



Multiplicity of solutions for a class of the quasilinear elliptic equations at resonance

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ABSTRACT

In this paper, by Morse theory we obtain the existence and multiplicity for a class of the quasilinear elliptic equations at resonance.

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1. Introduction

Let Ω be a bounded domain in \mathbf{R}^N with smooth boundary $\partial\Omega$, we study the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u - \mu \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $1 < p < \infty$, Δ_p denotes the p -Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $\Delta u = \Delta_2 u$, and $\mu > 0$ is a real parameter. If we assume that

(f₀) $f(x, 0) = 0$, $f \in C^1(\overline{\Omega} \times \mathbf{R}, \mathbf{R})$ and satisfies the following condition:

$$|f'(x, t)| \leq c(1 + |t|^{q-2}), \quad \forall t \in \mathbf{R}, x \in \Omega,$$

for some constants $c > 0$ and $q \in [2, s^*)$, where $s = \max\{p, 2\}$, $s^* = Ns/(N - s)$ if $s < N$ and $s^* = +\infty$ if $N \leq s$, then the weak solutions of (1.1) correspond to the critical points of the functional $I : W_0^{1,s}(\Omega) \rightarrow \mathbf{R}$ defined by

$$I(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{\mu}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F(x, u) dx,$$

where $F(x, u) = \int_0^u f(x, t) dt$, and $W_0^{1,s}(\Omega)$ is the Sobolev space endowed with the norm

$$\|u\| = \|\nabla u\|_s = \left(\int_{\Omega} |\nabla u|^s dx \right)^{\frac{1}{s}}.$$

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For the case of $p > 2$ and $\mu > 0$, there has been an increasing interest in looking at the existence of solutions of (1.1). Using the following conditions

$$\lambda_m < f'(x, 0) < \lambda_{m+1}, \quad F(x, u) < \frac{\mu_1}{p} |u|^p + C, \quad x \in \Omega,$$

where $m \geq 1$ and C is a constant, the authors in [1,2] prove that (1.1) has at least two nontrivial solutions by the three critical point theorems, here and in the sequel $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ denote the eigenvalues of $-\Delta_2$ in $W_0^{1,2}(\Omega)$, and μ_1 is the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ (see [3]). For Eq. (1.1) with right-hand side having p -linear growth at infinity, i.e.,

$$\lim_{|u| \rightarrow \infty} \frac{f(x, u)}{|u|^{p-2}u} = \lambda \notin \sigma(-\Delta_p), \quad \text{the spectrum of } -\Delta_p \text{ in } W_0^{1,p}(\Omega),$$

the papers [4,5] get the existence of one nontrivial solution. The critical groups computations via Morse theory for the solutions of (1.1) have been studied in [5,6], in particular, using condition (f_0) the paper [7] computes exactly the critical groups of the isolated mountain pass solution and obtains the existence of multiple nontrivial solutions of (1.1).

The first aim of this paper is to extend the results in [1,2]. We make the following assumptions:

(f_1) there exist $\alpha > 0$ and $m \geq 2$ such that

$$\frac{1}{2} \lambda_m u^2 < F(x, u) \leq \frac{1}{2} \lambda_{m+1} u^2, \quad \text{for } x \in \Omega, \quad 0 < |u| \leq \alpha,$$

(f_2) there is a constant $C > 0$ such that

$$\lim_{|u| \rightarrow \infty} \left(F(x, u) - \frac{1}{p} \mu_1 |u|^p \right) \leq C, \quad \text{for } x \in \Omega,$$

and our result is the following:

Theorem 1.1. *If $p > 2$, $\mu = 1$, and conditions (f_0) – (f_2) hold, then Eq. (1.1) has at least three nontrivial solutions.*

Remark 1. (1) Using condition (f_1) we will show that the functional I has a local linking at 0, and for the similar results with $p = 2$ and the following resonant condition

$$\frac{1}{2} \lambda_m u^2 \leq F(x, u) \leq \frac{1}{2} \lambda_{m+1} u^2, \quad \text{for } |u| \text{ small and } u \neq 0,$$

we refer to [8–10] and references therein.

(2) By condition (f_2) , our functional I is coercive on $W_0^{1,p}(\Omega)$. In the case of $p > 1$ and $\mu = 0$, using the condition

$$\lim_{|u| \rightarrow \infty} \left(F(x, u) - \frac{1}{p} \mu_1 |u|^p \right) = -\infty, \quad x \in \Omega, \quad (1.2)$$

the papers [11,12] prove that the functional I is also coercive. Obviously, our condition (f_2) is weaker than (1.2).

Next, we turn to the case of $1 < p < 2$. Let $f(x, 0) = 0$ and

$$g(x, u) = f(x, u) - \lambda_m u, \quad m \geq 2, \quad x \in \Omega,$$

and $G(x, u)$ be the primitive of $g(x, u)$. We consider the following hypotheses:

(g_1) $f \in C(\overline{\Omega} \times \mathbf{R}, \mathbf{R})$ and there is a constant $C > 0$ such that

$$|g(x, u)| \leq C |u|^{p-1} + C, \quad \forall u \in \mathbf{R}, \quad x \in \Omega,$$

(g_2) there is some $C > 0$ such that

$$\lim_{\|u\| \rightarrow \infty} \int_{\Omega} G(x, u) dx \leq C, \quad \text{where } u \in \ker(-\Delta - \lambda_m),$$

(g_3) there exists some $\alpha > 0$ such that

$$F(x, u) \leq \frac{\mu_1}{p} |u|^p, \quad \text{for } x \in \Omega, \quad |u| \leq \alpha,$$

and prove the following result:

Theorem 1.2. *If $1 < p < 2$, $\mu = 1$, and conditions (g_1) – (g_3) hold, then Eq. (1.1) has at least one nontrivial solution.*

Remark 2. For $p = 2$, the papers [13,14] assume that

$$|g(x, u)| \leq C|u|^r + C, \quad r \in (0, 1),$$

and

$$\lim_{\|v\| \rightarrow \infty} \frac{1}{\|v\|^{2r}} \int_{\Omega} G(x, v) dx \leq -C, \quad v \in \ker(-\Delta - \lambda_m), \quad (1.3)$$

where C is a constant, and prove that equation has multiple nontrivial solutions. Note that our condition (g_2) is weaker than (1.3).

This paper is organized as follows: In Section 2, the proof of Theorem 1.1 is given. Section 3 will prove Theorem 1.2. In the sequel, the letter C will be used to denote various positive constants whose exact values are irrelevant.

2. Proof of Theorem 1.1

The proofs of our theorems are based on the minimax methods and the Morse theory, then let us recall some results (see [15]). Let E be a real Banach space and $\Phi \in C^1(E, \mathbf{R})$.

Definition 2.1. The functional Φ is said to satisfy Palais–Smale (for short (PS)) condition if every sequence $\{u_n\} \subset E$ with

$$\Phi(u_n) \text{ being bounded,} \quad \Phi'(u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

possesses a convergent subsequence.

Let $u \in K = \{u \in E \mid \Phi'(u) = 0\}$ be an isolated critical point of Φ with $\Phi(u) = c \in \mathbf{R}$, and U be an isolated neighborhood of u , i.e., $K \cap U = \{u\}$. The group

$$C_*(\Phi, u) = H_*(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}), \quad * = 0, 1, 2, \dots,$$

is called the $*$ -th critical group of Φ at u , where $\Phi^c = \{u \in E \mid \Phi(u) \leq c\}$, and $H_*(\cdot, \cdot)$ are the singular relative homology groups with a coefficient group G . For $a < \inf \Phi(K)$, the $*$ -th critical group of Φ at infinity is defined by

$$C_*(\Phi, \infty) = H_*(E, \Phi^a), \quad * = 0, 1, 2, \dots$$

Now we want to prove Theorem 1.1. Since $p > 2$ in this theorem, the functional I is defined on $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \|\nabla u\|_p = \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

Let

$$H^- = \bigoplus_{i \leq m-1} \ker(-\Delta - \lambda_i),$$

$$H^0 = \ker(-\Delta - \lambda_m),$$

$$H^+ = \overline{\bigoplus_{j \geq m+1} \ker(-\Delta - \lambda_j)},$$

then we have

$$W_0^{1,2}(\Omega) = H^- \oplus H^0 \oplus H^+.$$

Set $V = H^- \oplus H^0$. Since $p > 2$, by the regularity theory (see [16]) we have

$$V \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

and $W_0^{1,p}(\Omega) \subset W_0^{1,2}(\Omega)$ continuously. Let $W = H^+ \cap W_0^{1,p}(\Omega)$, then we get the splitting

$$W_0^{1,p}(\Omega) = V \oplus W.$$

Lemma 2.2. Under conditions (f_0) and (f_1) , we have

$$C_d(I, 0) \neq 0, \quad \text{where } d = \dim(H^- \oplus H^0) \geq 2. \quad (2.1)$$

Proof. We first show that the functional I has a local linking at 0 with respect to $W_0^{1,p}(\Omega) = V \oplus W$, i.e., there exists $\rho > 0$ such that

$$\begin{cases} I(u) \leq 0, & \text{for } u \in V, \|u\| \leq \rho, \\ I(u) > 0, & \text{for } u \in W, 0 < \|u\| \leq \rho. \end{cases}$$

(1) By condition (f_1) , for $\varepsilon > 0$ and $u \in V$ which is a finite dimensional space, we have

$$\frac{1}{2}(\lambda_m + \varepsilon)u^2 \leq F(x, u), \quad \text{as } \|u\| \text{ is small,}$$

which implies that

$$\begin{aligned} I(u) &\leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda_m + \varepsilon}{2\lambda_m} \int_{\Omega} |\nabla u|^2 dx \\ &\leq -C\|u\|^2 + C\|u\|^p, \end{aligned}$$

then using $p > 2$ there exists $\rho > 0$ such that $I(u) \leq 0$ as $\|u\| \leq \rho$.

(2) By conditions (f_0) and (f_1) , there exists $p < \theta \leq p^*$ such that

$$F(x, u) \leq \frac{1}{2}\lambda_{m+1}u^2 + C|u|^\theta, \quad \forall u \in \mathbf{R}, x \in \Omega,$$

then for $u \in W$ we have

$$\begin{aligned} I(u) &\geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{1}{2} \int_{\Omega} \lambda_{m+1}u^2 dx - C \int_{\Omega} |u|^\theta dx \\ &\geq \frac{1}{p} \|u\|^p - C\|u\|^\theta, \end{aligned}$$

which implies that there exists $\rho > 0$ such that $I(u) > 0$ as $0 < \|u\| \leq \rho$.

Now using the results in [17], we complete the lemma. \square

Let

$$f_{\pm}(x, t) = \begin{cases} f(x, t), & \pm t \geq 0, \\ 0, & \pm t < 0, \end{cases}$$

and

$$I_{\pm}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} F_{\pm}(x, u) dx,$$

where $F_{\pm}(x, u) = \int_0^u f_{\pm}(x, s) ds$. It is well known that the critical points of I_{\pm} are exactly the weak solutions of Eq. (1.1).

Lemma 2.3. If conditions (f_0) and (f_2) hold, then

(1) I and I_{\pm} are coercive on $W_0^{1,p}(\Omega)$,

(2) I and I_{\pm} satisfy the (PS) condition.

Proof. (1) Our method is similar to the paper [12]. By contradiction, there is a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that

$$I(u_n) \leq C, \quad \text{as } \|u_n\| \rightarrow \infty. \quad (2.2)$$

Set $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and there is a $v_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v_0, & \text{weakly in } W_0^{1,p}(\Omega), \\ v_n \rightarrow v_0, & \text{strongly in } L^p(\Omega), \\ v_n(x) \rightarrow v_0(x), & \text{a.e. } x \in \Omega. \end{cases} \quad (2.3)$$

Using condition (f_2) and (2.2) we have

$$\frac{C}{\|u_n\|^p} \geq \frac{I(u_n)}{\|u_n\|^p} \geq \frac{1}{p} \int_{\Omega} (|\nabla v_n|^p - \mu_1 |v_n|^p) dx - \frac{C}{\|u_n\|^p},$$

it results

$$\limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^p dx \leq \mu_1 \int_{\Omega} |v_0|^p dx. \quad (2.4)$$

On the other hand, we get that

$$\mu_1 \int_{\Omega} |v_0|^p dx \leq \int_{\Omega} |\nabla v_0|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^p dx,$$

this together with (2.4) implies that

$$\int_{\Omega} |\nabla v_0|^p dx = \mu_1 \int_{\Omega} |v_0|^p dx, \quad \text{and} \quad v_n \rightarrow v_0 \quad \text{in } W_0^{1,p}(\Omega).$$

Since $\|v_0\| = 1$, we can set $v_0 = \pm \phi_1$, where $\phi_1 > 0$ is the eigenfunction of μ_1 , then by (2.3) we have

$$|u_n(x)| \rightarrow +\infty, \quad \text{a.e. } x \in \Omega. \quad (2.5)$$

From condition (f_2) and (2.5), the Fatou lemma implies that

$$\begin{aligned} I(u_n) &\geq \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx - \int_{\Omega} \left(F(x, u_n) - \frac{\mu_1}{p} |u_n|^p \right) dx \\ &\geq C \int_{\Omega} |u_n|^2 dx - \int_{\Omega} \left(F(x, u_n) - \frac{\mu_1}{p} |u_n|^p \right) dx \\ &\rightarrow +\infty, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

this is a contradiction. The case of I_+ (I_-) is similar.

(2) It follows from Theorem 3 in [1]. \square

Lemma 2.4. Assume (f_1) holds and let $e_1 > 0$ be the eigenfunction associated with λ_1 . Then there exists $t > 0$ such that $I_{\pm}(\pm te_1) < 0$.

Proof. By (f_1) , there exists $t > 0$ such that

$$F(te_1) \geq \frac{1}{2} \lambda_m |te_1|^2, \quad \text{for } |te_1| \leq \alpha,$$

which implies that

$$\begin{aligned} I_+(te_1) &\leq \frac{|t|^p}{p} \int_{\Omega} |\nabla e_1|^p dx + \frac{|t|^2}{2} \int_{\Omega} |\nabla e_1|^2 dx - \frac{\lambda_m}{2} \int_{\Omega} |te_1|^2 dx \\ &= \frac{|t|^p}{p} \int_{\Omega} |\nabla e_1|^p dx + \frac{\lambda_1 |t|^2}{2} \int_{\Omega} |e_1|^2 dx - \frac{\lambda_m}{2} \int_{\Omega} |te_1|^2 dx \\ &\leq C|t|^p - C|t|^2 < 0, \quad \text{as } t > 0 \text{ is small.} \end{aligned}$$

The case of I_- is similar. \square

Proof of Theorem 1.1. By Lemmas 2.3 and 2.4, Eq. (1.1) has at least a positive solution u_1 and a negative solution u_2 such that

$$C_*(I, u_1) = C_*(I, u_2) = \delta_{*,0} G.$$

Using the mountain pass lemma in [18], we can get that equation has a solution u_3 such that $C_1(I, u_3) \neq 0$. Without loss of generality, we assume that u_3 is isolated, then using condition (f_0) the result in [7] gives

$$C_*(I, u_3) = \delta_{*,1} G,$$

which implies that $u_3 \neq 0$ from (2.1). The proof is completed. \square

3. Proof of Theorem 1.2

In this section we will give the proof of Theorem 1.2, and in this theorem the functional I is defined on $W_0^{1,2}(\Omega)$ endowed with the norm

$$\|u\| = \|\nabla u\|_2 = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Lemma 3.1. *If conditions (g_1) and (g_2) hold, then the functional I satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset W_0^{1,2}(\Omega)$ be such that

$$|I(u_n)| \leq C, \quad I'(u_n) \rightarrow 0, \quad n \rightarrow \infty, \quad (3.1)$$

then from (g_1) we only need to prove that $\{u_n\}$ is bounded. By contradiction, suppose that $\|u_n\| \rightarrow \infty$, as $n \rightarrow \infty$.

Set $u_n = w_n^- + v_n + w_n^+ \in H^- \oplus H^0 \oplus H^+$, and $w_n = w_n^- + w_n^+$. Using (g_1) and (3.1), we have

$$\begin{aligned} \int_{\Omega} [|\nabla w_n^+|^2 - \lambda_m (w_n^+)^2] dx &\leq C \|w_n^+\| + \int_{\Omega} g(x, u_n) w_n^+ dx + \int_{\Omega} |\nabla u_n|^{p-1} |\nabla w_n^+| dx \\ &\leq C \|w_n^+\| + \int_{\Omega} (C|u_n|^{p-1} + C) |w_n^+| dx + \int_{\Omega} |\nabla u_n|^{p-1} |\nabla w_n^+| dx \\ &\leq C \|w_n^+\| + C \|w_n^+\| \|u_n\|^{p-1}, \end{aligned}$$

therefore,

$$\begin{aligned} \|w_n^+\|^2 &\leq C \|w_n^+\| + C \|w_n^+\| \|u_n\|^{p-1}, \\ \|w_n^+\| &\leq C \|u_n\|^{p-1} + C, \end{aligned} \quad (3.2)$$

and

$$\int_{\Omega} [|\nabla w_n^+|^2 - \lambda_m (w_n^+)^2] dx \leq C \|u_n\|^{2(p-1)} + C. \quad (3.3)$$

Similarly, we have

$$\|w_n^-\| \leq C \|u_n\|^{p-1} + C, \quad (3.4)$$

$$\left| \int_{\Omega} [|\nabla w_n^-|^2 - \lambda_m (w_n^-)^2] dx \right| \leq C \|u_n\|^{2(p-1)} + C. \quad (3.5)$$

Then (3.2) and (3.4) imply that

$$\frac{\|w_n\|}{\|u_n\|} \rightarrow 0, \quad \frac{\|v_n\|}{\|u_n\|} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (3.6)$$

From (3.3) and (3.5), we conclude

$$\begin{aligned} I(u_n) &\geq -C \|u_n\|^{2(p-1)} - C + \frac{1}{p} \int_{\Omega} |\nabla v_n|^p dx + \frac{1}{p} \int_{\Omega} [|\nabla u_n|^p - |\nabla v_n|^p] dx \\ &\quad - \int_{\Omega} G(x, v_n) dx - \int_{\Omega} [G(x, u_n) - G(x, v_n)] dx. \end{aligned} \quad (3.7)$$

Using the mean value theorem, there is a constant $\tau \in [0, 1]$ such that

$$\begin{aligned} \int_{\Omega} [|\nabla u_n|^p - |\nabla v_n|^p] dx &\leq \int_{\Omega} |\nabla v_n + \tau \nabla w_n|^{p-1} |\nabla w_n| dx \\ &\leq C \int_{\Omega} |\nabla v_n|^{p-1} |\nabla w_n| dx + C \int_{\Omega} |\nabla w_n|^p dx \\ &\leq C \|v_n\|^{p-1} \|w_n\| + C \|w_n\|^p \\ &\leq C \|v_n\|^{p-1} \|u_n\|^{p-1} + C \|v_n\|^{p-1} + C \|u_n\|^{p(p-1)} + C, \end{aligned} \quad (3.8)$$

similarly, we have

$$\begin{aligned}
 \int_{\Omega} [G(x, u_n) - G(x, v_n)] dx &= \int_{\Omega} g(x, v_n + \tau w_n) w_n dx \\
 &\leq \int_{\Omega} (C|v_n + \tau w_n|^{p-1} + C)|w_n| dx \\
 &\leq C\|v_n\|^{p-1}\|w_n\| + C\|w_n\|^p + C\|w_n\| \\
 &\leq C\|v_n\|^{p-1}\|u_n\|^{p-1} + C\|v_n\|^{p-1} + C\|u_n\|^{p(p-1)} + C.
 \end{aligned} \tag{3.9}$$

From (3.7) to (3.9), we get

$$\begin{aligned}
 I(u_n) &\geq -C\|u_n\|^{2(p-1)} - C + \frac{1}{p} \int_{\Omega} |\nabla v_n|^p dx - \int_{\Omega} G(x, v_n) dx - (C\|v_n\|^{p-1}\|u_n\|^{p-1} + \|v_n\|^{p-1} + \|u_n\|^{p(p-1)}) \\
 &\geq \|u_n\|^{2(p-1)} \left\{ -C - \frac{C}{\|u_n\|^{2(p-1)}} + \frac{C\|v_n\|^p}{\|u_n\|^{2(p-1)}} - \frac{C\|v_n\|^{p-1}\|u_n\|^{p-1}}{\|u_n\|^{2(p-1)}} - \frac{C\|v_n\|^{p-1}}{\|u_n\|^{2(p-1)}} - \frac{C\|u_n\|^{p(p-1)}}{\|u_n\|^{2(p-1)}} \right\} - C,
 \end{aligned}$$

then using $p > 2(p-1)$ and (3.6) we have

$$I(u_n) \rightarrow +\infty, \quad n \rightarrow \infty,$$

which is a contradiction. \square

Let

$$\begin{aligned}
 H^- &= \bigoplus_{i \leq m-1} \ker(-\Delta - \lambda_i), \\
 H^0 &= \ker(-\Delta - \lambda_m), \\
 H^+ &= \overline{\bigoplus_{j \geq m+1} \ker(-\Delta - \lambda_j)},
 \end{aligned}$$

then we have

$$W_0^{1,2}(\Omega) = H^- \oplus H^0 \oplus H^+.$$

Lemma 3.2. Let condition (g_1) hold. Then $I(u) \rightarrow -\infty$, as $\|u\| \rightarrow \infty$ and $u \in H^-$.

Proof. For $u \in H^-$, by (g_1) we get

$$\begin{aligned}
 I(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda_m}{2} \int_{\Omega} |u|^2 dx - \int_{\Omega} G(x, u) dx \\
 &\leq -C\|u\|^2 + C\|u\|^p + C\|u\|,
 \end{aligned}$$

then using $p < 2$, we complete the proof. \square

Lemma 3.3. Let conditions (g_1) and (g_2) hold. Then $I(u) \rightarrow \infty$ on $H^0 \oplus H^+$ as $\|u\| \rightarrow \infty$.

Proof. Let $u = v + w \in H^0 \oplus H^+$, by contradiction we assume that $I(u) \leq C$ as $\|u\| \rightarrow \infty$. Using (3.9) we get

$$\begin{aligned}
 C\|w\|^2 &\leq C + \int_{\Omega} G(x, u) dx \\
 &\leq C + \int_{\Omega} G(x, v) dx + \int_{\Omega} [G(x, u) - G(x, v)] dx \\
 &\leq C + C\|v\|^{p-1}\|w\| + C\|w\|^p + C\|w\|,
 \end{aligned}$$

then

$$\|w\| \leq C + C\|v\|^{p-1},$$

which implies that

$$\begin{aligned} I(u) &\geq C\|w\|^2 + \frac{1}{p} \int_{\Omega} |\nabla v|^p dx + \frac{1}{p} \int_{\Omega} [|\nabla u|^p - |\nabla v|^p] dx - \int_{\Omega} G(x, v) dx - \int_{\Omega} [G(x, u) - G(x, v)] dx \\ &\geq C\|w\|^2 + C\|v\|^p - C\|w\|\|v\|^{p-1} - C\|w\|^p - C\|w\| - C \\ &\geq C\|w\|^2 + C\|v\|^p - C\|v\|^{2(p-1)} - C\|v\|^{p-1} - C\|w\|^p - C \\ &\rightarrow +\infty, \quad \text{as } \|v\| + \|w\| \rightarrow \infty, \end{aligned}$$

this is a contradiction. \square

Lemma 3.4. *If conditions (g_1) and (g_3) hold, then $C_*(I, 0) = \delta_{*,0}G$.*

Proof. Using conditions (g_1) and (g_3) , there exists $2 < \theta \leq 2^*$ such that

$$F(x, t) \leq \frac{\mu_1}{p} |t|^p + C|t|^\theta, \quad \forall t \in \mathbf{R}, x \in \Omega,$$

then we have

$$\begin{aligned} I(u) &\geq \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\mu_1}{p} \int_{\Omega} |u|^p dx - C \int_{\Omega} |u|^\theta dx \\ &\geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - C \int_{\Omega} |u|^\theta dx \\ &\geq C\|u\|^2 - C\|u\|^\theta > 0, \quad \text{as } 0 < \|u\| \ll 1, \end{aligned}$$

which implies that 0 is a local minimum, thus $C_*(I, 0) = \delta_{*,0}G$. \square

Proof of Theorem 1.2. Using the results in [19], Lemmas 3.2 and 3.3 imply that

$$C_d(I, \infty) \neq 0, \quad \text{where } d = \dim(H^-) \geq 1,$$

then by Lemma 3.4, Eq. (1.1) has at least one nontrivial solution. The proof is completed. \square

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