



# $L^\infty$ a priori bounds for gradients of solutions to quasilinear inhomogeneous fast-growing parabolic systems

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## ABSTRACT

We prove boundedness of gradients of solutions to quasilinear parabolic systems, the main part of which is a generalization to the  $p$ -Laplacian and its right-hand side's growth depending on the gradient is not slower (and generally strictly faster) than  $p - 1$ . This result may be seen as a generalization to the classical notion of a controllable growth of the right-hand side, introduced by Campanato, over gradients of  $p$ -Laplacian-like systems. Energy estimates and a nonlinear iteration procedure of the Moser type are cornerstones of the used method.

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## 1. Introduction

### 1.1. General statement of the problem

We are interested in obtaining a local boundedness of gradients of solutions to the following parabolic system in  $\Omega \subset \mathbb{R}^n$

$$u_t^i - (A_\alpha^i(\nabla u))_{x_\alpha} = f^i(x, t, \nabla u) \quad i = 1, \dots, N$$

where the main part is a generalization of the  $p$ -Laplacian and the right-hand side grows as  $1 + |\nabla u|^w$  or  $|\nabla u|^w$  with  $w$  specified further. We say that a right-hand side is a fast-growing one, when  $w > p - 1$  holds.

The existing literature on the regularity issue of parabolic equations and systems is impressive. Let us recall that for equations the existing results are quite strong: even for the right-hand-side growth of  $1 + |\nabla u|^p$  one obtains  $C^{1,\alpha}$  regularity of solutions: see classic monograph Ladyzhenskaya et al. [1] for the case  $p = 2$  and DiBenedetto [2] for  $p \in (1, \infty)$ . Many further generalizations are possible: for instance in [3] the right-hand side takes the form  $e^u |\nabla u|^p$ , which suffices for a boundedness of  $\nabla u$ . Moreover, this growth condition seems to be optimal, because there are blow-up results for gradients of solutions to equations, the right-hand sides of which grow faster than  $p$ —compare Souplet [4]. In the case of systems, the regularity results are much weaker. One can construct irregular (i.e. unbounded or discontinuous) functions, which solve homogeneous parabolic systems. For  $n > 2$  it suffices for irregularity that the coefficients  $A(x, t)$  of the main part are discontinuous (and still bounded) or that there is a relevant non-diagonality of the main part—for details, consult Arkhipova [5]. Nevertheless, there are many classes of main part which allow for higher regularity (even  $C^{1,\alpha}$ ) in the homogeneous case; these are: having structure close to Laplacian or  $p$ -Laplacian, like those studied in [1] or [2],

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respectively, or having main part depending solely on  $\nabla u$ : see the well-known paper by Nečas and Šverák [6] or more extensive research done by Choe and Bae [7]. As these papers consider homogeneous systems, one may ask a natural question: what inhomogeneous counterparts of such systems remain, in a certain sense, regular? The general answer is unknown, but there are several hints: on one hand, for the right-hand side growing like  $1 + |\nabla u|^{p-1}$  the regularity of the homogeneous case seems to be retained—see [2]; on the other, unlike for equations, one cannot have the right-hand side growing as fast as  $|\nabla u|^p$  without further assumptions, even in the case of a system with the simplest main part, i.e. an inhomogeneous heat system. Recall the classical counterexample: for  $n \geq 3$  bounded but discontinuous function  $u(x) = \frac{x}{|x|}$

with unbounded weak derivatives solves  $u_t^i - \Delta u^i = u^i |\nabla u|^2 \left[ = (n-1) \frac{x^i}{|x|^3} \right]$ , for details—see [8]. It turns out that in the case of an inhomogeneous system for  $p = 2$  that one has to additionally assume a certain smallness in order to obtain regularity—for details, refer to Tolksdorf [9], Pinggen [10], Idone [11] or even the classical Ladyzhenskaya et al. [1]. The regularity issue for a general nonlinear inhomogeneous parabolic system with the right-hand side growing at the rate  $1 + |\nabla u|^w$  for  $w$  possibly close to  $p$ , homogeneous counterparts of which enjoy regularity, is not fully researched, especially for the case  $p \neq 2$ . There are several approaches to answering this question: some authors relax the notion of regularity by resorting to partial regularity—see for example classical papers of Italian school: Campanato [12], Giaquinta and Struwe [13] and newer ones: Fanciullo [14], Frehse and Specovius-Neugebauer [15], Misawa [16], Duzaar and Mingione [17]; or by demanding a high integrability-type regularity<sup>2</sup>, like in [18,19] or [20] (in the last paper the growth of the right-hand-side may be polynomially arbitrarily large!). Certain systems with peculiar structure or two-dimensional ones (or at least close to them in some sense) enjoy also high regularity, even if they are much more general than a Stokes-type system; for results in this direction compare the papers of Seregin, Arkhipova, Frehse, Kaplicky (and many others), for instance: Arkhipova [5], Naumann and Wolff [21], Kaplicky [22], Zajączkowski and Seregin [23].

In this note we focus on deriving a full regularity result, more precisely: the local boundedness of gradients, for a class of quasilinear parabolic inhomogeneous systems. Our goal is twofold: firstly to obtain results for a general inhomogeneous parabolic system, the main part of which is analogous to the system considered in [7], while retaining possibly general growth conditions for the right-hand side. Secondly, to sharpen these results with respect to growth of the right-hand side, restricting ourselves to less general systems, being close to  $p$ -Laplacian. For similar result on the level of solutions, compare Giorgi and O’Leary [24].

Let us emphasize that we proceed in a manner typical for the regularity approach: we assume existence of solution  $u$  in a given class, which is often a deep problem itself, from which we derive higher regularity. Moreover, we concentrate on a priori estimates while conducting the proofs: the rigorous version of computations is commented on in the conclusion.

## 1.2. General definitions and assumptions

Consider the parabolic problem in  $\Omega \subset \mathbb{R}^n$

$$u_t^i - (A_\alpha^i(\nabla u))_{x_\alpha} = f^i(x, t, \nabla u) \quad i = 1, \dots, N. \quad (1)$$

As all our results have a local character, any further specification of  $\Omega$  is irrelevant.

We say that a vector valued function  $u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$  is a weak solution to (1) iff

$$\int_{\Omega_T} -u^i \phi_t^i + A_\alpha^i(\nabla u) \phi_{x_\alpha}^i dx dt = \int_{\Omega_T} f^i(x, t, \nabla u) \phi^i dx dt \quad \forall \phi \in C_0^\infty(\Omega_T).$$

Globally, the following notions will be used:

- $\delta_\beta^\alpha$  denotes the Kronecker delta,
- $Q_R(x_0, t_0)$  denotes a parabolic cylinder, i.e.  $B_R(x_0) \times (t_0 - R^p, t_0)$ ; when possible, cut short to  $Q_R$ ,
- $\eta_{\rho,R} \in C_0^\infty(Q_R)$  denotes a standard parabolic cutoff function for  $Q_\rho \subset Q_R$ , when possible, cut short to  $\eta$ , which satisfies  $\eta = 1$  in  $Q_\rho$ ,  $\eta = 0$  outside  $Q_R$ ,  $|\nabla \eta| \leq c(R - \rho)^{-1}$ ,  $|\eta_{,t}| \leq c(R - \rho)^{-p}$ .

Throughout the article, summation over repeated indices is in use.

## 1.3. The structure of results

We show our results in the following order:

1. First, we derive a general result for inhomogeneous version of system analyzed in [7], where ellipticity assumptions for the main part are generalized by introducing exponent  $q$  (Theorem 2). Here, loosely speaking, admissible growth for the right-hand side is  $1 + |\nabla u|^{p-1}$ , so this result may be seen as parallel to DiBenedetto [2].

<sup>2</sup> Such results are especially interesting, as our result may be easily strengthened via higher regularity

- Next, we admit faster growths for the right-hand side, at the cost of assuming that the main part is closer to the  $p$ -Laplacian, in the sense that it is not enriched with terms involving  $q > p$  (Theorem 3).
- Finally, we state the result for the least general case, i.e. for the 3D  $p$ -Laplacian with the right-hand side growing as  $|\nabla u|^w$  (Theorem 1).

Since the last result seems to be the most traceable one, let us give an incentive to studying the technical remainder of this paper by stating Theorem 1 now: Consider  $u = (u^i) \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$  solving the 3D  $p$ -Laplace system

$$u_t^i - \operatorname{div}(|\nabla u|^{p-2} \nabla u^i) = f^i(t, x, u, \nabla u) \quad i = 1, 2, 3.$$

**Theorem 1.** Let  $\Omega$  be an arbitrary domain. Assume a growth condition:  $|f^i(x, t, \nabla u)| \leq |\nabla u|^w$ ,  $w \leq p$  and initial integrability<sup>3</sup>  $|\nabla u|_{L_{\text{loc}}^{\tilde{p}}} < \infty$ . If one of the following conditions is fulfilled:

- $w \leq p - 1$ ,  $\tilde{p} = p$ ;
- $w \in \left[\frac{p}{2}, \frac{\tilde{p}+4p-3}{5}\right)$ ;
- $w \leq \frac{p}{2}$  and  $p > 2 - \frac{2}{3}\tilde{p}$

then  $\nabla u$  is bounded:

$$|\nabla u|_{L_{\text{loc}}^\infty} < C |\nabla u|_{L_{\text{loc}}^{\tilde{p}}}.$$

For the proof, see the end of the next section.

Observe that for  $w > p - 1$  in the degenerate case (i.e.  $p \geq 2$ ) we have  $w \geq \frac{p}{2}$ , so point 2 applies. In such a case, merely from existence, i.e. for  $\tilde{p} = p$ , one has  $w < p - \frac{3}{5}$ . This corresponds for  $p = 2$  with Campanato's notion of controllable growth, stating that to obtain regularity, the growth of right-hand side must be smaller than  $\frac{7}{5}$  (see for example [12]). Therefore, this result may be seen as a generalization of classical results over gradients of  $p$ -Laplacian-like systems. Utilizing results on high integrability of certain systems, one may relax the growth condition further. For example, for the system analysed in [18] we have  $w < \varepsilon + \frac{9}{5}$ , because  $\nabla u \in L^{\varepsilon+4}(\Omega_T)$ , which can be taken as the initial integrability  $L^{\tilde{p}}$ .

In all our theorems there is no explicit assumption that  $w < p$ . In fact, the inequality  $w \leq p$  is enforced by the rigorous treatment of energy estimates. Simultaneously we know, from the counterexample recalled in the introduction, that  $w = p$  is generally not admissible. Therefore, our results can be viewed as a way to quantify the possible boundedness of gradients by means of higher integrability. As in the just mentioned case of Theorem 1, for  $p = 2$  one has  $w < p - \frac{3}{5}$ , which can be boosted in some cases to  $w < \varepsilon + \frac{9}{5}$ , because  $\nabla u \in L^{\varepsilon+4}$ . For  $w = p$  one would need  $\nabla u \in L^{\varepsilon+5}$ .

## 2. Boundedness of the gradient of the solution

As outlined in the introduction, first we prove the general theorem. As the cornerstone of the analysis is the energy method, we derive formal estimates for the sake of transparency. For a rigorous justification of the formal estimates please consult the conclusion of this note. We analyze solutions of

$$u_t^i - (A_\alpha^i(\nabla u))_{x_\alpha} = f^i(x, t, \nabla u) \quad i = 1, \dots, N \quad (2)$$

where the main part comes from [7] and the right-hand-side grows as  $1 + |\nabla u|^w$ ,  $w \leq \delta$  for a certain  $\delta \leq p$ , obtaining the boundedness of  $\nabla u$ . More precisely, one has the following:

**Theorem 2.** Under the following assumptions:

(A0) ellipticity-type:  $A_\alpha^i$  is given by potential  $F \in C^2(\mathbb{R})$ ,  $F'(0) \geq 0$ , as follows

$$A_\alpha^i(Q) = F(|Q|)_{Q_\alpha^i}$$

and  $F$  enjoys ellipticity

$$F(|Q|)_{Q_\alpha^i Q_\beta^j} \zeta_\alpha^i \zeta_\beta^j \in [\lambda |Q|^{p-2}, \lambda^{-1}(|Q|^{p-2} + |Q|^{q-2})] |\zeta|^2$$

where:

$$1 < p \leq q < p + 1 < \infty$$

(A1) growth-type

$$|f^i| \leq +c_1 |\nabla u|^w + c_2$$

where

$$p \geq w \geq 0, \quad c_i \in L^\infty(\Omega_T)$$

<sup>3</sup> From existence one has  $\tilde{p} = p$ , so this assumption may be void. It only helps to quantify the results when we have some additional knowledge on integrability.

(A2) *initial integrability*

$$\nabla u \in L_{\text{loc}}^{s_0+M}(\Omega_T), \quad M := \max(2, p, 2q - p, w + 1, 2w - p + 2)$$

with  $s_0$  satisfying

$$s_0 \geq 0, \quad s_0 + 2 + \frac{np}{2} - \frac{Mn}{2} > 0$$

and

$$\begin{cases} s_0 > p - 2 & \text{for } c_2 \neq 0 \\ s_0 > p - 2w - 2 & \text{for } c_2 = 0 \end{cases}$$

the gradient of the solution to (2) is locally bounded; moreover, for any  $Q_{R_0} \subset \Omega$  with  $R_0 < 1$  the following inequality holds

$$|\nabla u|_{L^\infty(Q_{\frac{R_0}{2}})} \leq C \left( \int_{Q_{R_0}} |\nabla u|^{s_0+M} dx dt \right)^{\frac{1}{s_0+2+\frac{np}{2}-\frac{Mn}{2}}} + C.$$

**Proof.** First we derive formal energy inequalities, then we implement an iteration scheme.

Differentiate formally system (2) to obtain

$$u_{tx_\gamma}^i - (A_{\alpha, u_{x_\beta}^j}^i (\nabla u) u_{x_\beta x_\gamma}^j)_{x_\alpha} = (f^i(x, t, \nabla u))_{x_\gamma} \quad i = 1, \dots, N \quad (3)$$

testing (3) by  $u_{x_\gamma}^i |\nabla u|^s \eta^2$  one gets

$$\begin{aligned} & \left[ \frac{1}{s+2} \frac{d}{dt} \int_{B_R} |\nabla u|^{s+2} \eta^2 dx \right] + \underbrace{\left[ \int_{Q_R} A_{\alpha, u_{x_\beta}^j}^i (\nabla u) u_{x_\beta x_\gamma}^j (|\nabla u|^{s-2} u_{x_\gamma x_\alpha}^i \eta^2 + s |\nabla u|^{s-2} u_{x_\gamma}^i u_{x_\alpha x_\delta}^k u_{x_\delta}^k \eta^2) dx \right]}_I \\ &= - \int_{B_R} f^i(x, t, \nabla u) [u_{x_\gamma x_\gamma}^i |\nabla u|^s \eta^2 + s u_{x_\gamma}^i |\nabla u|^{s-2} u_{x_\delta x_\gamma}^k u_{x_\delta}^k \eta^2 + 2 u_{x_\gamma}^i |\nabla u|^s \eta \eta_{x_\gamma}] dx \\ &+ \frac{2}{s+2} \int_{B_R} |\nabla u|^{s+2} \eta \eta_t dx - \int_{B_R} A_{\alpha, u_{x_\beta}^j}^i (\nabla u) u_{x_\beta x_\gamma}^k u_{x_\gamma}^k |\nabla u|^s \eta \eta_{x_\alpha} dx. \end{aligned} \quad (4)$$

Consider  $I$ . Utilizing ellipticity assumption (A0) with  $\zeta_\rho^l := u_{x_\beta x_\rho}^l$  one estimates the first summand of  $I$  as follows

$$\int_{B_R} A_{\alpha'(\nabla u) \beta}^i (\nabla u) u_{x_\beta x_\gamma}^j |\nabla u|^{s-2} u_{x_\gamma x_\alpha}^i \eta^2 dx \geq \lambda \int_{B_R} |\nabla u|^{p-2} |\nabla^2 u|^2 |\nabla u|^s \eta^2 dx \quad (5)$$

and because  $A_{\alpha'}^i$  is given by potential  $F$ , from differentiation we estimate the second summand of  $I$

$$\begin{aligned} & \int_{B_R} A_{\alpha'(\nabla u) \beta}^i (\nabla u) u_{x_\beta x_\gamma}^j s |\nabla u|^{s-2} u_{x_\gamma}^i u_{x_\alpha x_\delta}^k u_{x_\delta}^k \eta^2 dx \\ &= \int_{B_R} \left[ F''(|\nabla u|) \frac{u_{x_\alpha}^i u_{x_\beta}^j}{|\nabla u|^2} + F'(|\nabla u|) \left( \frac{\delta_{x_\alpha}^i \delta_{x_\beta}^j}{|\nabla u|} - \frac{u_{x_\alpha}^i u_{x_\beta}^j}{|\nabla u|^3} \right) \right] u_{x_\beta x_\gamma}^j s |\nabla u|^{s-2} u_{x_\gamma}^i u_{x_\alpha x_\delta}^k u_{x_\delta}^k \eta^2 dx \\ &= s \int_{B_R} \eta^2 F''(|\nabla u|) |\nabla u|^{s-4} \underbrace{(u_{x_\gamma}^i u_{x_\beta}^j u_{x_\beta x_\gamma}^i)}_{c^i} \underbrace{(u_{x_\alpha}^i u_{x_\delta}^k u_{x_\delta x_\alpha}^k)}_{c^i} dx \\ &+ s \int_{B_R} \eta^2 F'(|\nabla u|) |\nabla u|^{s-5} \underbrace{(u_{x_\delta}^k u_{x_\delta x_\alpha}^k u_{x_\gamma}^i u_{x_\gamma x_\alpha}^i)}_{= \frac{1}{4} (|\nabla u|^2)_{x_\alpha} (|\nabla u|^2)_{x_\alpha} |\nabla u|^2} - \underbrace{(u_{x_\alpha}^i u_{x_\gamma}^j u_{x_\beta}^j u_{x_\beta x_\gamma}^j u_{x_\delta}^k u_{x_\delta x_\alpha}^k)}_{= \frac{1}{4} u_{x_\alpha}^i (|\nabla u|^2)_{x_\alpha} u_{x_\gamma}^j (|\nabla u|^2)_{x_\gamma}} dx. \end{aligned} \quad (6)$$

From the ellipticity assumption (A0) one has  $F''(|s|) \geq 0$ ,  $F'(0) \geq 0$  therefore it holds:  $F'(|s|) \geq 0$ . This, in conjunction with the following computation:  $u_{x_\alpha}^i (|\nabla u|^2)_{x_\alpha} u_{x_\gamma}^j (|\nabla u|^2)_{x_\gamma} \leq |\nabla u|^2 |\nabla |\nabla u|^2|^2 = |\nabla u|^2 (|\nabla u|^2)_{x_\beta} (|\nabla u|^2)_{x_\beta}$ , implies that equation (6) takes the form

$$\int_{B_R} A^i_{\alpha'(\nabla u)^j_\beta} (\nabla u) u^i_{x_\beta x_\gamma} s |\nabla u|^{s-2} u^i_{x_\gamma} u^k_{x_\alpha x_\delta} u^k_{x_\delta} \eta^2 dx \geq 0. \quad (7)$$

Summing up (5) and (7) we conclude that  $I$  satisfies

$$\int_{B_R} A^i_{\alpha'(\nabla u)^j_\beta} (\nabla u) u^i_{x_\beta x_\gamma} (|\nabla u|^s u^i_{x_\gamma x_\alpha} \eta^2 + s |\nabla u|^{s-2} u^i_{x_\gamma} u^k_{x_\alpha x_\delta} u^k_{x_\delta} \eta^2) dx \geq \lambda \int_{B_R} |\nabla u|^{p-2} |\nabla^2 u|^2 |\nabla u|^s \eta^2 dx. \quad (8)$$

Inputting inequality (8) into (4) we arrive at

$$\begin{aligned} & \frac{1}{s+2} \frac{d}{dt} \int_{B_R} |\nabla u|^{s+2} \eta^2 dx + \lambda \int_{B_R} |\nabla u|^{p-2} |\nabla^2 u|^2 |\nabla u|^s \eta^2 dx \\ & \leq \int_{B_R} f^i(x, t, \nabla u) [u^i_{x_\gamma x_\gamma} |\nabla u|^s \eta^2 + s u^i_{x_\gamma} |\nabla u|^{s-2} u^k_{x_\delta x_\gamma} u^k_{x_\delta} \eta^2 + 2 u^i_{x_\gamma} |\nabla u|^s \eta \eta_{x_\gamma}] dx \\ & \quad + \frac{2}{s+2} \int_{B_R} |\nabla u|^{s+2} \eta \eta_t dx - \int_{B_R} A^i_{\alpha'(\nabla u)^j_\beta} (\nabla u) u^k_{x_\beta x_\gamma} u^k_{x_\gamma} |\nabla u|^s \eta \eta_{x_\alpha} dx \\ & \leq c \int_{B_R} [1 + |\nabla u|^w] [(1+s) |\nabla^2 u| |\nabla u|^s \eta^2 + 2 |\nabla u|^{s+1} \eta |\nabla \eta|] dx \\ & \quad + \frac{c}{s+2} \int_{B_R} |\nabla u|^{s+2} \eta |\eta_t| dx + \frac{c}{\lambda} \int_{B_R} [|\nabla u|^{p-2} + |\nabla u|^{q-2}] |\nabla^2 u| |\nabla u|^{s+1} \eta |\nabla \eta| dx \end{aligned} \quad (9)$$

where the last inequality is valid in view of growth (A1) and ellipticity (A0) assumptions.

Absorb  $|\nabla^2 u|$  from the right-hand-side of (9) using Young's inequality and integrate with respect to time

$$\begin{aligned} & \frac{1}{s+2} \sup_t \int_{B_R} |\nabla u|^{s+2} \eta^2 dx + (\lambda - \varepsilon) \int_{Q_R} |\nabla u|^{p-2} |\nabla^2 u|^2 |\nabla u|^s \eta^2 dx dt \\ & \leq c \int_{Q_R} (1 + |\nabla u|^w) |\nabla u|^{s+1} \eta |\nabla \eta| dx dt + \frac{c}{s+2} \int_{Q_R} |\nabla u|^{s+2} \eta |\eta_t| dx dt \\ & \quad + c \int_{Q_R} |\nabla u|^s [|\nabla u|^p + |\nabla u|^{2q-p}] \eta^2 |\nabla \eta|^2 dx dt + c(1+s) \int_{Q_R} \eta^2 |\nabla u|^s [|\nabla u|^{2w-p+2} + |\nabla u|^{2-p}] dx dt. \end{aligned} \quad (10)$$

By estimates for derivatives of the cutoff function  $\eta$  we obtain

$$\begin{aligned} & \frac{1}{s+2} \sup_t \int_{B_R} |\nabla u|^{s+2} \eta^2 dx + (\lambda - \varepsilon) \int_{Q_R} |\nabla u|^{p-2} |\nabla^2 u|^2 |\nabla u|^s \eta^2 dx dt \\ & \leq \frac{c}{(R-\rho)^{\max(2,p)}} \int_{Q_R} |\nabla u|^s [|\nabla u|^p + |\nabla u| + |\nabla u|^{w+1} \\ & \quad + (2+s)^{-1} |\nabla u|^2 + |\nabla u|^p + |\nabla u|^{2q-p} + (1+s) (|\nabla u|^{2w-p+2} + |\nabla u|^{2-p})] dx dt \end{aligned} \quad (11)$$

for  $0 < \rho < R < 1$ . Since for some  $w$ ,  $p$  the exponents  $2w - p + 2$ ,  $2 - p$  may be nonpositive, we estimate respective powers of  $|\nabla u|$  using  $|\nabla u|^s$  as follows

$$\int_{Q_R} |\nabla u|^s [(1+s) (|\nabla u|^{2w-p+2} + |\nabla u|^{2-p})] dx dt \leq (1+s) \int_{Q_R} [1 + |\nabla u|^{\max(s+2-p, s+2w+2-p)}] dx dt. \quad (12)$$

For the last inequality to hold, we must assume

$$s > \max(p - 2w - 2, p - 2). \quad (13)$$

Because summand  $|\nabla u|^{2-p}$  occurs only if  $c_2 \neq 0$  in the growth condition (A1):  $|f^i| \leq c_1 |\nabla u|^w + c_2$ , the above assumption (13) can be written as

$$\begin{cases} s > p - 2 & \text{for } c_2 \neq 0 \\ s > p - 2w - 2 & \text{for } c_2 = 0. \end{cases}$$

In the forthcoming iteration scheme we construct a growing sequence of  $s_i$ , therefore it is sufficient to assume

$$\begin{cases} s_0 > p - 2 & \text{for } c_2 \neq 0 \\ s_0 > p - 2w - 2 & \text{for } c_2 = 0 \end{cases}$$

which coincides with our initial integrability assumption (A2).

By computation, the following inequality holds

$$\int_{Q_R} |\nabla(|\nabla u|^{\frac{p+s}{2}} \eta)|^2 dx dt \leq \int_{Q_R} |\nabla u|^{p+s} |\nabla \eta|^2 dx dt + (p+s)^2 \int_{Q_R} |\nabla u|^{p+s-2} |\nabla^2 u|^2 \eta^2 dx dt. \quad (14)$$

Adding to both sides  $\frac{\lambda-\varepsilon}{(p+s)^2} \int_{Q_R} |\nabla u|^{p+s} |\nabla \eta|^2 dx dt$  and considering properties of  $\eta$ , as  $(s+p)^2 \leq c(1+s^2)$ , we arrive from (11), by virtue of (14), at

$$\sup_t \int_{B_R} |\nabla u|^{s+2} \eta^2 dx + \int_{Q_R} |\nabla(|\nabla u|^{\frac{p+s}{2}} \eta)|^2 dx dt \leq \frac{c(1+s^3)}{(R-\rho)^M} \int_{Q_R} 1 + |\nabla u|^{s+M} dx dt \quad (15)$$

taking into account (12) if necessary. Recall that by definition  $M = \max(2, p, 2q-p, w+1, 2w-p+2)$ .

By Hölder and critical-Sobolev inequalities (respectively), one gets

$$\begin{aligned} \int_{Q_\rho} |\nabla u|^{p+s+(s+2)\frac{2}{n}} dx dt &\leq \int_{t_0-R^2}^{t_0} \left[ \int_{B_R} |\nabla u|^{s+2} \eta^2 dx \right]^{\frac{2}{n}} \left[ \int_{B_R} |\nabla u|^{(s+p)\frac{n}{n-2}} \eta^{\frac{2n}{n-2}} dx \right]^{\frac{n-2}{n}} dt \\ &\leq \left[ \sup_t \int_{B_R} |\nabla u|^{s+2} \eta^2 dx \right]^{\frac{2}{n}} \int_{t_0-R^2}^{t_0} \left[ \int_{B_R} (|\nabla u|^{\frac{p+s}{2}} \eta)^{\frac{2n}{n-2}} dx \right]^{\frac{n-2}{n}} dt \\ &\leq \left[ \sup_t \int_{B_R} |\nabla u|^{s+2} \eta^2 dx \right]^{\frac{2}{n}} \int_{Q_R} |\nabla(|\nabla u|^{\frac{p+s}{2}} \eta)|^2 dx dt \\ &\stackrel{(15)}{\leq} \left[ \frac{c(1+s^3)}{(R-\rho)^M} \int_{Q_R} 1 + |\nabla u|^{s+M} dx dt \right]^{1+\frac{2}{n}}. \end{aligned} \quad (16)$$

Inequality (16) is our desired energy estimate, which we now iterate. Define recursively numbers  $s_i : s_{i+1} + M = p + s_i + (s_i + 2)\frac{2}{n}$ , then

$$s_i = \left(1 + \frac{2}{n}\right)^i \left[ s_0 + n + 2 - \frac{n(p-M)}{2} \right] - \left[ 2 - \frac{n(p-M)}{2} \right].$$

Utilizing the initial-integrability assumption, i.e.  $s_0 + 2 + n - \frac{n(p-M)}{2} > 0$ , we have

$$s_i \xrightarrow{i \rightarrow \infty} \infty; \quad \frac{s_i}{(1 + \frac{2}{n})^i} \xrightarrow{i \rightarrow \infty} s_0 + 2 + n - \frac{n(p-M)}{2}. \quad (17)$$

Let

$$\psi_i = \int_{S_{R_i}} |\nabla u|^{s_i+M} dx dt$$

then (16) with  $\eta_{R_{i+1}, R_i}$  can be written as

$$\begin{aligned} |S_{R_{i+1}}| \psi_{i+1} &\leq \left[ C(1+s_i^p) \left( \frac{2^{i+2}}{R_0} \right)^M |S_{R_i}| (1 + \psi_i) \right]^{1+\frac{p}{n}} \\ &\implies R_{i+1}^{n+2} \psi_{i+1} \leq \left[ C(1+s_i^p) \left( \frac{2^{i+2}}{R_0} \right)^M R_{i+1}^{n+2} (1 + \psi_i) \right]^\beta \\ &\implies \psi_{i+1} \leq [C(1+s_i^p)^\beta 2^i (1 + \psi_i)]^\beta \end{aligned} \quad (18)$$

with  $\beta := 1 + \frac{2}{n}$ ; the last inequality given by  $R_i := \frac{R_0}{2}(1 + 2^{-i})$ . As we know from (17) that asymptotically  $s_i$  behaves like  $\beta^i$ , finally (18) folds to

$$\psi_{i+1} \leq C^i \psi_i^\beta + C^i$$

which, by a standard computation (see [7] for details), gives

$$\psi_{i+1} \leq C^{\beta^{i+1}} \psi_0^{\beta^{i+1}} + (i+1)C^{\beta^{i+1}}. \quad (19)$$

From the above considerations one gets, using the definition of  $\psi$

$$\begin{aligned} R_0^{-\frac{n+2}{s_{i+1}+M}} |\nabla u|_{L^{s_{i+1}+M}\left(Q_{\frac{R_0}{2}}\right)} &\leq \left( \int_{S_{R_{i+1}}} |u|^{s_{i+1}+M} \right)^{\frac{1}{s_{i+1}+M}} = \psi_{i+1}^{\frac{1}{s_{i+1}+M}} \\ &\stackrel{(19)}{\leq} (C^{\beta^{i+1}} \psi_0^{\beta^{i+1}})^{\frac{1}{s_{i+1}+M}} + ((i+1)C^{\beta^{i+1}})^{\frac{1}{s_{i+1}+M}} \xrightarrow{i \rightarrow \infty} C \psi_0^{\frac{1}{s_0+2+n-\frac{Mn}{p}}} + C \end{aligned} \quad (20)$$

in view of (17).

As  $s_i + M \xrightarrow{i \rightarrow \infty} \infty$ , (20) in tandem with the initial integrability assumption gives the following uniform bound

$$|\nabla u|_{L^\infty\left(Q_{\frac{R_0}{2}}\right)} \leq C \left( \int_{Q_{R_0}} |\nabla u|^{s_0+M} dx dt \right)^{\frac{1}{s_0+2+n-\frac{Mn}{p}}} + C. \quad \square$$

In the next theorem we neglect the term possessing  $q > p$  in the ellipticity assumption. This allows us, in turn, to obtain bigger growths of the right-hand side, as now it is possible to derive estimates for negative  $s > -\frac{\lambda}{\Lambda}$ .

**Theorem 3.** *The gradient of the solution to (2) is locally bounded, under the following assumptions:*

(A0) *ellipticity-type:  $A_\alpha^i$  is given by potential  $F \in C^2(\mathbb{R})$  as follows*

$$A_\alpha^i(Q) = F(|Q|)_{Q_\alpha^i}$$

*and  $F$  enjoys ellipticity*

$$F(|Q|)_{Q_\alpha^i Q_\beta^j} \zeta_\alpha^i \zeta_\beta^j \in [\lambda |Q|^{p-2}, \Lambda |Q|^{p-2}] |\zeta|^2$$

(A1) *growth-type*

$$|f^i| \leq c |\nabla u|^w, \quad p \geq w \geq 0, \quad c \in L^\infty(\Omega_T)$$

(A2) *initial integrability*

$$\nabla u \in L_{\text{loc}}^{s_0+M}(\Omega_T), \quad M := \max(2, p, w+1, 2w-p+2)$$

*with  $s_0$  satisfying*

$$\begin{cases} s_0 > \max\left(-\frac{\lambda}{\Lambda}, p-2w-2\right) \\ s_0+2+\frac{np}{2}-\frac{Mn}{2} > 0 \end{cases}$$

*moreover, for any  $Q_{R_0} \subset \Omega$  with  $R_0 < 1$  following inequality holds*

$$|\nabla u|_{L^\infty(Q_{\frac{R_0}{2}})} \leq C \left( \int_{Q_{R_0}} |\nabla u|^{s_0+M} dx dt \right)^{\frac{1}{s_0+2+\frac{np}{2}-\frac{Mn}{2}}} + C.$$

**Proof.** For  $s \geq 0$  Theorem 3 is a special case of Theorem 2, therefore it suffices to show it in the case of negative  $s$ . The only difference in the energy estimates is the lack of positivity of the left-hand side term of (4), where the sign of  $s$  plays a role

$$\int_{B_R} A_{\alpha'(\nabla u)_{\beta}^j}^i (\nabla u) u_{x_\beta x_\gamma}^j s |\nabla u|^{s-2} u_{x_\gamma}^i u_{x_\gamma x_\delta}^k u_{x_\delta}^k \eta^2 dx$$

it can be however estimated as follows

$$\begin{aligned} \int_{B_R} A_{\alpha'(\nabla u)_{\beta}^j}^i (\nabla u) u_{x_\beta x_\gamma}^j s |\nabla u|^{s-2} u_{x_\gamma}^i u_{x_\gamma x_\delta}^k u_{x_\delta}^k \eta^2 dx &= s \int_{B_R} \eta^2 F''(|\nabla u|) |\nabla u|^{s-4} u_{x_\gamma}^i u_{x_\beta}^j u_{x_\beta x_\gamma}^j u_{x_\alpha}^i u_{x_\delta}^k u_{x_\delta x_\alpha}^k dx \\ &\geq s \Lambda \int_{B_R} \eta^2 |\nabla u|^{p+s-2} |\nabla^2 u|^2 dx \end{aligned}$$

which allows for a following counterpart of (8)

$$\int_{B_R} A_{\alpha'(\nabla u)_{\beta}^j}^i (\nabla u) u_{x_\beta x_\gamma}^j (|\nabla u|^{s_{x_\gamma x_\alpha}} \eta^2 + s |\nabla u|^{s-2} u_{x_\gamma}^i u_{x_\gamma x_\delta}^k u_{x_\delta}^k \eta^2) dx \geq (\lambda + s \Lambda) \int_{B_R} |\nabla u|^{p-2} |\nabla^2 u|^2 |\nabla u|^s \eta^2 dx.$$

From this inequality on, one proceeds identically as in the proof of Theorem 2.  $\square$

As a consequence of Theorem 3 we obtain Theorem 1.

**Proof of Theorem 1.** Point 1 stems from the theory in [2] and it implies that in points 2 and 3 one can assume  $w > p - 1$ . The rest of the proof follows from Theorem 3. Indeed, for a  $p$ -Laplace system  $\lambda = \Lambda$ , so the ellipticity condition is fulfilled and the remaining assumptions of Theorem 3 can be rewritten with  $s_0 = \tilde{p} - M$  as follows:

$$\begin{aligned} M &:= \max(2, 2w - p + 2) \\ \nabla u &\in L^{\tilde{p}}_{\text{loc}}(\Omega_T) \\ \tilde{p} &> \max\left(M - 1, \frac{5M}{2} - \frac{3p}{2} - 2\right). \end{aligned} \quad (21)$$

In the case  $w \geq \frac{p}{2}$ :  $M = 2w - p + 2$  and (21) gives  $w < \frac{\tilde{p}+p-1}{2}$  for  $w \leq p - \frac{2}{3}$  and  $w < \frac{\tilde{p}+4p-3}{5}$  for  $w \geq p - \frac{2}{3}$ ; as in view of  $w > p - 1$  and  $\tilde{p} \geq p$  the first condition is void, we obtain assumption  $w < \frac{\tilde{p}+4p-3}{5}$ .

In the case  $w \leq \frac{p}{2}$ :  $M = 2$  and (21): for  $p \geq \frac{4}{3}$  takes the form  $\tilde{p} > 1$ , which always holds, and for  $p < \frac{4}{3}$  it reads  $\tilde{p} > 3 - \frac{3p}{2}$ . These two conditions are:  $p > 2 - \frac{2}{3}\tilde{p}$ .  $\square$

Please recall, that by point 2, merely from existence, i.e. for  $\tilde{p} = p$ , one has  $w < p - \frac{3}{5}$ .

### 3. Conclusion

#### 3.1. Note on the rigorous estimates

The above computations are formal. To perform them rigorously, transform the considered problem using Steklov averages with respect to time and use finite differences instead of differentiating it with respect to space. This procedure has been presented in [2,25] for homogeneous systems. In our case we need to deal additionally with a quasilinear right-hand side, for which the testing function  $u_{x_j}^i |\nabla u|^s \eta^2$  may not be admissible. In order to begin iteration, we need to have  $s_0 + w + 1 \leq p$  and to perform it at the  $i$ -th step:  $s_i + w + 1 \leq s_i + M$ . However, although the latter inequality holds from the definition of  $M$ , the former may sometimes prove troublesome. In such cases one can resort to testing with  $F_n(u_{x_j}^i |\nabla u|^s \eta^2)$ , where  $F_n(x)$  is a Lipschitz truncation at the level  $n$ . As the estimates are valid for every  $n$ , we can proceed as before. Observe however, that we do not encounter these difficulties during computations for Theorem 1. For additional rigorous treatment (especially for  $s$  nonpositive, consult Choe and Bae [7] and references therein<sup>4</sup> as well as Choe [26,27]).

#### 3.2. Further research

There are many possible generalizations to the result. The most obvious one is to allow for faster growths of the right-hand side at the cost of additional assumptions, especially as some of them, like boundedness of the solution, appear naturally in the existence theory. It would be interesting to obtain a general result for a critically growing right-hand side (i.e. like  $1 + |\nabla u|^p$ ), with some smallness assumption, which would generalize the classical results for the heat system mentioned in the introduction.

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<sup>4</sup> Observe, however, that in [7] it is allowed that  $s_0 > -2$ , which seems to be incorrect as far as  $s \in (-2, -1]$  are concerned.

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