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Multidimensional fixed point theorems in partially ordered complete metric spaces

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ABSTRACT

In this paper we propose a notion of *coincidence point* between mappings in any number of variables and we prove some existence and uniqueness fixed point theorems for nonlinear mappings verifying different kinds of contractive conditions and defined on partially ordered metric spaces. These theorems extend and clarify very recent results that can be found in [T. Gnana-Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (7)(2006) 1379–1393], [V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 4889–4897] and [M. Berzig, B. Samet, An extension of coupled fixed point's concept in higher dimension and applications, Comput. Math. Appl. 63 (8) (2012) 1319–1334].

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1. Introduction

The notion of *coupled fixed point* was introduced by Guo and Lakshmikantham [1] in 1987. In a recent paper, Gnana-Bhaskar and Lakshmikantham [2] introduced the concept of *mixed monotone property* for contractive operators of the form $F : X \times X \to X$, where X is a partially ordered metric space, and then established some coupled fixed point theorems. They also illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later, Lakshmikantham and Ćirić [3] proved coupled coincidence and coupled common fixed point results for nonlinear mappings satisfying certain contractive conditions in partially ordered complete metric spaces. After that many results appeared on coupled fixed point theory in different contexts (see e.g. [4,3,5–7]).

The concept of *tripled fixed point* has been introduced by Berinde and Borcut [8]. In their manuscript, some new tripled point theorems are obtained using mixed monotone mappings. Their results generalize and extend the Gnana-Bhaskar and Lakshmikantham's research for nonlinear mappings. Moreover, these results could be used to study the existence of solutions of periodic boundary value problem involving $y'' = f(t, y, y')$. In [9], Turinici proved some product fixed point theorems in ordered metric spaces involving Picard operators and normal matrices. Then, he obtained a tripled fixed point theorem in Berinde and Borcut's sense. In fact, he proposed a contractivity condition using the maximum in order to improve Berinde and Borcut's argument.

Very recently, Berzig and Samet [10] have extended and generalized the mentioned fixed point results to higher dimensions. However, they used permutations of variables and distinguished between the first and the last variables. Furthermore, it is not clear the odd-dimensional case. Some authors [11] only refer Berinde and Borcut's paper, where we cannot deduce a simple way to choose the variables, for instance, in dimension five.

In this paper, our main aim is to obtain some existence and uniqueness theorems that extend the mentioned previous results for nonlinear mappings of any number of arguments, not necessarily permuted or ordered, in the framework of

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partially ordered complete metric spaces, using a weaker contraction condition. In order to do that, we propose a notion of *coincidence point* between mappings in any number of variables. Different kinds of contractive conditions are studied and we use a distinct methodology to prove our results. When there are two, three or four variables, particular cases of these results are already known under some contractive conditions. Finally, examples to support our results are also given.

2. Preliminaries

Let n be a positive integer. Henceforth, X will denote a non-empty set and X^n will denote the product space $X^n = X \times X \times \overset{n}{\cdots} \times X$. We represent the identity mapping on X as I_X. Throughout this manuscript, m and p will denote non-negative integers and $i, j, s \in \{1, 2, \ldots, n\}$. Unless otherwise stated, "for all m" will mean "for all $m \geq 0$" and "for all i" will mean "for all $i \in \{1, 2, \ldots, n\}$".

A *metric on* X is a mapping $d : X \times X \to \mathbb{R}$ satisfying, for all $x, y, z \in X$:

$$\text{(i) } d(x, y) = 0 \quad \text{if, and only if,} \quad x = y; \qquad \text{(ii) } d(x, y) \leq d(z, x) + d(z, y).$$

From these properties we can easily deduce that $d(x, y) \geq 0$ and $d(y, x) = d(x, y)$ for all $x, y \in X$. The last requirement is called the *triangle inequality*. If d is a metric on X, we say that (X, d) is a *metric space*.

Definition 1 ([12]). A triple $(X, d, \leq)$ is called an *ordered metric space* if (X, d) is a metric space and $(X, \leq)$ is a partially ordered set.

Definition 2 ([2]). Let $g : X \to X$ be a mapping. If $(X, d, \leq)$ is an ordered metric space, then X is said to have the *sequential g-monotone property* if it verifies the following properties:

(i) If $\{x_m\}$ is a non-decreasing sequence and $\lim_{m\to\infty} x_m = x$, then $gx_m \leq gx$ for all m.
(ii) If $\{y_m\}$ is a non-increasing sequence and $\lim_{m\to\infty} y_m = y$, then $gy_m \geq gy$ for all m.

If g is the identity mapping, then X is said to have the *sequential monotone property*.

Some authors introduced the concept of *coincidence point* in different ways and with different names. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. For brevity, $g(x)$ will be denoted by gx.

Definition 3. A point $(x_1, x_2, \ldots, x_n) \in X^n$ is

- a *coupled fixed point* [2] if $n = 2$, $F(x_1, x_2) = x_1$ and $F(x_2, x_1) = x_2$.
- a *tripled fixed point* [8] if $n = 3$, $F(x_1, x_2, x_3) = x_1$, $F(x_2, x_1, x_2) = x_2$ and $F(x_3, x_2, x_1) = x_3$.
- a *quartet fixed point* [13] if $n = 4$, $F(x_1, x_2, x_3, x_4) = gx_1$, $F(x_2, x_3, x_4, x_1) = gx_2$, $F(x_3, x_4, x_1, x_2) = gx_3$ and $F(x_4, x_1, x_2, x_3) = gx_4$.

3. The mixed g-monotone property and Φ-coincidence points

The following definitions extend previous considerations from other authors [2,8,13]. Let $F : X^n \to X$ and $g : X \to X$ be two mappings.

Definition 4. We say that F and g are *commuting* if $gF(x_1, x_2, \ldots, x_n) = F(gx_1, gx_2, \ldots, gx_n)$ for all $x_1, \ldots, x_n \in X$.

Henceforth, fix a partition $\{A, B\}$ of $\Lambda_n = \{1, 2, \ldots, n\}$, that is, $A \cup B = \Lambda_n$ and $A \cap B = \varnothing$. We will denote

$$\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\}, \quad \text{and} \quad \Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\}.$$

If $(X, \leq)$ is a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, we will use the following notation

$$x \leq_i y \Leftrightarrow \begin{cases} x \leq y, & \text{if } i \in A, \\ x \geq y, & \text{if } i \in B. \end{cases}$$

Definition 5. Let $(X, \leq)$ be a partially ordered space. We say that F has the *mixed g-monotone property* if F is g-monotone non-decreasing in arguments of A and g-monotone non-increasing in arguments of B, i.e., for all $x_1, x_2, \ldots, x_n, y, z \in X$ and all i,

$$gy \leq gz \Rightarrow F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \leq_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n).$$

Henceforth, let $\sigma_1, \sigma_2, \ldots, \sigma_n, \tau : \Lambda_n \to \Lambda_n$ be $n + 1$ mappings from Λ_n into itself and let Φ be the $(n + 1)$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_n, \tau)$.

Definition 6. A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a *Φ-coincidence point of the mappings F and g* if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_{\tau(i)} \quad \text{for all } i.$$

If g is the identity mapping on X, then $(x_1, x_2, \ldots, x_n) \in X^n$ is called a *Φ-fixed point of the mapping F*.

Remark 7. If F and g are commuting and $(x_1, x_2, \ldots, x_n) \in X^n$ is a Φ-coincidence point of F and g, then $(gx_1, gx_2, \ldots, gx_n)$ also is a Φ-coincidence point of F and g.

If we represent a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, i.e., $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$, then

- Gnana-Bhaskar and Lakshmikantham's election in $n = 2$ is $\sigma_1 = \tau = (1, 2)$ and $\sigma_2 = (2, 1)$;
- Berinde and Borcut's election in $n = 3$ is $\sigma_1 = \tau = (1, 2, 3)$, $\sigma_2 = (2, 1, 2)$ and $\sigma_2 = (3, 2, 1)$;
- Karapinar's election in $n = 4$ is $\sigma_1 = \tau = (1, 2, 3, 4)$, $\sigma_2 = (2, 3, 4, 1)$, $\sigma_3 = (3, 4, 1, 2)$ and $\sigma_4 = (4, 1, 2, 3)$.

These cases consider A as the odd numbers in $\{1, 2, \ldots, n\}$ and B as its even numbers. However, Berzig and Samet [10] use $A = \{1, 2, \ldots, m\}$, $B = \{m + 1, \ldots, n\}$ and arbitrary mappings.

Example 8. The existence of Φ-coincidence points of the mappings F and g is not always true. For instance, let $X = \{a, b\}$ with $a \neq b$. For $x, y \in X$ define $x \leq y \Leftrightarrow x = y$ and the metric $d(a, b) = 1$. Therefore (X, d) is complete. Let $g = I_X$ and let $F : X^2 \to X$ be defined by $F(a, b) = F(a, a) = b$ and $F(b, a) = F(b, b) = a$. Clearly, F has the mixed monotone property and verifies condition (1) in Theorem 9. If $\tau = \sigma_1 = (1, 2)$ and $\sigma_2 = (2, 1)$, then F and g have no Φ-coincidence points.

4. Main results

Theorem 9. *Let $(X, d, \leq)$ be a complete ordered metric space. Let $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n, \tau)$ be a $(n+1)$-tuple of mappings from $\{1, 2, \ldots, n\}$ into itself such that $\tau \in \Omega_{A,B}$ is a permutation and verifying that $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property on X, $F(X^n) \subseteq g(X)$ and g commutes with F. Assume that there exists $k \in [0, 1)$ verifying*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq k \max_{1 \leq i \leq n} d(gx_i, gy_i) \tag{1}$$

for which $gx_i \leq_i gy_i$ for all i. Suppose either F is continuous or X has the sequential g-monotone property. If there exist $x_0^1, x_0^2, \ldots, x_0^n \in X$ verifying

$$gx_0^{\tau(i)} \leq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}), \quad \text{for all } i,$$

then F and g have, at least, one Φ-coincidence point.

Proof. As τ is a permutation of $\{1, 2, \ldots, n\}$, then $\{\tau(1), \tau(2), \ldots, \tau(n)\} = \{1, 2, \ldots, n\}$, i.e., $\{\tau(i)\}_{i=1}^n = \{i\}_{i=1}^n$. We also observe that, since $\tau \in \Omega_{A,B}$, then $\tau(i) \in A$ (respectively, $\tau(i) \in B$) if, and only if, $i \in A$ (respectively, $i \in B$). The proof is divided in six steps.

Step 1. *Definition of the sequences* $\{x_m^1\}_{m\geq 0}$, $\{x_m^2\}_{m\geq 0}, \ldots, \{x_m^n\}_{m\geq 0}$. Since $F(X^n) \subseteq g(X)$, we can choose $x_1^1, x_1^2, \ldots, x_1^n \in X$ such that $gx_1^{\tau(i)} = F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)})$ for all i. Again, from $F(X^n) \subseteq g(X)$, we can choose $x_2^1, x_2^2, \ldots, x_2^n \in X$ such that $gx_2^{\tau(i)} = F(x_1^{\sigma_i(1)}, x_1^{\sigma_i(2)}, \ldots, x_1^{\sigma_i(n)})$ for all i. Continuing this process, we can consider sequences $\{x_m^1\}_{m\geq 0}, \{x_m^2\}_{m\geq 0}, \ldots, \{x_m^n\}_{m\geq 0}$ such that $gx_{m+1}^{\tau(i)} = F(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_i(n)})$ for all m and all i.

Step 2. *We claim that* $gx_m^i \leq_i gx_{m+1}^i$ for all m and all i. (2)

We prove it by induction methodology in $m \geq 0$. For $m = 0$, we have $gx_0^{\tau(i)} \leq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}) = gx_1^{\tau(i)}$ for all i (and $\tau(i)$ and i take all values in A or B at the same time). Suppose that (2) is true for m and we are going to prove it for $m + 1$. The induction hypothesis is

$$gx_m^j \leq_j gx_{m+1}^j \quad \text{for all } j \Leftrightarrow \begin{cases} gx_m^j \leq gx_{m+1}^j, & \text{if } j \in A, \\ gx_m^j \geq gx_{m+1}^j, & \text{if } j \in B. \end{cases} \tag{3}$$

Now we want to prove that $gx_{m+1}^{\tau(i)} \leq_i gx_{m+2}^{\tau(i)}$ for all i, i.e., $gx_{m+1}^{\tau(i)} \leq gx_{m+2}^{\tau(i)}$ if $i \in A$ and $gx_{m+1}^{\tau(i)} \geq gx_{m+2}^{\tau(i)}$ if $i \in B$. Therefore, we have to distinguish between whether $i \in A$ or $i \in B$. Suppose that $i \in A$. In this case, $\tau(i) \in A$. As F has the mixed g-monotone property, we apply that F is g-monotone non-decreasing in A-arguments with the first inequalities of (3) and deduce that, for all $a_1, a_2, \ldots, a_n \in X$:

$$gx_m^j \leq gx_{m+1}^j \Rightarrow F(a_1, \ldots, a_{s-1}, x_m^j, a_{s+1}, \ldots, a_n) \leq F(a_1, \ldots, a_{s-1}, x_{m+1}^j, a_{s+1}, \ldots, a_n), \quad \text{if } j, s \in A,$$

and that F is g-monotone non-increasing in B-arguments with the second inequalities of (3):

$$gx_m^j \geq gx_{m+1}^j \Rightarrow F(a_1,\ldots,a_{s-1},x_{m+1}^j,a_{s+1},\ldots,a_n) \leq F(a_1,\ldots,a_{s-1},x_m^j,a_{s+1},\ldots,a_n), \quad \text{if } j,s\in B.$$

This means that, if $j,s\in\{1,2,\ldots,n\}$ verify $j,s\in A$ or $j,s\in B$, then

$$F(a_1,\ldots,a_{s-1},x_m^j,a_{s+1},\ldots,a_n) \leq F(a_1,\ldots,a_{s-1},x_{m+1}^j,a_{s+1},\ldots,a_n)$$

for all $a_1,a_2,\ldots,a_n\in X$. As $\sigma_i\in\Omega_{A,B}$:

$$\begin{aligned} gx_{m+1}^{\tau(i)} &= F(x_m^{\sigma_i(1)},x_m^{\sigma_i(2)},x_m^{\sigma_i(3)},\ldots,x_m^{\sigma_i(n)}) \leq (1,\sigma_i(1)\in A \text{ or } 1,\sigma_i(1)\in B)\\ &\leq F(x_{m+1}^{\sigma_i(1)},x_m^{\sigma_i(2)},x_m^{\sigma_i(3)},\ldots,x_m^{\sigma_i(n)}) \leq (2,\sigma_i(2)\in A \text{ or } 2,\sigma_i(2)\in B)\\ &\leq F(x_{m+1}^{\sigma_i(1)},x_{m+1}^{\sigma_i(2)},x_m^{\sigma_i(3)},\ldots,x_m^{\sigma_i(n)}) \leq (3,\sigma_i(3)\in A \text{ or } 3,\sigma_i(3)\in B)\\ &\leq \cdots \leq F(x_{m+1}^{\sigma_i(1)},x_{m+1}^{\sigma_i(2)},x_{m+1}^{\sigma_i(3)},\ldots,x_{m+1}^{\sigma_i(n)}) = gx_{m+2}^{\tau(i)}. \end{aligned}$$

Hence $gx_{m+1}^{\tau(i)}\leq gx_{m+2}^{\tau(i)}$ when i (or $\tau(i)$) is in A and (2) is true if $i\in A$. Now suppose that $i\in B$ (so $\tau(i)\in B$). In this case, we apply that F is g-monotone non-decreasing in A-arguments with the second inequalities of (3) and that F is g-monotone non-increasing in B-arguments with the first inequalities of (3), and we deduce, for all $a_1,a_2,\ldots,a_n\in X$, that, if $j,s\in\{1,2,\ldots,n\}$ verify $j\in A, s\in B$ or $j\in B, s\in A$, then

$$F(a_1,\ldots,a_{s-1},x_m^j,a_{s+1},\ldots,a_n) \geq F(a_1,\ldots,a_{s-1},x_{m+1}^j,a_{s+1},\ldots,a_n).$$

Since $\sigma_i\in\Omega'_{A,B}$, therefore

$$\begin{aligned} gx_{m+1}^{\tau(i)} &= F(x_m^{\sigma_i(1)},x_m^{\sigma_i(2)},x_m^{\sigma_i(3)},\ldots,x_m^{\sigma_i(n)}) \geq (1\in A,\sigma_i(1)\in B \text{ or } 1\in B,\sigma_i(1)\in A)\\ &\geq F(x_{m+1}^{\sigma_i(1)},x_m^{\sigma_i(2)},x_m^{\sigma_i(3)},\ldots,x_m^{\sigma_i(n)}) \geq (2\in A,\sigma_i(2)\in B \text{ or } 2\in B,\sigma_i(2)\in A)\\ &\leq F(x_{m+1}^{\sigma_i(1)},x_{m+1}^{\sigma_i(2)},x_m^{\sigma_i(3)},\ldots,x_m^{\sigma_i(n)}) \geq (3\in A,\sigma_i(3)\in B \text{ or } 3\in B,\sigma_i(3)\in A)\\ &\geq \cdots \geq F(x_{m+1}^{\sigma_i(1)},x_{m+1}^{\sigma_i(2)},x_{m+1}^{\sigma_i(3)},\ldots,x_{m+1}^{\sigma_i(n)}) = gx_{m+2}^{\tau(i)}. \end{aligned}$$

Hence $gx_{m+1}^{\tau(i)}\geq gx_{m+2}^{\tau(i)}$ when $i\in B$, so $gx_{m+1}^{\tau(i)}\leq_i gx_{m+2}^{\tau(i)}$ and (2) is true in any case.

Now define $\delta_{mp}=\max_{1\leq j\leq n} d(gx_m^j,gx_p^j)=\max_{1\leq j\leq n} d(gx_m^{\tau(j)},gx_p^{\tau(j)})$ for all $p>m\geq 0$.

Step 3. *We claim that* $d(gx_m^i,gx_{m+p}^i)\leq\delta_{m(m+p)}\leq\dfrac{k^m}{1-k}\delta_{01}$ for all $m\geq 0$, all $p\geq 1$ and all i. (4)

As $gx_m^{\tau(i)}\leq_i gx_{m+1}^{\tau(i)}$ for all m and all i, then condition (1) implies that, for all $m\geq 1$ and all i:

$$\begin{aligned} d(gx_m^{\tau(i)},gx_{m+1}^{\tau(i)}) &= d(F(x_{m-1}^{\sigma_i(1)},x_{m-1}^{\sigma_i(2)},\ldots,x_{m-1}^{\sigma_i(n)}),F(x_m^{\sigma_i(1)},x_m^{\sigma_i(2)},\ldots,x_m^{\sigma_i(n)}))\\ &\leq k\max_{1\leq j\leq n} d(gx_{m-1}^{\sigma_i(j)},gx_m^{\sigma_i(j)}) \leq k\max_{1\leq j\leq n} d(gx_{m-1}^j,gx_m^j) = k\,\delta_{(m-1)m}. \end{aligned}$$

As $\{\tau(i)\}_{i=1}^n=\{i\}_{i=1}^n$, taking maximum, $\delta_{m(m+1)}=\max_{1\leq j\leq n} d(gx_m^j,gx_{m+1}^j)\leq k\,\delta_{(m-1)m}$. Repeating this process,

$$d(gx_m^j,gx_{m+1}^i)\leq\delta_{m(m+1)}\leq k\,\delta_{(m-1)m}\leq k^2\,\delta_{(m-2)(m-1)}\leq\cdots\leq k^m\,\delta_{01} \quad \text{for all } m \text{ and all } i. \tag{5}$$

As $1\leq 1/(1-k)$, for all i and all m

$$d(gx_m^i,gx_{m+1}^i)\leq\delta_{m(m+1)}\leq k^m\,\delta_{01}\leq\frac{k^m}{1-k}\,\delta_{01},$$

so (4) is true if $p=1$. By the triangle inequality and (5):

$$\begin{aligned} d(gx_m^j,gx_{m+p}^i) &\leq d(gx_m^j,gx_{m+1}^i)+d(gx_{m+1}^j,gx_{m+2}^i)+\cdots+d(gx_{m+p-1}^i,gx_{m+p}^i)\\ &\leq k^m\,\delta_{01}+k^{m+1}\,\delta_{01}+\cdots+k^{m+p-1}\,\delta_{01}=k^m(1+k+\cdots+k^{p-1})\delta_{01}\\ &= k^m\frac{1-k^p}{1-k}\delta_{01}\leq\frac{k^m}{1-k}\delta_{01}. \end{aligned}$$

Step 4. *Every sequence* $\{gx_m^i\}_{m\geq 0}$ *is Cauchy*. Let $\varepsilon>0$. As $\{k^m\}\searrow 0$, we can find $m_0\in\mathbb{N}$ such that $(k^{m_0}\,\delta_{01})/(1-k)<\varepsilon$. If $m\geq m_0$ and $p\geq 1$, then

$$d(gx_m^j,gx_{m+p}^i)\leq\frac{k^m}{1-k}\delta_{01}\leq\frac{k^{m_0}}{1-k}\delta_{01}<\varepsilon,$$

so $\{gx_m^i\}_{m\geq 0}$ is Cauchy. As X is d-complete, there exist $x_1, x_2, \ldots, x_n \in X$ such that $x_i = \lim_{m\to\infty} gx_m^i$ for all i (in other words, $x_{\tau(i)} = \lim_{m\to\infty} gx_m^{\tau(i)}$ for all i). As g is continuous, $gx_{\tau(i)} = \lim_{m\to\infty} ggx_{m+1}^{\tau(i)}$, but

$$ggx_{m+1}^{\tau(i)} = gF(x_m^{\sigma_i(1)}, x_m^{\sigma_i(2)}, \ldots, x_m^{\sigma_n(n)}) = F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)}). \tag{6}$$

Step 5. *Suppose that F is continuous*. Letting $m \to \infty$ in (6), we find, for all i,

$$gx_{\tau(i)} = \lim_{m\to\infty} ggx_{m+1}^{\tau(i)} = \lim_{m\to\infty} F(gx_m^{\sigma_i(1)}, gx_m^{\sigma_i(2)}, \ldots, gx_m^{\sigma_i(n)}) = F(x_{\sigma_i(1)}, gx_{\sigma_i(2)}, \ldots, gx_{\sigma_i(n)}),$$

so $(x_1, x_2, \ldots, x_n)$ is a Φ-coincidence point of F and g.

Step 6. *Suppose that X has the sequential g-monotone property*. In this case, by (2) we know that $gx_m^i \leq_i gx_{m+1}^i$ for all m and all i. This means that the sequence $\{gx_m^i\}_{m\geq 0}$ is monotone. As $x_i = \lim_{m\to\infty} gx_m^i$, we deduce that $ggx_m^i \leq_i gx_i$ for all m and all i. This condition implies that, for all m and all j,

$$\text{either } \left[ggx_m^{\sigma_j(i)} \leq_i gx_{\sigma_j(i)} \text{ for all } i\right] \quad \text{or} \quad \left[gx_{\sigma_j(i)} \leq_i ggx_m^{\sigma_j(i)} \text{ for all } i\right]$$

(the first case occurs when $j \in A$ and the second one when $j \in B$). Then by (1), for all j

$$d(F(gx_m^{\sigma_j(1)}, gx_m^{\sigma_j(2)}, \ldots, gx_m^{\sigma_j(n)}), F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(n)})) \leq k \max_{1\leq i\leq n} d(ggx_m^{\sigma_j(i)}, gx_{\sigma_j(i)}).$$

Letting $m \to \infty$, we deduce, by (6)

$$F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(n)}) = \lim_{m\to\infty} F(gx_m^{\sigma_j(1)}, gx_m^{\sigma_j(2)}, \ldots, gx_m^{\sigma_j(n)}) = \lim_{m\to\infty} ggx_{m+1}^{\tau(j)} = gx_{\tau(j)}$$

for all j, so $(x_1, x_2, \ldots, x_n)$ is a Φ-coincidence point of F and g. □

The following corollary is almost the main result in [10] when g is the identity mapping on X.

Corollary 10. *Theorem 9 also is true if we replace condition (1) by*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} \alpha_i d(gx_i, gy_i) \quad \text{for which } gx_i \leq_i gy_i \text{ for all } i,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1)$ verify $\alpha_1 + \alpha_2 + \cdots + \alpha_n < 1$.

Proof. If $k = \alpha_1 + \alpha_2 + \cdots + \alpha_n < 1$, then

$$\begin{aligned} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) &\leq \sum_{i=1}^{n} \alpha_i d(gx_i, gy_i) \leq \sum_{i=1}^{n} \alpha_i \max_{1\leq j\leq n} d(gx_j, gy_j) \\ &= k \max_{1\leq i\leq n} d(gx_i, gy_i). \quad \square \end{aligned}$$

5. Uniqueness of Φ-coincidence points

The unicity of the Φ-coincidence point of F and g in Theorem 9 is not always true. For instance, if $F \equiv x_0$ is constant and $g \equiv x_0$ also is constant, then every $(x_1, x_2, \ldots, x_n) \in X^n$ is a Φ-coincidence point of F and g. However, we are going to study some conditions in order to determine a special Φ-coincidence point or to guarantee that this point is unique.

Consider on the product space X^n the following partial order: for $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$,

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \Leftrightarrow x_i \leq_i y_i, \quad \text{for all } i.$$

We say that $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are *comparable* if $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$ or $(x_1, x_2, \ldots, x_n) \geq (y_1, y_2, \ldots, y_n)$.

Theorem 11. *Under the hypothesis of Theorem 9, assume that for all Φ-coincidence points $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ of F and g there exists $(u_1, u_2, \ldots, u_n) \in X^n$ such that $(gu_1, gu_2, \ldots, gu_n)$ is comparable, at the same time, to $(gx_1, gx_2, \ldots, gx_n)$ and to $(gy_1, gy_2, \ldots, gy_n)$.*

Then F and g have a unique Φ-coincidence point $(z_1, z_2, \ldots, z_n) \in X^n$ such that $gz_i = z_i$ for all i.

Proof. From Theorem 9, the set of Φ-coincidence points of F and g is non-empty. The proof is divided in two steps.

Step 1. *We claim that if* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ *are two* Φ*-coincidence points of* F *and* g, *then*

$$gx_i = gy_i \quad \text{for all } i. \tag{7}$$

Let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ be two Φ-coincidence points of F and g and let $(u_1, u_2, \ldots, u_n) \in X^n$ be a point such that $(gu_1, gu_2, \ldots, gu_n)$ is comparable, at the same time, to $(gx_1, gx_2, \ldots, gx_n)$ and to $(gy_1, gy_2, \ldots, gy_n)$. Using $(u_1, u_2, \ldots, u_n)$ define the following sequences. Let $u_0^i = u_i$ for all i. Reasoning as in Theorem 9, we can determine sequences $\{u_m^1\}_{m\geq 0}, \{u_m^2\}_{m\geq 0}, \ldots, \{u_m^n\}_{m\geq 0}$ such that $gu_{m+1}^{\tau(i)} = F(u_m^{\sigma_i(1)}, u_m^{\sigma_i(2)}, \ldots, u_m^{\sigma_i(n)})$ for all m and all i. We are going to prove that $gx_i = \lim_{m\to 0} gu_m^i = gy_i$ for all i, so (7) will be true.

First, we reason with $(gu_1, gu_2, \ldots, gu_n)$ and $(gx_1, gx_2, \ldots, gx_n)$, and the same argument will be true for $(gu_1, gu_2, \ldots, gu_n)$ and $(gy_1, gy_2, \ldots, gy_n)$. As $(gu_1, gu_2, \ldots, gu_n)$ and $(gx_1, gx_2, \ldots, gx_n)$ are comparable, we can suppose that $(gu_1, gu_2, \ldots, gu_n) \leq (gx_1, gx_2, \ldots, gx_n)$ (the other case is similar), i.e., $gu_i \leq_i gx_i$ for all i. Since $i, \tau(i) \in A$ or $i, \tau(i) \in B$, then $gu_{\tau(i)} \leq_i gx_{\tau(i)}$ for all i, which is equivalent to $gu_0^{\tau(i)} \leq_i gx_{\tau(i)}$ for all i. Using that F has the mixed g-monotone property and reasoning as in Theorem 9, it is possible to prove that

$$gu_m^i \leq_i gx_i \quad (\text{or equivalently, } gu_m^{\tau(i)} \leq_i gx_{\tau(i)}) \quad \text{for all } m \geq 1 \text{ and all } i. \tag{8}$$

This condition implies that, for all j and all $m \geq 1$

$$\text{either } \left[gu_m^{\sigma_j(i)} \leq_i gx_{\sigma_j(i)} \text{ for all } i\right] \quad \text{or} \quad \left[gx_{\sigma_j(i)} \leq_i gu_m^{\sigma_j(i)} \text{ for all } i\right].$$

If we define $\beta_m = \max_{1\leq i\leq n} d(gu_m^i, gx_i)$ for all m, we can deduce from (1) that, for all j and all m,

$$\begin{aligned} d(gu_{m+1}^{\tau(j)}, gx_{\tau(j)}) &= d(F(u_m^{\sigma_j(1)}, u_m^{\sigma_j(2)}, \ldots, u_m^{\sigma_j(n)}), F(x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(n)})) \\ &\leq k \max_{1\leq i\leq n} d(gu_m^{\sigma_j(i)}, gx_{\sigma_j(i)}) \leq k \max_{1\leq i\leq n} d(gu_m^i, gx_i) = k\beta_m. \end{aligned} \tag{9}$$

Taking maximum in j, we deduce $\beta_{m+1} = \max_{1\leq j\leq n} d(gu_{m+1}^{\tau(j)}, gx_{\tau(j)}) \leq k\beta_m$ for all m. Repeating this process, $\beta_{m+1} \leq k\beta_m \leq k^2\beta_{m-1} \leq \cdots \leq k^{m+1}\beta_0$ for all m. As $\{k^m\} \to 0$, we deduce that $\{\beta_m\} \to 0$. Letting $m \to \infty$ in (9), we deduce $\lim_{m\to\infty} d(gu_{m+1}^{\tau(j)}, gx_{\tau(j)}) = 0$ for all j, i.e.,

$$\lim_{m\to\infty} gu_m^i = gx_i \quad \text{for all } i. \tag{10}$$

If we had supposed that $(gx_1, gx_2, \ldots, gx_n) \leq (gu_1, gu_2, \ldots, gu_n)$, we would have obtained the same property (10). In addition, as $(gu_1, gu_2, \ldots, gu_n)$ also is comparable to $(gy_1, gy_2, \ldots, gy_n)$, we can prove that $gy_i = \lim_{m\to 0} gu_m^i = gx_i$ for all i reasoning in the same way.

Let $(x_1, x_2, \ldots, x_n) \in X^n$ be a Φ-coincidence point of F and g and define $z_i = gx_i$ for all i. As $(z_1, z_2, \ldots, z_n) = (gx_1, gx_2, \ldots, gx_n)$, Remark 7 ensures us that $(z_1, z_2, \ldots, z_n)$ also is a Φ-coincidence point of F and g.

Step 2. *We claim that* $(z_1, z_2, \ldots, z_n)$ *is the unique* Φ*-coincidence point of* F *and* g *such that* $gz_i = z_i$ *for all* i. Indeed, by step 1 we observe that $gz_i = gx_i = z_i$ for all i. Suppose that $(z_1', z_2', \ldots, z_n') \in X^n$ is another Φ-coincidence point of F and g such that $gz_i' = z_i'$ for all i. As $(z_1, z_2, \ldots, z_n)$ and $(z_1', z_2', \ldots, z_n')$ are two Φ-coincidence point of F and g, by step 1 we deduce that $gz_i = gz_i'$ for all i, so $z_i = gz_i = gz_i' = z_i'$ for all i. Therefore, $(z_1, z_2, \ldots, z_n)$ is the unique Φ-coincidence point of F and g such that $gz_i = z_i$ for all i. □

Remark 12. For $(u_1, u_2, \ldots, u_n) \in X^n$, denote $F(u_{\sigma_i(1)}, u_{\sigma_i(2)}, \ldots, u_{\sigma_i(n)})$ by $\mathcal{U}_{\tau(i)}$ for all i. Then the previous theorem also is true if we suppose that for all Φ-coincidence points $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ of F and g there exists $(u_1, u_2, \ldots, u_n) \in X^n$ such that $(\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n)$ is comparable, at the same time, to $(gx_1, gx_2, \ldots, gx_n)$ and to $(gy_1, gy_2, \ldots, gy_n)$. Indeed, we observe that $gu_1^{\tau(i)} = F(u_0^{\sigma_i(1)}, u_0^{\sigma_i(2)}, \ldots, u_0^{\sigma_i(n)}) = F(u_{\sigma_i(1)}, u_{\sigma_i(2)}, \ldots, u_{\sigma_i(n)}) = \mathcal{U}_{\tau(i)} \leq_i gx_{\tau(i)}$ for all i, and the induction process is allowed to start.

Remark 13. If $\tau = (2, 1)$, $\sigma_1 = (1, 1)$ and $\sigma_2 = (2, 2)$ in Example 8, then (a, b) and (b, a) are Φ-coincidence points of F and g, but they are not of the kind (z, z).

It is natural to say that g is *injective* on the set of all Φ-coincidence points of F and g when $gx_i = gy_i$ for all i implies $x_i = y_i$ for all i when $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$ are two Φ-coincidence points of F and g. For example, this is true is g is injective on X.

Corollary 14. *In addition to the hypotheses of Theorem* 11, *suppose that* g *is injective on the set of all* Φ*-coincidence points of* F *and* g. *Then* F *and* g *have a unique* Φ*-coincidence point.*

Proof. If $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are two coincidence points of F and g, we have proved in (7) that $gx_i = gy_i$ for all i. As g is injective on these points, then $x_i = y_i$ for all i. □

Corollary 15. *In addition to the hypotheses of Theorem 11, suppose that* $(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)})$ *is comparable to* $(z_{\sigma_j(1)}, z_{\sigma_j(2)}, \ldots, z_{\sigma_j(n)})$ *for all* i, j. *Then* $z_1 = z_2 = \cdots = z_n$.

In particular, there exists a unique $z \in X$ *such that* $F(z, z, \ldots, z) = z$, *which verifies* $gz = z$.

Proof. Let $M = \max_{1\le i,j\le n} d(z_i, z_j) = \max_{1\le i,j\le n} d(z_{\tau(i)}, z_{\tau(j)})$. Fix $j, s \in \{1, 2, \ldots, n\}$. As $(z_{\sigma_j(1)}, z_{\sigma_j(2)}, \ldots, z_{\sigma_j(n)})$ is comparable to $(z_{\sigma_s(1)}, z_{\sigma_s(2)}, \ldots, z_{\sigma_s(n)})$, then either $z_{\sigma_j(i)} \le_i z_{\sigma_s(i)}$ for all i or $z_{\sigma_s(i)} \le_i z_{\sigma_j(i)}$ for all i. Since $gz_i = z_i$ for all i, we know that either $gz_{\sigma_j(i)} \le_i gz_{\sigma_s(i)}$ for all i or $gz_{\sigma_s(i)} \le_i gz_{\sigma_j(i)}$ for all i. In any case, applying (1),

$$\begin{aligned} d(z_{\tau(j)}, z_{\tau(s)}) &= d(gz_{\tau(j)}, gz_{\tau(s)}) = d(F(z_{\sigma_j(1)}, z_{\sigma_j(2)}, \ldots, z_{\sigma_j(n)}), F(z_{\sigma_s(1)}, z_{\sigma_s(2)}, \ldots, z_{\sigma_s(n)})) \\ &\le k \max_{1\le i\le n} d(gz_{\sigma_j(i)}, gz_{\sigma_s(i)}) \le k \max_{1\le i,j\le n} d(z_i, z_j) = kM. \end{aligned}$$

Taking the supremum on j and s, we deduce $0 \le M \le kM$, so $M = 0$ and $z_i = z_j$ for all i, j. □

6. Consequences

We particularize Theorems 9 and 11 and Corollaries 10 and 14 assuming that g is the identity mapping on X.

Corollary 16. *Let* $(X, d, \le)$ *be a complete ordered metric space. Let* $\Phi = (\sigma_1, \sigma_2, \ldots, \sigma_n, \tau)$ *be a* $(n+1)$*-tuple of mappings from* $\{1, 2, \ldots, n\}$ *into itself such that* $\tau \in \Omega_{A,B}$ *is a permutation and verifying that* $\sigma_i \in \Omega_{A,B}$ *if* $i \in A$ *and* $\sigma_i \in \Omega'_{A,B}$ *if* $i \in B$. *Let* $F : X^n \to X$ *be a mapping having the mixed monotone property on* X. *Assume that* F *verifies some of the following conditions.*

(a) *There exists* $k \in [0, 1)$ *verifying*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \le k \max_{1\le i\le n} d(x_i, y_i) \quad \text{for which } x_i \le_i y_i \text{ for all } i.$$

(b) *There exist* $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1)$ *such that* $\alpha_1 + \alpha_2 + \cdots + \alpha_n < 1$ *and*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \le \sum_{i=1}^{n} \alpha_i d(x_i, y_i) \quad \text{for which } x_i \le_i y_i \text{ for all } i.$$

Suppose either F *is continuous or* X *has the sequential monotone property. If there exist* $x_0^1, x_0^2, \ldots, x_0^n \in X$ *verifying*

$$x_0^{\tau(i)} \le_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \ldots, x_0^{\sigma_i(n)}), \quad \text{for all } i,$$

then there exist $(z_1, z_2, \ldots, z_n) \in X^n$ *such that*

$$F(z_{\sigma_i(1)}, z_{\sigma_i(2)}, \ldots, z_{\sigma_i(n)}) = z_{\tau(i)} \quad \text{for all } i. \tag{11}$$

Additionally, let Δ *be the subset of all points of* X^n *verifying (11) and suppose that for all* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \Delta$ *there exists* $(u_1, u_2, \ldots, u_n) \in X^n$ *such that* $(u_1, u_2, \ldots, u_n)$ *(or* $(\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n)$*) is comparable, at the same time, to* $(x_1, x_2, \ldots, x_n)$ *and to* $(y_1, y_2, \ldots, y_n)$. *Then* Δ *is reduced to a single point.*

Example 17. If $X = \mathbb{R}$ provided with its usual order, $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ and $A_1, A_2, \ldots, A_n, B, M \in \mathbb{R}$ are such that $M > |A_1| + |A_2| + \cdots + |A_n|$, the mapping $F : \mathbb{R}^n \to \mathbb{R}$, defined as $F(x_1, x_2, \ldots, x_n) = (A_1x_1 + A_2x_2 + \cdots + A_nx_n + B)/M$ for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$, verifies the hypothesis of Corollary 16(b). It is easy to check that $(x_0, x_0, \ldots, x_0)$, where $x_0 = B/(M - A_1 - A_2 - \cdots - A_n)$, is the unique Φ-coincidence point of F and the identity mapping on X and verifies $F(x_0, x_0, \ldots, x_0) = x_0$.

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Appendix. Applications

As usual in other papers, throughout this Appendix subscripts are used to indicate the arguments of the function. For instance, $F(x_1, x_2, \ldots, x_n)$ will be denoted by $F_{x_1,x_2,\ldots,x_n}$.

A.1. Application 1

Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be *Lipschitzian* if there exists $k \geq 0$ such that $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$. The smallest k (denoted by k_f) for which this inequality holds is said to be the *Lipschitz constant for* f. If $X = \mathbb{R}$ provided with the Euclidean metric, examples of Lipschitzian mapping $f_i : \mathbb{R} \to \mathbb{R}$ are the following:

- $f_1(x) = K, \quad k_{f_1} = 0$
- $f_2(x) = \alpha\, x, \quad k_{f_2} = \alpha$
- $f_3(x) = \sin x, \quad k_{f_3} = 1$
- $f_4(x) = \cos x, \quad k_{f_4} = 1$
- $f_5(x) = \ln(|x| + 1), \quad k_{f_5} = 1$
- $f_6(x) = \arctan x, \quad k_{f_6} = 1$
- $f_7(x) = \dfrac{1}{1+x^2}, \quad k_{f_7} = \dfrac{3\sqrt{3}}{8}.$

A Lipschitzian mapping $f : X \to X$ is a *contraction* if $k_f < 1$. The *Banach Contraction Principle* ensures that every contraction from a complete metric space into itself has a unique fixed point.

Let $\{f_i : \mathbb{R} \to \mathbb{R}\}_{i=1}^n$ be a family of Lipschitzian mappings. Let $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$ be real numbers and define $h : \mathbb{R} \to \mathbb{R}$ as

$$h(x) = \sum_{i=1}^{n} \beta_i f_i(x), \quad \text{for all } x \in \mathbb{R}.$$

Then h is another Lipschitzian mapping and $k_h \leq \sum_{i=1}^n |\beta_i|\, k_{f_i}$. Obviously, if $K = \sum_{i=1}^n |\beta_i|\, k_{f_i} < 1$, then h is a contraction, so there exists a unique $x_0 \in \mathbb{R}$ such that $h_{x_0} = x_0$.

Next, define $F : \mathbb{R}^n \to \mathbb{R}$ as

$$F_{x_1 x_2 \cdots x_n} = \sum_{i=1}^{n} \beta_i f_i(x_i), \quad \text{for all } x_1, x_2, \ldots, x_n \in \mathbb{R}. \tag{A.1}$$

It is clear that $F_{xx\cdots x} = h_x$ for all $x \in \mathbb{R}$. Furthermore

$$\begin{aligned} d(F_{x_1 x_2 \cdots x_n}, F_{y_1 y_2 \cdots y_n}) &= \left| \sum_{i=1}^{n} \beta_i (f_i(x_i) - f_i(y_i)) \right| \leq \sum_{i=1}^{n} |\beta_i|\, |f_i(x_i) - f_i(y_i)| \\ &\leq \sum_{i=1}^{n} |\beta_i|\, k_{f_i}\, |x_i - y_i| \leq K \max_{1 \leq j \leq n} d(x_j, y_j). \end{aligned}$$

If $K < 1$, then F verifies (1) with $gx = x$ for all $x \in \mathbb{R}$.

Corollary 18. *Let $\{f_i : \mathbb{R} \to \mathbb{R}\}_{i=1}^n$ be a family of Lipschitzian mappings on $\mathbb{R}$ (provided with the Euclidean metric) and let $\{\beta_i\}_{i=1}^n$ be a family of real numbers such that $\sum_{i=1}^n |\beta_i|\, k_{f_i} < 1$. Then the system*

$$(S) \quad \begin{cases} \beta_1 f_1(x_1) + \beta_2 f_2(x_2) + \cdots + \beta_n f_n(x_n) = x_1, \\ \beta_1 f_1(x_2) + \beta_2 f_2(x_3) + \cdots + \beta_n f_n(x_1) = x_2, \\ \vdots \\ \beta_1 f_1(x_n) + \beta_2 f_2(x_1) + \cdots + \beta_n f_n(x_{n-1}) = x_n, \end{cases}$$

has a unique solution, which is $(x_0, x_0, \ldots, x_0)$, where x_0 is the only real solution of

$$\beta_1 f_1(x) + \beta_2 f_2(x) + \cdots + \beta_n f_n(x) = x.$$

Example 19. Consider the system

$$(S_1) \quad \begin{cases} 180x + 36 \sin y - \dfrac{40}{1+z^2} + 75 \arctan t - 1260 = 0, \\ 180y + 36 \sin z - \dfrac{40}{1+t^2} + 75 \arctan x - 1260 = 0, \\ 180z + 36 \sin t - \dfrac{40}{1+x^2} + 75 \arctan y - 1260 = 0, \\ 180t + 36 \sin x - \dfrac{40}{1+y^2} + 75 \arctan z - 1260 = 0. \end{cases}$$

If we choose $f_1(x) = 1, f_2(x) = \sin x$, $f_3(x) = 1/(1 + x^2)$ and $f_4(x) = \arctan x$, then f_1, f_2, f_3 and f_4 are Lipschitzian mappings and $k_{f_1} = 0, k_{f_2} = k_{f_4} = 1$ and $k_{f_3} = 3\sqrt{3}/8$. Let $\beta_1 = 7, \beta_2 = -1/5, \beta_3 = 2/9$ and $\beta_4 = -5/12$. Then

$|\beta_1|\,k_{f_1}+|\beta_2|\,k_{f_2}+|\beta_3|\,k_{f_3}+|\beta_4|\,k_{f_4}=(37+5\sqrt{3})/60<1$. As system (S_1) is equal to (S), then (S_1) has a unique solution, which is of the form $(x_0,x_0,\dots,x_0)$, where x_0 is the only solution of

$$180x+36\sin x-\frac{40}{1+x^2}+75\arctan x-1260=0.$$

Approximately, $x_0=6.393447506802$.

A.2. Application 2

Let $a,b\in\mathbb{R}$ with $a<b$ and let $I=[a,b]$. Consider $X=\mathcal{L}^1(I)$ with the distance

$$d_1(f,g)=\int_I |f(t)-g(t)|\,dt,$$

where $\int$ represents Lebesgue's integral. It is well-known that $(\mathcal{L}^1(I),d_1)$ is a complete MS. Let $k,\beta_1,\beta_2,\dots,\beta_n\in\mathbb{R}$ be real numbers and let $G:\mathbb{R}^n\to\mathbb{R}$ be a mapping verifying $G(0,0,\dots,0)=0$ and

$$\left|G_{x_1x_2\cdots x_n}-G_{y_1y_2\cdots y_n}\right|\le k\sum_{i=1}^{n}\beta_i\,|x_i-y_i|\quad\text{for all }(x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$$

(a simple way to consider a kind of function like this is (A.1)). If $A\in\mathbb{R}$, is it possible to find functions $f_1,f_2,\dots,f_n\in\mathcal{L}^1(I)$ such that

$$\left\{f_i(x)=A+\int_a^x G(f_i(t),f_{i+1}(t),\dots,f_{i-1}(t))dt,\quad\text{for all } x\in I\right\}_{i=1}^{n}\ ? \tag{A.2}$$

Define $F:\mathcal{L}^1(I)^n\to\mathcal{F}(I)$ for all $f_1,f_2,\dots,f_n\in\mathcal{L}^1(I)$ and all $x\in I$ as

$$F_{f_1f_2\cdots f_n}(x)=A+\int_a^x G(f_1(t),f_2(t),\dots,f_n(t))dt.$$

On the one hand,

$$\begin{aligned}
\int_I\left|F_{f_1f_2\cdots f_n}(x)\right|dx&=\int_I\left|A+\int_a^x G(f_1(t),f_2(t),\dots,f_n(t))dt\right|dx\\
&\le\int_I\left(|A|+\left|\int_a^x G(f_1(t),f_2(t),\dots,f_n(t))dt\right|\right)dx\le\\
&=|A|\,(b-a)+\int_I\left(\int_a^x|G(f_1(t),f_2(t),\dots,f_n(t))-G(0,0,\dots,0)|\,dt\right)dx\\
&\le|A|\,(b-a)+\int_I\left(\int_a^x k\sum_{i=1}^{n}\beta_i\,|f_i(t)-0|\;dt\right)dx\\
&=|A|\,(b-a)+k\sum_{i=1}^{n}\beta_i\int_I\left(\int_a^x|f_i(t)|\;dt\right)dx\\
&=|A|\,(b-a)+k\sum_{i=1}^{n}\beta_i\int_I\left(\int_I|f_i(t)|\;dt\right)dx\\
&=|A|\,(b-a)+k\sum_{i=1}^{n}\beta_i\int_I\|f_i\|_1\,dx=|A|\,(b-a)+k(b-a)\sum_{i=1}^{n}\beta_i\,\|f_i\|_1<\infty.
\end{aligned}$$

This means that $F_{f_1f_2\cdots f_n}\in\mathcal{L}^1(I)$ for all $f_1,f_2,\dots,f_n\in\mathcal{L}^1(I)$ and $F:\mathcal{L}^1(I)^n\to\mathcal{L}^1(I)$ is well-defined. On the other hand,

$$\begin{aligned}
d_1(F_{f_1f_2\cdots f_n},F_{g_1g_2\cdots g_n})&=\int_I\left|F_{f_1f_2\cdots f_n}(x)-F_{g_1g_2\cdots g_n}(x)\right|dx\\
&=\int_I\left|\left(A+\int_a^x G(f_1(t),f_2(t),\dots,f_n(t))dt\right)-\left(A+\int_a^x G(g_1(t),g_2(t),\dots,g_n(t))dt\right)\right|dx\\
&=\int_I\left|\int_a^x\left(G(f_1(t),f_2(t),\dots,f_n(t))-G(g_1(t),g_2(t),\dots,g_n(t))\right)dt\right|dx\\
&\le\int_I\left(\int_a^x|(G(f_1(t),f_2(t),\dots,f_n(t))-G(g_1(t),g_2(t),\dots,g_n(t)))|\,dt\right)dx
\end{aligned}$$

$$\leq \int_I \left(\int_a^x k \sum_{i=1}^n \beta_i \left| f_i(t) - g_i(t) \right| dt \right) dx = k \int_I \sum_{i=1}^n \beta_i \left(\int_a^x \left| f_i(t) - g_i(t) \right| dt \right) dx \leq$$

$$= k \int_I \sum_{i=1}^n \beta_i \left(\int_I \left| f_i(t) - g_i(t) \right| dt \right) dx = k \int_I \sum_{i=1}^n \beta_i d_1(f_i, g_i) dx$$

$$= k \sum_{i=1}^n \beta_i d_1(f_i, g_i) \int_I dx = k(b-a) \sum_{i=1}^n \beta_i d_1(f_i, g_i).$$

If we suppose that $K = k(b-a) \sum_{i=1}^n \beta_i < 1$, then F verifies (1) with $g(f) = f$ for all $f \in \mathcal{L}^1(I)$. Then system (A.2) has a unique solution, which is of the form $(f_0, f_0, \ldots, f_0)$, where $f_0 \in \mathcal{L}^1(I)$ is the only solution of the equation

$$f_0(x) = A + \int_a^x G(f_0(t), f_0(t), \ldots, f_0(t)) dt, \quad \text{for all } x \in I$$

(this exists as a simple application of the *Banach Contraction Principle*).

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