb{Z})$, $V^-(t,x)=\max\{-V(t,x),0\}$. Our results are the following theorems.

Theorem 1.1. *Assume that V satisfies (V_1)–(V_4). Let $k\geqslant 1$ be fixed. Then there is $\delta>0$, such that when $\sup_{(t,x)\in[0,2\pi]\times\mathbb{R}^N}V^-(t,x)\leqslant\delta$, for $\lambda\in(\lambda_k-\delta,\lambda_k)$, $(\mathrm{HS})_\lambda$ has at least three nontrivial 2π-periodic solutions.*

Theorem 1.2. *Assume that V satisfies (V_1)–(V_3) and (V_5). Let $k\geqslant 1$ be fixed. Then there is $\delta>0$, such that when $\sup_{(t,x)\in[0,2\pi]\times\mathbb{R}^N}V^-(t,x)\leqslant\delta$, for $\lambda\in(\lambda_k,\lambda_k+\delta)$, $(\mathrm{HS})_\lambda$ has at least three nontrivial 2π-periodic solutions.*

Theorem 1.3. *Assume that V satisfies (V_1)–(V_3) and $V\leqslant 0$ for $|x|>0$ small, $t\in[0,2\pi]$. Let $k\geqslant 1$ be fixed. Then there is $\delta>0$, such that when $\sup_{(t,x)\in[0,2\pi]\times\mathbb{R}^N}V^-(t,x)\leqslant\delta$, for $\lambda\in(\lambda_k-\delta,\lambda_k]$, $(\mathrm{HS})_\lambda$ has at least two nontrivial 2π-periodic solutions.*

We now give some comments and comparisons. In a remarkable paper [10] of Rabinowitz, $(\mathrm{HS})_0$ was studied by applying a critical point theorem, which is now well known as the generalized mountain pass theorem, built by Rabinowitz in [9] for the case that $A\equiv 0$ and one nonconstant periodic solution was obtained when the potential V was of class C^1 and satisfied (V_2), (V_3) and global sign condition $V\geqslant 0$. In [4] the author extended the existence result in [10] by studying $(\mathrm{HS})_0$ with A being a constant symmetric matrix via local linking [5] argument and one nontrivial periodic solution was obtained when the potential V was of class C^1 and satisfied (V_2), (V_3) and local sign conditions (see [4]) near the origin. Motivated by a recent work of Rabinowitz, Su and Wang [12], in the current paper, we obtain multiplicity results of $(\mathrm{HS})_\lambda$ when λ is very close to any a fixed eigenvalue of (LHS). These results are new, since, to the best of our knowledge, there are less multiplicity results for Hamiltonian systems with superlinear terms in the literature if the even assumption on V was absent. These results are valid for A being constant even zero. We emphasize here that the origin $x=0$ acts as a local saddle point of the energy functional Φ of $(\mathrm{HS})_\lambda$ defined below when $\lambda\geqslant\lambda_1$. When λ is close to λ_1 from the left side, $x=0$ acts as a local minimizer of Φ.

The approach to the existence results is variational which means that we look for critical points of the energy functional of $(\mathrm{HS})_\lambda$:

$$\Phi(x)=\frac{1}{2}\int_0^{2\pi}|\dot{x}|^2-\big(A(t)x,x\big)-\lambda|x|^2\,dt-\int_0^{2\pi}V(t,x)\,dt$$

which is defined on the Hilbert space

$$E:=H^1\big(S^1;\mathbb{R}^N\big)=\left\{x\colon S^1\to\mathbb{R}^N \;\Big|\; \int_0^{2\pi}|\dot{x}|^2+|x|^2\,dt<\infty\right\}$$

with the inner product

$$\langle x,y\rangle=\int_0^{2\pi}\big(\dot{x}(t),\dot{y}(t)\big)+\big(x(t),y(t)\big)\,dt$$

and the corresponding norm $\|x\|^2=\langle x,x\rangle$. It follows from (V_1) and the compact embedding $E\hookrightarrow C([0,2\pi],\mathbb{R}^N)$ that $\Phi\in C^2(E,\mathbb{R})$ with derivatives given by, for $x,y,z\in E$,

$$\big\langle\Phi'(x),y\big\rangle=\int_0^{2\pi}(\dot{x},\dot{y})-\big(A(t)x,y\big)-\lambda(x,y)\,dt-\int_0^{2\pi}\big(V'_x(t,x),y\big)\,dt,$$

$$\big\langle\Phi''(x)y,z\big\rangle=\int_0^{2\pi}(\dot{y},\dot{z})-\big(A(t)y,z\big)-\lambda(y,z)\,dt-\int_0^{2\pi}\big(V''_x(t,x)y,z\big)\,dt.$$

We will follow the ideas in [12] by combining Morse theory, topological linking and bifurcation arguments to prove these theorems. In Section 2, we prove two solutions for $(\mathrm{HS})_\lambda$ by bifurcation methods in [11] and then give the estimation of the

Morse index of these solutions. In Section 3 we get a solution by linking argument and give partial estimates of the Morse index via homological linking idea. The proofs of Theorems 1.1–1.3 are finished in Section 4.

2. Solutions near zero with their Morse indices

In this section we get two solutions for $(\mathrm{HS})_\lambda$ via bifurcation arguments [11]. We first cite a bifurcation theorem in [11].

Proposition 2.1. *(See Theorem 11.35 in [11].) Let E be a Hilbert space and $\Psi \in C^2(E,\mathbb{R})$ with*

$$\nabla\Psi(u) = Lu + H(u)$$

where $L \in \mathcal{L}(E,E)$ is symmetric and $H(u) = o(\|u\|)$ as $\|u\| \to 0$. Consider the equation

$$Lu + H(u) = \lambda u. \tag{2.1}$$

Let $\mu \in \sigma(L)$ be an isolated eigenvalue of finite multiplicity. Then either

(i) *$(\mu, 0)$ is not an isolated solution of (2.1) in $\{\mu\} \times E$, or*
(ii) *there is an one-sided neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, (2.1) has at least two distinct nontrivial solutions, or*
(iii) *there is a neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, (2.1) has at least one nontrivial solution.*

We now apply Proposition 2.1 to get two nontrivial 2π-periodic solutions of $(\mathrm{HS})_\lambda$ for λ close to an eigenvalue of (LHS) and then give the estimates of the Morse index. We use some notations. For $j = 1, 2, 3, \dots$, denote

$$E(\lambda_j) = \ker\left(\frac{d^2}{dt^2} + A(t) + \lambda_j\right), \qquad E_j = \bigoplus_{i=1}^{j} E(\lambda_i), \qquad E_j^{\perp} = \left\{x \in E \;\middle|\; \langle x, y\rangle = 0,\ y \in E_j\right\}.$$

Note that $\dim E_0 = 0$. Set $\nu_j = \dim E(\lambda_j)$, $\ell_j = \dim E_j = \sum_{i=1}^{j} \nu_i$. For a critical point x of a functional $\Phi \in C^2(E,\mathbb{R})$, we denote by $\mu(x)$ and $\nu(x)$ the Morse index and nullity of Φ at $x \in E$.

Theorem 2.2. *Assume that V satisfies (V_1) and (V_2). Let $k \geqslant 1$ be fixed. Then there exists $\delta > 0$, such that $(\mathrm{HS})_\lambda$ has at least two nontrivial 2π-periodic solutions for*

(1) *every $\lambda \in (\lambda_k - \delta, \lambda_k)$ if $V_x''(t,x) > 0$ for $|x| > 0$ small, $t \in [0, 2\pi]$;*
(2) *every $\lambda \in (\lambda_k, \lambda_k + \delta)$ if $V_x''(t,x) < 0$ for $|x| > 0$ small, $t \in [0, 2\pi]$.*

Furthermore, the Morse index and nullity of such a solution x_λ satisfy

$$\ell_{k-1} \leqslant \mu(x_\lambda) \leqslant \mu(x_\lambda) + \nu(x_\lambda) \leqslant \ell_k, \quad \text{for } 0 < |\lambda - \lambda_k| < \delta. \tag{2.2}$$

Proof. We first prove the existence results by verifying that case (ii) of Proposition 2.1 occurs under the given conditions. Under (V_1) and (V_2), every eigenvalue λ_j of (LHS) gives rise to a bifurcation point $(\lambda_j, 0)$ of $(\mathrm{HS})_\lambda$ (see [11]). Let $(\lambda, x) \in \mathbb{R} \times E$ be a solution of $(\mathrm{HS})_\lambda$ near $(\lambda_k, 0)$ which satisfies

$$\begin{cases} -\ddot{x} - A(t)x = \lambda x + V_x'(t,x), \\ x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi). \end{cases} \tag{2.3}$$

By (V_1) and (V_2), we have

$$V_x'(t,x) = V_x'(t,0) + V_x''(t,\eta x)x = V_x''(t,\eta x)x, \quad \text{for some } 0 < \eta < 1. \tag{2.4}$$

We consider the case (1). By the embedding $E \hookrightarrow C([0,2\pi],\mathbb{R}^N)$, for $\|x\| > 0$ small, $\|x\|_C > 0$ small. Then by the assumption (V_4), we have that

$$V_x''\big(t, \eta x(t)\big) > 0, \quad t \in [0, 2\pi].$$

It follows that

$$A(t) + V_x''\big(t, \eta x(t)\big) > A(t), \quad t \in [0, 2\pi].$$

Now consider the linear Hamiltonian systems

$$\begin{cases} -\ddot{y} - \big(A(t) + V_x''\big(t, \eta x(t)\big)\big)y = \mu y, \\ y(0) = y(2\pi), \quad \dot{y}(0) = \dot{y}(2\pi). \end{cases} \tag{2.5}$$

We denote the distinct eigenvalues of (2.5) by $\mu_1(x) < \mu_2(x) < \cdots < \mu_i(x) < \cdots$ as $x \neq 0$. Notice that by (V_2), if we take $x = 0$ then for each $i \in \mathbb{N}$, there is $j \in \mathbb{N}$ such that $\mu_i(0) = \lambda_j$. Thus the standard variational characterization of the eigenvalues of (2.5) shows that $\mu_i(x)$ is less than the corresponding j-th ordered eigenvalue λ_j of (LHS), and furthermore, $\mu_i(x) \to \lambda_j$ as $x \to 0$ in E. By (2.3) and (2.4), we see that x is a solution of (2.5) with eigenvalue λ. It follows that $\lambda < \lambda_k$ since λ is close to λ_k. This proves the case (1). The existence for case (2) is proved in a similar way.

Now we estimate of the Morse indices for the solutions obtained above. Let x_λ be a bifurcation solution of $(\mathrm{HS})_\lambda$. Then

$$\|x_\lambda\| \to 0, \quad \text{as } \lambda \to \lambda_k.$$

By the embedding $E \hookrightarrow C([0,2\pi], \mathbb{R}^N)$, we have

$$\|x_\lambda\|_C \to 0, \quad \lambda \to \lambda_k. \tag{2.6}$$

For each $y \in E$, we have

$$\big\langle \Phi''(x_\lambda)y, y\big\rangle = \int_0^{2\pi} |\dot{y}|^2 - \big(A(t)y, y\big) - \lambda|y|^2\, dt - \int_0^{2\pi} \big(V''_x(t, x_\lambda)y, y\big)\, dt.$$

Therefore, for $y \in E_{k-1}$,

$$\big\langle \Phi''(x_\lambda)y, y\big\rangle \leqslant (\lambda_{k-1} - \lambda) \int_0^{2\pi} |y|^2\, dt + \int_0^{2\pi} \big|\big(V''_x(t, x_\lambda)y, y\big)\big|\, dt,$$

and for $z \in E_k^\perp$,

$$\big\langle \Phi''(x_\lambda)z, z\big\rangle \geqslant (\lambda_{k+1} - \lambda) \int_0^{2\pi} |z|^2\, dt - \int_0^{2\pi} \big|\big(V''_x(t, x_\lambda)z, z\big)\big|\, dt.$$

By (V_2) and (2.6), we see that there is $\delta > 0$ such that when $0 < |\lambda - \lambda_k| < \delta$,

$$\big\langle \Phi''(x_\lambda)y, y\big\rangle < 0, \quad 0 \neq y \in E_{k-1}, \qquad \big\langle \Phi''(x_\lambda)z, z\big\rangle > 0, \quad 0 \neq z \in E_k^\perp.$$

Thus (2.2) holds. The proof is complete. □

For the later use we now give the computations of critical groups of Φ at zero. Recall that the q-th critical group of Φ at its isolated critical point x is defined as $C_q(\Phi, x) := H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{x\})$ (see [2,8]). Here $c = \Phi(x)$ and $H_q(A, B)$ is the q-th relative singular homological group of the topological pair (A, B) with the coefficients in a field $\mathbb{F}$.

Remark 2.3. Due to the compactness of the embedding $E \hookrightarrow C([0,2\pi], \mathbb{R}^N)$, the functional Φ satisfies the bounded (PS) condition, i.e. any bounded sequence $\{x_n\}$ in E with $\Phi'(x_n) \to 0$ as $n \to \infty$ has a convergent subsequence (see [8]).

Indeed, for such a sequence $\{x_n\}$ in E, up to a subsequence if necessary, we can assume that $x_n \rightharpoonup x$ in E and $x_n \to x$ in $C([0,2\pi], \mathbb{R}^N)$, and furthermore, $x_n \to x$ in $L^2([0,2\pi], \mathbb{R}^N)$. Since

$$\begin{aligned}\big\langle \Phi'(x_n) - \Phi'(x), x_n - x\big\rangle &= \int_0^{2\pi} |\dot{x_n} - \dot{x}|^2\, dt - \int_0^{2\pi} \big(\big(A(t) + \lambda\big)(x_n - x), x_n - x\big)\, dt \\ &\quad - \int_0^{2\pi} \big(V'_x(t, x_n) - V'_x(t, x), x_n - x\big)\, dt,\end{aligned}$$

and as $n \to \infty$,

$$\big\langle \Phi'(x_n) - \Phi'(x), x_n - x\big\rangle \to 0,$$

$$\left|\int_0^{2\pi} \big(V'_x(t, x_n), x_n - x\big)\, dt\right| \leqslant \big\|V'_x(t, x_n)\big\|_2 \|x_n - x\|_2 \to 0,$$

$$\int_0^{2\pi} \big(V_x'(t,x), x_n - x\big)\,dt \to 0,$$

$$\int_0^{2\pi} \big(\big(A(t)+\lambda\big)(x_n - x), x_n - x\big)\,dt \to 0,$$

we see that

$$\int_0^{2\pi} |\dot{x}_n - \dot{x}|^2\,dt \to 0, \quad n \to \infty$$

and then $x_n \to x$ in E.

It suffices to use the above local compactness in describing the critical groups of Φ at an isolated critical point [2,8]. We have

Proposition 2.4. *Assume that V satisfies (V_1) and (V_2), $k \geqslant 1$. Then for $\lambda \in (\lambda_{k-1}, \lambda_k)$, $C_q(\Phi, 0) \cong \delta_{q,\ell_{k-1}}\mathbb{F}$. For $\lambda = \lambda_k$, if the trivial solution $x = 0$ of $(\mathrm{HS})_\lambda$ is isolated, then*

(1) *$C_q(\Phi, 0) \cong \delta_{q,\ell_{k-1}}\mathbb{F}$ provided $V(t,x) \leqslant 0$ for $|x| > 0$ small;*
(2) *$C_q(\Phi, 0) \cong \delta_{q,\ell_k}\mathbb{F}$ provided $V(t,x) \geqslant 0$ for $|x| > 0$ small.*

Proof. For $\lambda \in (\lambda_{k-1}, \lambda_k)$, 0 is a non-degenerate critical point of Φ with the Morse index $\mu(0) = \ell_{k-1}$. It follows that $C_q(\Phi, 0) \cong \delta_{q,\ell_{k-1}}\mathbb{F}$ (see [2,8]).

Let $\lambda = \lambda_k$. Then 0 is a degenerate critical point of Φ with the Morse index $\mu(0) = \ell_{k-1}$ and nullity $\nu(0) = \nu_k$. We prove the case (1). For $k > 1$, we want to verify that the functional Φ has a local linking structure [5,7] at 0 with respect to the direct sum decomposition $E = E_{k-1} \oplus E_{k-1}^\perp$ when V satisfies $V(t,x) \leqslant 0$ for $|x| > 0$ small. That is for some $r > 0$,

$$\Phi(u) > 0, \quad \text{for } u \in E_{k-1}^\perp,\ 0 < \|u\| \leqslant r, \qquad \Phi(u) \leqslant 0, \quad \text{for } u \in E_{k-1},\ \|u\| \leqslant r.$$

By the continuity of embedding $E \hookrightarrow C([0,2\pi], \mathbb{R}^N)$, when $x \in E_{k-1}$ and $\|x\| \leqslant r$ with $r > 0$ small, $\|x\|_C$ must be small, therefore by (V_2) we have

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\int_0^{2\pi} \big(|\dot{x}|^2 - \big(A(t)x, x\big) - \lambda_k |x|^2\big)\,dt - \int_0^{2\pi} V(t,x)\,dt \\ &\leqslant \frac{1}{2}(\lambda_{k-1} - \lambda_k)\|x\|_2^2 + o\big(\|x\|_2^2\big) \leqslant 0.\end{aligned}$$

For $x \in E_{k-1}^\perp$, we write $x = y + z$ where $y \in E(\lambda_k)$, $z \in E_k^\perp$. Then

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\int_0^{2\pi} \big(|\dot{x}|^2 - \big(A(t)x, x\big) - \lambda_k |x|^2\big)\,dt - \int_0^{2\pi} V(t,x)\,dt \\ &\geqslant \frac{1}{2}(\lambda_{k+1} - \lambda_k)\int_0^{2\pi} |z|^2\,dt - \int_0^{2\pi} V(t,x)\,dt.\end{aligned}$$

For $y \in E(\lambda_k)$ and $0 < \|y\| \leqslant r$ with $r > 0$ small, we must have $\Phi(y) = -\int_0^{2\pi} V(t,y)\,dt > 0$ as otherwise we would have that $V(t, y(t)) \equiv 0$ for all $t \in [0, 2\pi]$ and then 0 is not isolated. Hence we get the conclusion that $\Phi(x) > 0$ for $0 < \|x\| \leqslant r$ with $r > 0$ small. We apply Proposition 2.2 in [13] to get the result of $C_q(\Phi, 0)$.

When $k = 1$, $\ell_{k-1} = 0$, the above arguments show that 0 is a local minimizer of Φ and then $C_q(\Phi, 0) \cong \delta_{q,0}\mathbb{F}$.

In the case (2), a similar way shows that Φ has a local linking structure at 0 with respect to the decomposition $E := E_k \oplus E_k^\perp$ and then Proposition 2.2 in [13] is applied to get the result of $C_q(\Phi, 0)$. The proof is complete. □

3. Linking solutions with homological information

In this section we get one nontrivial 2π-periodic solution for $(\mathrm{HS})_\lambda$ by using the generalized linking arguments [11] and then give the homological information via the homological linking [7]. We recall the abstract generalized linking theorem which follows from [11,7,2].

Proposition 3.1. *(See [11,7,2].) Let E be a real Banach space with $E = X \oplus Y$ and $\ell = \dim X$ is finite. Suppose that $\Phi \in C^1(E, \mathbb{R})$, satisfies (PS) and*

(Φ_1) *there exist $\rho > 0$, $\alpha > 0$ such that*

$$\Phi(u) \geqslant \alpha, \quad u \in S_\rho = Y \cap \partial B_\rho, \tag{3.1}$$

where $B_\rho = \{u \in E \mid \|u\| \leqslant \rho\}$;

(Φ_2) *there exist $R > \rho > 0$, and $e \in Y$ with $\|e\| = 1$ such that*

$$\Phi(u) < \alpha, \quad u \in \partial Q, \tag{3.2}$$

where $Q = \{u = v + se \mid \|u\| \leqslant R,\ v \in X,\ 0 \leqslant s \leqslant R\}$.

Then Φ has a critical point u_* with $\Phi(u_*) = c_* \geqslant \alpha$ and

$$C_{\ell+1}\big(\Phi, u^*\big) \neq 0. \tag{3.3}$$

We note here that under the framework of Proposition 3.1, S_ρ and ∂Q homotopically link with respect to the direct sum decomposition $E = X \oplus Y$. S_ρ and ∂Q also homologically linking [2,7]. The conclusion (3.3) follows from Theorems 1.1′ and 1.5 of Chapter II in [2] (see also [7]) and c_* can be characterized as $c_* := \inf_{\gamma \in \Gamma} \sup_{x \in |\gamma|} \Phi(x)$ where $\Gamma = \{\gamma \mid \text{singular } \ell + 1 \text{ chains with } \partial\gamma = \partial Q\}$.

We will apply the above abstract result to get a nontrivial solutions for $(\mathrm{HS})_\lambda$. We first verify (PS).

Lemma 3.2. *Assume that V satisfies (V_1) and (V_3), then for any fixed $\lambda \in \mathbb{R}$, the functional Φ satisfies the (PS) condition.*

Proof. According to Remark 2.3, we only need to show that any sequence $\{x_n\} \subset E$ with

$$\big|\Phi(x_n)\big| \leqslant C, \quad n \in \mathbb{N}, \qquad \Phi'(x_n) \to 0, \quad n \to \infty \tag{3.4}$$

is bounded, here and below we use C to denote various positive constants. The argument is standard (see [9,4]). Choosing a positive number $\beta \in (\theta^{-1}, 2^{-1})$. We have for n large that

$$C + \beta\|x_n\| \geqslant \Phi(x_n) - \beta\big\langle \Phi'(x_n), x_n\big\rangle.$$

According to (V_3),

$$V(t, x) \geqslant C|x|^\theta - C, \quad x \in \mathbb{R}^N. \tag{3.5}$$

Therefore,

$$C + \beta\|x_n\| \geqslant \left(\frac{1}{2} - \beta\right)\|\dot{x}_n\|_2^2 - \left(\frac{1}{2} - \beta\right)\big(\Lambda + |\lambda|\big)\|x_n\|_2^2 + (\theta\beta - 1)C\|x_n\|_\theta^\theta - C.$$

Where $\Lambda = \max_{t \in [0, 2\pi]} \|A(t)\|_{\mathbb{R}^N}$. By the Hölder inequality and the Young inequality we get for any $\epsilon > 0$ that

$$\|x_n\|_2^2 \leqslant (2\pi)^{\frac{\theta-2}{\theta}}\|x_n\|_\theta^2 \leqslant \frac{2\pi(\theta - 2)}{\theta}\epsilon^{\frac{2}{2-\theta}} + \frac{2}{\theta}\epsilon\|x_n\|_\theta^\theta. \tag{3.6}$$

Thus for a fixed $\epsilon > 0$ small enough, we have by (3.6) that

$$\begin{aligned} C + \beta\|x_n\| &\geqslant \left(\frac{1}{2} - \beta\right)\|\dot{x}_n\|_2^2 - C\epsilon\|x_n\|_\theta^\theta + C(\theta\beta - 1)\|x_n\|_\theta^\theta \\ &\geqslant \left(\frac{1}{2} - \beta\right)\|\dot{x}_n\|_2^2 + \frac{1}{2}(\theta\beta - 1)C\|x_n\|_\theta^\theta \\ &\geqslant C\big(\|\dot{x}_n\|_2^2 + \|x_n\|_2^2\big) = C\|x_n\|^2. \end{aligned}$$

Therefore $\{x_n\}$ is bounded in E. The proof is complete. □

From the above arguments and the continuous embedding $E \hookrightarrow C([0,2\pi],\mathbb{R}^N)$, we see that the set of critical points of Φ is uniformly bounded in E. Therefore we may assume that the potential V satisfies the growth condition

$$\bigl|V(t,x)\bigr| \leqslant C\bigl(1+|x|^q\bigr), \quad x \in \mathbb{R}^N \tag{V}$$

for any fixed number $q > \theta$. Otherwise, we can modify V in a similar way as in [11] to a new function satisfying the required growth condition and then work on the problem with the modified potential (see [11] for details). We will use (V) directly in the following estimates.

Lemma 3.3. *Assume that V satisfies (V_1), (V_2) and (V_3) and $k \geqslant 1$. Then there exist constants $\rho > 0$, $\alpha > 0$ such that for all $\lambda \leqslant \frac{\lambda_k+\lambda_{k+1}}{2}$, such that*

$$\Phi(x) \geqslant \alpha, \quad \textit{for } x \in E_k^{\perp} \textit{ with } \|x\| = \rho. \tag{3.7}$$

Proof. By (V_1), (V_2) and (V), for $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$V(t,x) \leqslant \frac{1}{2}\epsilon|x|^2 + C_\epsilon|x|^q.$$

For $\lambda \in \mathbb{R}$, the operator K_λ defined, using the Riesz representation theorem, by

$$\langle K_\lambda x, y\rangle = \int_0^{2\pi} (x,y) + \bigl(\bigl(A(t)+\lambda\,\mathrm{id}\bigr)x, y\bigr)\,dt, \quad x, y \in E$$

is compact. For $\lambda_* = \frac{\lambda_k+\lambda_{k+1}}{2}$, there is $\eta > 0$ such that

$$\Psi_{\lambda_*}(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \bigl(\bigl(A(t)+\lambda_*\,\mathrm{id}\bigr)x, x\bigr)\,dt \geqslant \frac{1}{2}\eta\|x\|^2, \quad x \in E_{\lambda_*}^+,$$

where E_λ^+ is the positively definite invariant subspace of $\mathrm{id} - K_\lambda$ (see [8]). Since for all $\lambda \leqslant \lambda_*$, $E_k^\perp \subset E_{\lambda_*}^+ \subset E_\lambda^+$, we have that

$$\Psi_\lambda(x) = \frac{1}{2}\int_0^{2\pi} |\dot{x}|^2 - \bigl(\bigl(A(t)+\lambda\,\mathrm{id}\bigr)x, x\bigr)\,dt \geqslant \frac{1}{2}\eta\|x\|^2, \quad x \in E_k^\perp$$

and furthermore the constant η is independent of $\lambda \leqslant \lambda_*$. Now for $x \in E_k^\perp$, we have

$$\begin{aligned}
\Phi(x) &= \frac{1}{2}\int_0^{2\pi} \bigl(|\dot{x}|^2 - \bigl(A(t)x, x\bigr) - \lambda|x|^2\bigr)\,dt - \int_0^{2\pi} V(t,x)\,dt \\
&\geqslant \frac{1}{2}\eta\|x\|^2 - \frac{1}{2}\epsilon\int_0^{2\pi} |x|^2\,dt - C_\epsilon\int_0^{2\pi} |x|^q\,dt \\
&\geqslant \frac{1}{2}(\eta-\epsilon)\|x\|^2 - \widehat{C}_\epsilon\|x\|^q \\
&\geqslant \frac{1}{4}\eta\|x\|^2 - C_*\|x\|^q,
\end{aligned}$$

when we take $\epsilon = \eta/2$, where C_* is independent of λ. Since $q > 2$ and the function $g(r) = \frac{1}{4}\eta r^2 - C_* r^q$ achieves its maximum

$$g_{\max} = \frac{q-2}{2q}\bigl(2^{-1}\eta\bigr)^{\frac{q}{q-2}}(qC_*)^{\frac{2}{2-q}} := \alpha$$

on $(0,\infty)$ at $\rho = (\frac{\eta}{2qC_*})^{\frac{1}{q-2}}$, we see that

$$\Phi(x) \geqslant \alpha, \quad \text{for } x \in E_k^\perp \text{ with } \|x\| = \rho.$$

The constant α is independent of $\lambda \leqslant \lambda_*$. The proof is complete. □

Lemma 3.4. *Assume that V satisfies (V_1), (V_3) and $k \geqslant 1$. Then there exist $R > 0$, $\delta > 0$ and $\sigma \in \mathbb{R}$, all independent of λ, such that when $\lambda \in (\lambda_k - \delta, \lambda_k + \delta)$ and $\sup_{(t,x)\in[0,2\pi]\times\mathbb{R}^N} V^-(t,x) \leqslant \delta$,*

$$\Phi(x) \leqslant \sigma < \alpha, \quad \text{for } x \in \partial Q, \tag{3.8}$$

where $Q = \{x \in E_k \oplus \operatorname{span}\{\phi_{k+1}\} \mid \|x\| \leqslant R,\ x = y + s\phi_{k+1},\ y \in E_k,\ s \geqslant 0\}$ and ϕ_{k+1} is a normalized eigenfunction of (LHS) *corresponding to λ_{k+1}.*

Proof. From (V_3) we deduce (3.5) with the positive constant C independent of λ. For $x \in V_k := E_k \oplus \operatorname{span}\{\phi_{k+1}\}$, write $x = y + z$, $y \in E_{k-1}$, $z \in E(\lambda_k) \oplus \operatorname{span}\{\varphi_{k+1}\}$. Then

$$\Phi(x) \leqslant \frac{1}{2}(\lambda_{k-1} - \lambda)\|y\|_2^2 + \frac{1}{2}(\lambda_{k+1} - \lambda)\|z\|_2^2 - C\|u\|_\theta^\theta + C. \tag{3.9}$$

Since $\theta > 2$ and V_k is finite dimensional, (3.9) shows there exists $R > 0$ independent of λ, such that

$$\Phi(x) \leqslant 0, \quad \text{for } x \in V_k \text{ with } \|x\| = R. \tag{3.10}$$

Now fixing such an $R > 0$ and assuming $\lambda \in (\lambda_{k-1}, \lambda_{k+1})$. Set $M := \sup_{(t,x)\in[0,2\pi]\times\mathbb{R}^N} V^-(t,x)$. For $y \in E_k$ with $\|y\| \leqslant R$, we write $y = w + z$, $w \in E_{k-1}$, $z \in E(\lambda_k)$. Then we have that

$$\begin{aligned}
\Phi(y) &= \frac{1}{2}\int_0^{2\pi} \left(|\dot{y}|^2 - \left(A(t)y, y\right) - \lambda|y|^2\right) dt - \int_0^{2\pi} V(t,y)\, dt \\
&\leqslant \frac{1}{2}(\lambda_{k-1} - \lambda)\|w\|_2^2 + \frac{1}{2}(\lambda_k - \lambda)\|z\|_2^2 + \int_{\{t\in[0,2\pi]:\ V\leqslant 0\}} V^-(t,y)\, dt \\
&\leqslant \frac{1}{2}|\lambda_k - \lambda| R^2 + 2\pi M.
\end{aligned} \tag{3.11}$$

Notice that

$$\partial Q = \left\{x = y + s\phi_{k+1} \;\middle|\; \|x\| = R,\ y \in E_k,\ s \geqslant 0\right\} \cup \left\{y \in E_k \;\middle|\; \|y\| \leqslant R\right\}, \tag{3.12}$$

it follows from (3.10) and (3.11) that (3.8) holds by taking $\delta = \frac{\alpha}{R^2 + 4\pi}$ and $\sigma = \alpha/2$. The proof is complete. □

Now we are ready to get the following existence theorem by applying Proposition 3.1.

Theorem 3.5. *Assume that V satisfies (V_1), (V_2) and (V_3) and $k \geqslant 1$. Then there exists $\delta > 0$ such that when $\sup_{(t,x)\in[0,2\pi]\times\mathbb{R}^N} V^-(t,x) \leqslant \delta$, for each $\lambda \in (\lambda_k - \delta, \lambda_k + \delta)$, $(\mathrm{HS})_\lambda$ has a nontrivial 2π-periodic solution x^λ with positive energy and such that*

$$C_{\ell_k+1}\left(\Phi, x^\lambda\right) \ncong 0. \tag{3.13}$$

Proof. By Lemmas 3.3 and 3.4, for each $\lambda \in (\lambda_k - \delta, \lambda_k + \delta)$, the functional Φ verifies (Φ_1) and (Φ_2) with respect to the decomposition $E = E_k \oplus E_k^\perp$ and $\dim E_k = \ell_k$:

$$\inf_{x\in S_\rho} \Phi(x) \geqslant \alpha > \frac{\alpha}{2} \geqslant \max_{y\in\partial Q} \Phi(y).$$

By Lemma 3.2, Φ verifies (PS). As $R > \rho > 0$, S_ρ and ∂Q homologically link, it follows from Proposition 3.1 that Φ has a critical point x^λ satisfying (3.13). The proof is finished. □

We close this section with some remarks. The existence of one nontrivial 2π-periodic solution of $(\mathrm{HS})_\lambda$ can be obtained by applying directly the homotopic linking [11] in the way that Φ has a critical value $c^\lambda \geqslant \alpha$ which can be characterized as

$$c^\lambda := \inf_{h\in\Gamma} \max_{x\in Q} \Phi\left(h(x)\right)$$

where $\Gamma := \{h \in C(Q, E) \mid h = \mathrm{id} \text{ on } \partial Q\}$. This critical value c^λ has a uniform positive bound from below with respect to all $\lambda \leqslant \lambda_*$. The homological information for critical points obtained are completely new and will be used in proving the main theorems. The existence results in Theorem 3.5 are also new and did not coincided with the results in [9,4] when A is constant or zero since the construction of linking is rather different from that used in [9,4]. Indeed, under the assumptions (V_1)–(V_3) in Theorem 3.5, for any fixed $\lambda \in \mathbb{R}$, say, $\lambda \in [\lambda_{k-1}, \lambda_k)$ with $k \geqslant 2$, one can construct a critical value of Φ via *a homotopic linking with respect to the direct sum decomposition $E = E_{k-1} \oplus E_{k-1}^\perp$* and the existence results in [9,4] were obtained in this way.

4. Proofs of main results

In the final section we give the proofs of Theorems 1.1–1.3.

Proof of Theorem 1.1. By Theorem 2.2(1), $(\mathrm{HS})_\lambda$ has two nontrivial 2π-periodic solutions x_λ^i $(i=1,2)$ with their Morse indices satisfying

$$\ell_{k-1} \leqslant \mu\big(x_\lambda^i\big) \leqslant \mu\big(x_\lambda^i\big) + \nu\big(x_\lambda^i\big) \leqslant \ell_k, \quad i=1,2.$$

From the Gromoll–Meyer Theorem [2,3], we have that

$$C_q\big(\Phi, x_\lambda^i\big) \cong 0, \quad q \notin [\ell_{k-1}, \ell_k],\ i=1,2. \tag{4.1}$$

By Theorem 3.5 for the part $\lambda \in (\lambda_k - \delta, \lambda_k)$, $(\mathrm{HS})_\lambda$ has a nontrivial 2π-periodic solution x^λ satisfying

$$C_{\ell_k+1}\big(\Phi, x^\lambda\big) \ncong 0. \tag{4.2}$$

From (4.1) and (4.2) we see that $x^\lambda \neq x_\lambda^i$ $(i=1,2)$. The proof is complete. □

Proof of Theorem 1.2. With the same argument as above, it follows from Theorem 2.2(2) and Theorem 3.5 for the part $\lambda \in (\lambda_k, \lambda_k + \delta)$. We omit the details. □

We remark here that the conclusions of Theorems 1.1 and 1.2 can also be proved by comparing the energies (see [12]). Now we give

Proof of Theorem 1.3. By Theorem 3.5 for the part $\lambda \in (\lambda_k - \delta, \lambda_k]$, $(\mathrm{HS})_\lambda$ has a 2π-periodic solution x^λ with its energy $\Phi(x^\lambda) \geqslant \alpha > 0$ and

$$C_{\ell_k+1}\big(\Phi, x^\lambda\big) \ncong 0. \tag{4.3}$$

By Proposition 2.4, we have that

$$C_q(\Phi, 0) \cong \delta_{q,\ell_{k-1}}\mathbb{F}. \tag{4.4}$$

Under (V_1)–(V_3), for any fixed $\lambda \in \mathbb{R}$, by Proposition 4.1 below, we have that

$$C_q(\Phi, \infty) \cong 0, \quad \text{for } q \in \mathbb{Z}. \tag{4.5}$$

Assume that $(\mathrm{HS})_\lambda$ has only two 2π-periodic solutions 0 and x^λ. Choose $a, b \in \mathbb{R}$ such that $a < 0 < b < \Phi(x^\lambda)$. Then by the deformation and excision properties of singular homology (see [1,2]), we have

$$C_q(\Phi, \infty) \cong H_q\big(E, \Phi^a\big), \qquad C_q(\Phi, 0) \cong H_q\big(\Phi^b, \Phi^a\big), \qquad C_q\big(\Phi, x^\lambda\big) \cong H_q\big(E, \Phi^b\big).$$

From (4.5) and the exact sequence for the topological triple (E, Φ^b, Φ^a) we deduce that

$$C_{q+1}\big(\Phi, x^\lambda\big) \cong C_q(\Phi, 0), \quad \text{for } q \in \mathbb{Z}. \tag{4.6}$$

There is a contradiction from (4.3) and (4.4) when we take $q = \ell_k$ in (4.6). The proof is complete. □

We recall that the notion $C_q(\Phi, \infty)$ was introduced in [1] in describing the global homological information of the topological pair (E, Φ^a) where $a \ll -1$ such that the functional Φ possesses the deformation property and has no critical points on Φ^a. Now we present a proof for (4.5). The idea is essentially from a famous paper [16] where (4.5) was first built to obtain the existence of a third solution for superlinear elliptic problem via Morse theory.

Proposition 4.1. *Assume that V satisfies (V_1)–(V_3). Then for any a fixed $\lambda \in \mathbb{R}$, it holds*

$$C_q(\Phi, \infty) \cong 0, \quad \textit{for } q \in \mathbb{Z}.$$

Proof. Given $\lambda \in \mathbb{R}$. Denote $B_1 = \{x \in E:\ \|x\| \leqslant 1\}$. For each $x \in \partial B_1$, $\xi > 0$, we have by (3.5) that

$$\begin{aligned}
\Phi(\xi x) &= \frac{1}{2}\xi^2 \int_0^{2\pi} |\dot{x}|^2 - \big((A(t)+\lambda)x, x\big)\,dt - \int_0^{2\pi} V(t, \xi x)\,dt \\
&\leqslant \frac{1}{2}\xi^2 \int_0^{2\pi} |\dot{x}|^2 - \big((A(t)+\lambda)x, x\big)\,dt - C\xi^\theta \int_0^{2\pi} |x|^\theta\,dt + C.
\end{aligned}$$

As $\theta > 2$, we see that

$$\Phi(\xi x) \to -\infty, \quad \text{as } \xi \to +\infty. \tag{4.7}$$

For each $x \in \partial B_1$, $\xi > 0$, by (V_3), we have

$$\begin{aligned}
\frac{d}{d\xi}\Phi(\xi x) &= \bigl\langle \Phi'(\xi x), x \bigr\rangle \\
&= \xi \int_0^{2\pi} |\dot{x}|^2 - \bigl(\bigl(A(t)+\lambda\bigr)x, x\bigr)\,dt - \int_0^{2\pi} \bigl(V'_x(t,\xi x), x\bigr)\,dt \\
&= \xi^{-1}\left(2\Phi(\xi x) + 2\int_0^{2\pi} V(t,\xi x)\,dt - \int_0^{2\pi} \bigl(V'_x(t,\xi x), \xi x\bigr)\,dt\right) \\
&\leqslant \frac{1}{\xi}\left(2\Phi(\xi x) + \int_{\{t:\ |\xi x(t)| \leqslant \bar{r}\}} 2V(t,\xi x) - \bigl(V'_x(t,\xi x), \xi x\bigr)\,dt\right) \\
&\leqslant \frac{1}{\xi}\bigl(2\Phi(\xi x) + M\bigr)
\end{aligned}$$

where

$$M := 2\pi \max_{t\in[0,2\pi],\ |y|\leqslant \bar{r}} \bigl(2\bigl|V(t,y)\bigr| + \bar{r}\bigl|V'_x(t,y)\bigr|\bigr).$$

Therefore, for any a fixed $a < -\frac{M}{2}$, we get that

$$\Phi(\xi x) \leqslant a \quad \Longrightarrow \quad \frac{d}{d\xi}\Phi(\xi x) < 0. \tag{4.8}$$

Notice that $\Phi(0) = 0$, it follows from (4.7) and (4.8) that for any $x \in \partial B_1$, there is a *unique* $\omega(x) > 0$ such that

$$\Phi\bigl(\omega(x)x\bigr) = a, \quad x \in \partial B_1. \tag{4.9}$$

By (4.9) and the implicit function theorem we have that $\omega \in C(\partial B_1, \mathbb{R})$. Now define

$$h(x) = \begin{cases} 1, & \text{if } \Phi(x) \leqslant a, \\ \|x\|^{-1}\omega(\|x\|^{-1}x), & \text{if } \Phi(x) > a,\ x \neq 0. \end{cases}$$

Then $h \in C(E \setminus \{0\}, \mathbb{R})$. Define a map $\psi : [0,1] \times E \setminus \{0\} \to E \setminus \{0\}$ by

$$\psi(\sigma, x) = (1-\sigma)x + \sigma h(x)x. \tag{4.10}$$

Clearly, ψ is continuous, and for all $x \in E \setminus \{0\}$ with $\Phi(x) > a$, by (4.9)

$$\Phi\bigl(\psi(1,x)\bigr) = \Phi\bigl(\omega\bigl(\|x\|^{-1}x\bigr)\|x\|^{-1}x\bigr) = a.$$

Therefore

$$\psi(1,x) \in \Phi^a \quad \text{for all } x \in E \setminus \{0\}, \qquad \psi(\sigma,x) = x \quad \text{for all } \sigma \in [0,1],\ x \in \Phi^a,$$

and so Φ^a is a strong deformation retract of $E \setminus \{0\}$. Hence

$$C_q(\Phi, \infty) := H_q\bigl(E, \Phi^a\bigr) \cong H_q\bigl(E, E \setminus \{0\}\bigr) \cong H_q(B_1, \partial B_1) \cong 0, \quad q \in \mathbb{Z}$$

since ∂B_1 is contractible which follows from the fact that $\dim E = \infty$. □

We close up the paper with further remarks for theorems in a special case $N = 1$. In this case, Theorem 1.3 states that $(\mathrm{HS})_\lambda$ has at least three nontrivial 2π-periodic solutions. This result is also new and the existence of a third solution can be obtained by further applying Morse theory. Moreover, in this case, Theorems 1.1 and 1.2 can be proved in the same way as that of Theorem 1.3. We refer to [6,15] for the arguments and leave the details for the interested readers. In this paper we have used some basic ideas and tools about Morse theory to which one refers to [2,8] and to [13,14] for a brief summary.

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