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Lower bound of L^2 decay of the Navier–Stokes flow in the half-space $\mathbb{R}^n_+$ and its asymptotic behavior in the frequency space

Takahiro Okabe

Hirosaki University, 1 Bunkyo-cho, Hirosaki 036-8560, Japan

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ABSTRACT

We consider the asymptotic behavior of weak solutions of the Navier–Stokes equations in the half-space $\mathbb{R}^n_+$. We obtain the lower bound of the energy decay of the Navier–Stokes flow, by means of the profile of the initial data. Indeed, we construct a class of the initial data which causes the slow decay of the Navier–Stokes flow, with an explicit rate. Furthermore, we investigate the asymptotic behavior of concentration in the frequency space.

1. Introduction

In this paper, we consider an asymptotic behavior in L^2 of weak solutions of the Navier–Stokes equations in the half-space $\mathbb{R}^n_+$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \mathbb{R}^n_+ \times (0,\infty) \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n_+ \times (0,\infty) \\ u = 0 \quad \text{on } \partial\mathbb{R}^n_+ \times (0,\infty) \\ u(0) = a \quad \text{in } \mathbb{R}^n_+, \end{cases} \tag{N–S}$$

where $n \geq 3$, $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n; x_n > 0\}$ denotes the upper half-space. Here, $u = u(x,t) = (u^1(x,t), \ldots, u^n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and pressure of the fluid at point $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$, respectively, while $a = a(x) = (a^1(x), \ldots, a^n(x))$ is the given initial velocity.

In his celebrated paper [11], Leray proposed the problem whether or not weak solutions of (N–S) tend to zero in L^2 as the time goes to infinity. Masuda [12] first gave a partial answer to Leray's problem and clarified that the energy inequality of strong type plays an important role in L^2 decay of weak solutions. Here, we mean by the energy inequality of strong type:

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2\, d\tau \leq \|u(s)\|_2^2 \tag{1.1}$$

for almost all $s \geq 0$, including $s = 0$, and for all $t \geq s$. Leray called a weak solution with (1.1) a *turbulent solution*. Schonbek [15] first obtained an explicit rate of the temporal energy decay for weak solutions with large initial data in $L^1 \cap L^2$.

E-mail address: okabe@cc.hirosaki-u.ac.jp.

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Later on, the decay rate was precisely investigated by Kajikiya and Miyakawa [9], Schonbek [16], and Wiegner [26]. Moreover, it was clarified that the asymptotic behavior of the linear Stokes flow is important for the study on the energy decay of the nonlinear Navier–Stokes flow. On the other hand, Schonbek [16–18] observed that if $a \in L^r \cap L^2$, for some $1 < r < 2$ satisfies the average $\int a(x)\,dx \neq 0$, then it holds that

$$C(1+t)^{-\frac{3}{4}} \le \|u(t)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

Furthermore, she proved that if the initial data a fulfilled $\int a(x)\,dx = 0$, in addition to $\int |x|\,|a(x)|^2\,dx < \infty$, together with some restrictions on the profile of the lower frequency part a, we have also that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \ge C(1+t)^{-\frac{n+2}{4}}, \tag{1.2}$$

for $n = 2, 3$. See also Miyakawa and Schonbek [13].

In this direction, the author established more precise behavior of solutions of the lower bound in $L^2(\mathbb{R}^n)$. Indeed, introducing a class $K^m_{\alpha,\delta}(\mathbb{R}^n)$, defined by

$$K^m_{\alpha,\delta}(\mathbb{R}^n) := \{\phi \in L^2(\mathbb{R}^n);\ |\hat{\phi}(\xi)| \ge \alpha|\xi|^m,\ |\xi| \le \delta\}, \quad m \ge 0,\ \alpha, \delta > 0, \tag{1.3}$$

he [14] proved that if $a \in K^m_{\alpha,\delta}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ with $1 < r < 2$, then the weak solution $u(t)$ of (N–S) satisfies

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \ge C(1+t)^{-\frac{n+2m}{4}}, \tag{1.4}$$

for $n = 2, 3, 4$. We note that the set $K^m_{\alpha,\delta}(\mathbb{R}^n)$ has a different character of the initial profile from that of [17,18,13], and that in particular, our characterization covers the results of [16–18], when $0 \le m < 1$.

In the half-space, there are many results for the upper bound of the temporal decay of the Stokes flow and the Navier–Stokes flow. See, for instance, Borchers and Miyakawa [3], Fujigaki and Miyakawa [6,7], Bae and Choe [2], Bae [1], Choe and Jin [5], and Han [8]. However, up to now, it seems that there are few results for the *lower bound* of the energy decay. In such a situation, [6,7] obtained the same lower bound as (1.2) under some condition on initial data. Especially, in [7], it was clarified that the strong solution $u(t)$ of (N–S) satisfies $\|u(t)\|_2 \ge Ct^{-n/4}$ if and only if the Stokes flow $v(t)$ satisfies $\|v(t)\|_2 \ge Ct^{-n/4}$. As is mentioned in [7], it seems to be an interesting problem to characterize a class of the initial data which exhibits a lower bound of the Stokes flow in the half-space $\mathbb{R}^n_+$.

In the present paper, focusing on the profile of initial data, we investigate the lower bound such as (1.4) for weak solutions of (N–S) which satisfy the energy inequality of strong type (1.1) in the half-space $\mathbb{R}^n_+$. Our rate as in (1.4) improves the rate given by [6] like (1.2). Furthermore, we give a positive answer to the question of [7] for the slow decay of the Stokes flow by the concrete characterization of the initial data in $\mathbb{R}^n_+$ which is similar to (1.3).

To study on the asymptotic behavior of the Navier–Stokes flow in the half-space, we first consider the Stokes flow and establish the estimate from below in terms of the explicit solution formula given by Ukai [25]. In the whole space $\mathbb{R}^n$, a number of decay properties of lower bounds relies heavily on the Fourier transform, in [17,18,13,14], whose method is difficult to be applicable to the other domains. However, in order to overcome such difficulty, we split the variables of the initial data a with the following form:

$$a(x) = a'(x')\eta(x_n),$$

where $x = (x', x_n) \in \mathbb{R}^n$ and $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Moreover, under some restriction (see Assumption in Section 2) on a' and η, we notice that the property of a' is dominant to the slow decay of the Stokes flow. By this form, the problem is reduced to that on the lower dimensional whole space $\mathbb{R}^{n-1}$. Conversely, we see that the 2-dimensional initial data can be embedded in the 3-dimensional half-space $\mathbb{R}^3_+$ and also the whole space $\mathbb{R}^3$, where the slow decay properties are preserved. In the same manner, for every $n \in \mathbb{N}$, we find out a hierarchy structure between $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ for the decay of the lower bounds of solutions with respect to the initial data. On the other hand, instead of $K^m_{\alpha,\delta}(\mathbb{R}^n)$ as in (1.3), we introduce a more general profile on the lower frequency part on initial data such as

$$T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n) := \{\phi \in L^2(\mathbb{R}^n);\ |\hat{\phi}(\xi)| \ge \alpha|\xi_n|^m,\ |\xi_n| \le \gamma,\ |\xi'| \le \delta\}, \tag{1.5}$$

for $m \ge 0, \alpha, \gamma, \delta > 0$, where $\xi = (\xi', \xi_n) \in \mathbb{R}^n$ and $\xi' := (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. It should be noted that the class $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ can be characterized in terms of the estimate from below of the low frequency $\xi = (\xi', \xi_n)$ in the ξ_n direction. It turns out that such a profile of initial data only in one direction to ξ_n dominates the asymptotic behavior in time from below of the Stokes flow. We also note that by making use of $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$, we can improve the previous result in [14] for the whole space $\mathbb{R}^n$. By the virtue of Ukai's solution formula of the Stokes flow, the profile of initial data can be directly applicable to the exact exponent of the decay in (1.4). If we take $m = 0$ in (1.4) and (1.5), then we obtain such a lower bound as:

$$\|u(t)\|_2 \ge Ct^{-\frac{n}{4}} \quad \text{for large } t. \tag{1.6}$$

In addition, if $|\widehat{a'}(\xi')| \le M$ for near $\xi' = 0$, it is easy to see that

$$\|u(t)\|_2 \le C(1+t)^{-\frac{n}{4}}. \tag{1.7}$$

Therefore, (1.7) gives the optimal decay rate of the energy of the Navier–Stokes flow in the half-space $\mathbb{R}^n_+$ for such a initial data. Indeed, we construct an initial data which causes both (1.6) and (1.7), as an example in $T^0_{\alpha,\gamma,\delta}(\mathbb{R}^n)$.

As an application of the lower bound of (1.4), we consider the asymptotic behavior of weak solutions of (N–S) in the following sense:

$$\lim_{t\to\infty} \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} = 1, \tag{1.8}$$

for $\lambda > 0$, where E_λ is the resolution of identity of the Stokes operator. In the whole space $\mathbb{R}^n$, (1.8) corresponds to the energy concentration phenomenon within the lower frequency region, since $\widehat{E_\lambda u}(\xi) = \chi_{\{|\xi|\le\sqrt{\lambda}\}}\hat{u}(\xi)$, where $\chi_{\{|\xi|\le\sqrt{\lambda}\}}$ is the characteristic function on the set $\{\xi; |\xi| \le \sqrt{\lambda}\}$. Skalák [21–24] observed (1.8) and proved that the decay properties of solutions determine a band of frequency λ where (1.8) holds. On the other hand, in [14], giving the lower bound (1.4) with the profile $K^m_{\alpha,\delta}(\mathbb{R}^n)$ of initial data, we obtained (1.8) for all $\lambda > 0$. Furthermore, the explicit convergence rate of $u(t)$ in (1.8) was shown. In this paper, we derive (1.8) in terms of the class $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ in (1.5) for the half-space $\mathbb{R}^n_+$ and give the convergence rate of (1.8) for both t and $\lambda > 0$. In comparison with the problem in the whole space $\mathbb{R}^n$, it is difficult to see what (1.8) implies. In particular, the spectral resolution E_λ for the Stokes operator A should be characterized by some quantities depending on the domain, or its eigenfunctions, which will be discussed in a forthcoming paper.

In Section 2, we shall give our main results. Section 3 is devoted to preparing some propositions related to the solution formula of the Stokes flow. In Section 4, we give the lower bound of the Stokes flow with the class (1.5). Then we obtain the lower bound of the Navier–Stokes flow in Section 5. In Section 6, we deal with the asymptotic behavior such as (1.8). Furthermore, at the end of the present paper, we construct an example of the initial data which belongs to $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ for each $n \ge 2$.

2. Results

Before stating our results, we introduce the following notations. Let $C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ denote the set of all C^∞-solenoidal vectors ϕ with compact support in $\mathbb{R}^n_+$, i.e., div $\phi = 0$ in $\mathbb{R}^n_+$. $L^r_\sigma(\mathbb{R}^n_+)$ is the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ with respect to the L^r-norm $\|\cdot\|_r$, $1 < r < \infty$; $(\cdot,\cdot)$ is the duality pairing between $L^r(\mathbb{R}^n_+)$ and $L^{r'}(\mathbb{R}^n_+)$, where $1/r + 1/r' = 1$, $1 \le r \le \infty$. $L^r(\mathbb{R}^n_+)$ and $W^{m,r}(\mathbb{R}^n_+)$ stand for the usual (vector-valued) Lebesgue L^r-space and L^r-Sobolev space over $\mathbb{R}^n_+$, respectively. Let $W^{m,r}_0(\mathbb{R}^n_+)$ be the closure in $W^{m,r}(\mathbb{R}^n_+)$ of the set $C^\infty_0(\mathbb{R}^n_+)$ of smooth functions with compact support in $\mathbb{R}^n_+$. In the special case $r = 2$, let $H^1_{0,\sigma}(\mathbb{R}^n_+)$ denote the closure of $C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ with respect to the norm $\|\phi\|_{H^1} := \|\phi\|_2 + \|\nabla\phi\|_2$, where $\nabla\phi = (\partial\phi_i/\partial x_j)_{i,j=1,\dots,n}$. When X is a Banach space, $\|\cdot\|_X$ denotes the norm on X. $C^m([t_1, t_2]; X)$ and $L^r(t_1, t_2; X)$ are the usual Banach spaces, where $m = 0, 1, \dots,$ and t_1 and t_2 are real numbers such that $t_1 < t_2$.

Let us define our weak solutions of (N–S):

Definition 2.1. Let $a \in L^2_\sigma(\mathbb{R}^n_+)$. A measurable function u defined on $\mathbb{R}^n_+ \times (0,\infty)$ is called a weak solution of (N–S), if

(i) $u \in L^\infty\big(0,\infty; L^2_\sigma(\mathbb{R}^n_+)\big) \cap L^2\big(0,T; H^1_{0,\sigma}(\mathbb{R}^n_+)\big)$ for all $0 < T < \infty$,
(ii) the relation

$$\int_0^\infty \left[-\left(u(\tau), \frac{\partial\phi}{\partial t}(\tau) \right) + \big(\nabla u(\tau), \nabla\phi(\tau)\big) + \big(u\cdot\nabla u(\tau), \phi(\tau)\big) \right] d\tau = \big(a, \phi(0)\big) \tag{2.1}$$

holds for all $\phi \in C^1\big([0,\infty); H^1_{0,\sigma}(\mathbb{R}^n_+) \cap L^n(\mathbb{R}^n_+)\big)$ vanishing near $t = \infty$.

It is well known that we can redefine any weak solution $u(t)$ of (N–S) on a set of measure zero of the time interval $(0,\infty)$ so that $u(t)$ is weakly continuous in t with values in $L^2_\sigma(\mathbb{R}^n_+)$. Moreover, such a redefined weak solution u satisfies for each $0 \le s \le t$

$$\int_s^t \left[-\left(u(\tau), \frac{\partial\phi}{\partial t}(\tau) \right) + \big(\nabla u(\tau), \nabla\phi(\tau)\big) + \big(u\cdot\nabla u(\tau), \phi(\tau)\big) \right] d\tau = -\big(u(t), \phi(t)\big) + \big(u(s), \phi(s)\big) \tag{2.2}$$

for all $\phi \in C^1\big([s,t]; H^1_{0,\sigma}(\mathbb{R}^n_+) \cap L^n(\mathbb{R}^n_+)\big)$, see Serrin [19, Theorem 4, p.79].

Let us define the Stokes operator A_r in $L^r_\sigma(\mathbb{R}^n_+)$, $1 < r < \infty$. We have the following Helmholtz decomposition:

$$L^r(\mathbb{R}^n_+) = L^r_\sigma(\mathbb{R}^n_+) \oplus G^r(\mathbb{R}^n_+) \quad \text{(direct sum)}$$

where $G^r(\mathbb{R}^n) = \{\nabla p \in L^r(\mathbb{R}^n_+); p \in L^r_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})\}$, see Borchers and Miyakawa [3] and also Simader and Sohr [20]. Let P denote the projection operator from $L^r(\mathbb{R}^n_+)$ to $L^r_\sigma(\mathbb{R}^n_+)$. The Stokes operator A_r is defined by $A_r := -P\Delta$ with domain $D(A_r) := W^{2,r}(\mathbb{R}^n_+) \cap W^{1,r}_0(\mathbb{R}^n_+) \cap L^r_\sigma(\mathbb{R}^n_+)$. Since A_2 is a nonnegative self-adjoint operator on $L^2_\sigma(\mathbb{R}^n_+)$, A_2 admits the spectral decomposition, i.e., there uniquely exists a resolution of identity $\{E_\lambda\}_{\lambda\ge0}$ such that

$$A = \int_0^\infty \lambda\, dE_\lambda.$$

For simplicity, we abbreviate A_r to A when r is known from the context.

In this paper we establish a lower bound of the energy decay of weak solutions with (1.1) and characterize the initial data which causes such a slow decay. For this purpose, let us recall the definition of the set $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$:

$$T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n) := \{\phi \in L^2(\mathbb{R}^n); |\hat{\phi}(\xi)| \geq \alpha|\xi_n|^m,\ |\xi_n| \leq \gamma,\ |\xi'| \leq \delta\},$$

for $m \geq 0, \alpha, \gamma, \delta > 0$. In this definition, it should be noted that the direction to ξ_n does not have a special meaning. We may replace ξ_n by any one direction ξ_j for $j = 1, \ldots, n$.

Hence we impose on the initial data a the following assumption:

Assumption. We consider the initial data with the following form:

(A1) $a = (a'(x', x_n), 0) \in L^2_\sigma(\mathbb{R}^n_+)$.

(A2) $a'(x', x_n) = (a_1(x')\eta(x_n), \ldots, a_{n-1}(x')\eta(x_n)) =: a''(x')\eta(x_n)$ with $\eta \in L^2(\mathbb{R}_+)$ satisfy $|\widehat{\eta^*}(\xi_n)| > C$ for almost all $|\xi_n| \leq \tilde{\delta}$ for some $C > 0$ and $\tilde{\delta} > 0$, where η^* denotes the odd extension with respect to x_n, i.e.,

$$\eta^*(x_n) := \begin{cases} \eta(x_n), & x_n > 0, \\ -\eta(-x_n) & x_n < 0. \end{cases}$$

(A3) $a'' \in T^m_{\alpha,\gamma,\delta}(\mathbb{R}^{n-1})$ for some $m \geq 0$ and $\alpha, \gamma, \delta > 0$.

Remark 2.1. As an example of the initial data which satisfies (A1) and (A2), it suffices to take $a'' \in L^2_\sigma(\mathbb{R}^{n-1})$. Furthermore, we can find such an $\eta \in L^2(\mathbb{R}_+)$ as in (A2), see also the Appendix.

Now our results read:

Theorem 2.1. *Let $n \geq 3$, and let r and m satisfy either* (i) *or* (ii)*:*

(i) $1 < r \leq 2n/(n+2), 0 \leq m < 1$,

(ii) $2n/(n+2) < r < 2n/(n+1), 0 \leq m < 2n/r - n - 1$.

If $a \in L^r(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$ satisfies the assumptions (A1)–(A3) *for some $\alpha, \gamma, \delta > 0$, then there exist $T > 1$ and a constant $C > 0$ such that every weak solution $u(t)$ of (N–S) with* (1.1) *fulfills the estimate,*

$$\|u(t)\|_2 \geq Ct^{-\frac{n+2m}{4}} \tag{2.3}$$

for all $t \geq T$.

Remark 2.2. (i) We note that (2.3) improves the result in [7] when $0 \leq m < 1$.

(ii) The estimate (2.3) inspires us that the optimal decay rate for such an initial data seems to be $n/4$. Indeed, by taking $m = 0$ in (2.3), we obtain

$$Ct^{-\frac{n}{4}} \leq \|u(t)\|_2 \leq C_r(1+t)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}, \quad t > T, \tag{2.4}$$

for $a \in L^r(\mathbb{R}^n_+) \cap L^2(\mathbb{R}^n_+)$, $1 < r < 2$. Letting $r \to 1$ in (2.4) formally, we may expect an exact estimate both from below and above such that

$$Ct^{-\frac{n}{4}} \leq \|u(t)\|_2 \leq C(1+t)^{-\frac{n}{4}}, \quad t \geq T.$$

However, up to now, we do not establish any uniform estimate with respect to $1 < r < 2$ on the constant C_r in (2.4).

(iii) In addition to the case $m = 0$, if $|\widehat{a''}(\xi')| \leq M$ for near $\xi' = 0$ and $|\widehat{\eta^*}(\xi_n)| \leq M$ for near $\xi_n = 0$ then we obtain the optimal decay rate $n/4$ for such an initial data, since it holds that

$$Ct^{-\frac{n}{4}} \leq \|u(t)\|_2 \leq C(1+t)^{-\frac{n}{4}}, \quad t \geq T.$$

As an application of Theorem 2.1, we can show the asymptotic behavior in terms of the frequency or spectrum.

Theorem 2.2. *Let $n = 3, 4$. Let $r > 1$ and $m \geq 0$ be*

$$1 < r < \frac{n}{n-1}, \qquad 0 \leq m < \frac{n}{r} - n + 1.$$

If $a \in L^r(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^2_+)$ satisfies assumptions (A1)–(A3) *for some $\alpha, \gamma, \delta > 0$, then for every weak solution $u(t)$ of (N–S) with* (1.1) *there exist $T > 0$ and $C = C(n, r, m, \alpha, \gamma, \delta, a, T) > 0$ such that*

$$\left| \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} - 1 \right| \leq \frac{C}{\lambda} t^{-(\frac{n}{r}-n+1-m)} \tag{2.5}$$

holds for all $\lambda > 0$ and for all $t \geq T$.

Remark 2.3. (i) Since λ appears in the denominator in the right hand side in (2.5), it takes much time for the energy of weak solutions of (N–S) to concentrate onto the small λ. (ii) For $n \geq 5$, Theorem 2.2 still hold for the strong solution with the small initial data.

3. Preliminaries

We first introduce some specific properties of solutions, $v = (v', v^n)$, $v' = (v^1, \ldots, v^{n-1})$, of the Stokes equations:

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + \nabla p = 0 \quad \text{in } \mathbb{R}^n_+ \times (0, \infty) \\ \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^n_+ \times (0, \infty) \\ v = 0 \quad \text{on } \partial\mathbb{R}^n_+ \times (0, \infty) \\ v(0) = a \quad \text{in } \mathbb{R}^n_+. \end{cases} \tag{S}$$

Ukai [25] gave a explicit solution formula for (S). To state Ukai's formula we prepare some notations. Let $R = (R', R_n)$ with $R' = (R_1, \ldots, R_{n-1})$ and $S = (S_1, \ldots, S_{n-1})$ denote the Riesz transform over $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. Each R_j (resp. S_j) is a bounded linear operator on $L^r(\mathbb{R}^n)$ (resp. $L^r(\mathbb{R}^{n-1})$), $1 < r < \infty$. For a function $f(x', x_n)$, we understand that S_j acts as a convolution with respect to the variables x', so S_j is regarded as a bounded operator on both $L^r(\mathbb{R}^n)$ and $L^r(\mathbb{R}^n_+)$, $1 < r < \infty$. Let $B = B_r = -\Delta$ be the Laplacian on $\mathbb{R}^n_+$ with domain $D(B) := W^{2,r}(\mathbb{R}^n_+) \cap W^{1,r}_0(\mathbb{R}^n_+)$. It is well known that $-B$ generates a bounded analytic semigroup $\{e^{-tB}\}_{t\geq 0}$ on $L^r(\mathbb{R}^n_+)$, $1 < r < \infty$. More precisely, we have

$$e^{-tB}f = e^{t\Delta}f^*\big|_{\mathbb{R}^n_+}, \quad \text{for } f \in L^r(\mathbb{R}^n_+),\ 1 < r < \infty,$$

where $e^{t\Delta}$ is the usual heat operator on $\mathbb{R}^n$ and f^* denotes the odd extension with respect to variable x_n, i.e.,

$$f^*(x', x_n) := \begin{cases} f(x', x_n), & x_n > 0, \\ -f(x', -x_n), & x_n < 0. \end{cases}$$

The solution formula of Ukai [25] is now read:

Proposition 3.1 (*Ukai [25]*). *For $a \in L^r_\sigma(\mathbb{R}^n_+)$, $1 < r < \infty$, the solution $v = (v', v^n)$ of (S) is expressed as*

$$v^n(t) = Ue^{-tB}[a^n + S \cdot a'], \qquad v'(t) = e^{-tB}[a' - Sa^n] + Sv^n$$

where U is the bounded operator on $L^r(\mathbb{R}^n_+)$, indeed, $Uf = R' \cdot S(R' \cdot S - R_n)ef\big|_{\mathbb{R}^n_+}$, which is also expressed with the Fourier transform on $\mathbb{R}^{n-1}$ as

$$\widehat{Uf}(\xi', x_n) = |\xi'| \int_0^{x_n} e^{-|\xi'|(x_n - y)}\hat{f}(\xi', y)\, dy.$$

Here, ef denotes the zero extension of f from $\mathbb{R}^n_+$ over $\mathbb{R}^n$:

$$ef(x', x_n) = \begin{cases} f(x', x_n) & x_n > 0 \\ 0 & x_n < 0. \end{cases} \tag{3.1}$$

Remark 3.1. In this paper, we use the Fourier transform with the following form:

$$\hat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \quad i := \sqrt{-1}.$$

Furthermore, we note that the symbols of Riesz's operator R_j and S_j are

$$\sigma(R_j) = -i\xi_j/|\xi|, \quad j = 1, \ldots, n,$$
$$\sigma(S_j) = -i\xi_j/|\xi'|, \quad j = 1, \ldots, n-1,$$

which have opposite signs of ones in [25,3].

See [25] for the proof.

Let $a \in L^r_\sigma(\mathbb{R}^n_+)$ and put $e^{-tA}a = v(t)$ with the solution $v(t)$ of (S). From Proposition 3.1, deriving the resolvent estimate of the Stokes operator A, Borchers and Miyakawa [3] gave the following propositions:

Proposition 3.2 (*Borchers and Miyakawa [3]*). *The family $\{e^{-tA}\}_{t\geq 0}$ defines a bounded analytic semigroup of class C_0 on $L^r_\sigma(\mathbb{R}^n_+)$, $1 < r < \infty$.*

Proposition 3.3 (*[3]*).

(i) *The estimate*

$$\|\nabla^2 u\|_r \leq C\|A_r u\|_r, \quad u \in D(A_r),\ 1 < r < \infty,$$

holds with C independent of u.

(ii) $D(A_r^{1/2}) = W_0^{1,r}(\mathbb{R}^n_+) \cap L^r_\sigma(\mathbb{R}^n_+)$, $1 < r < \infty$, *and we have*

$$\|\nabla u\|_r \le C\|A_r^{1/2}u\|_r, \quad u \in D(A_r^{1/2})$$

with C independent of u.

(iii) *If* $u \in D(A_r^\alpha)$, $1 < r < \infty$, $0 < \alpha < 1$, *and if* $0 < 1/q = 1/r - 2\alpha/n < 1$, *then* $u \in L^q(\mathbb{R}^n_+)$ *and we have the estimate*

$$\|u\|_q \le C\|A_r^\alpha u\|_r, \quad u \in D(A_r^\alpha)$$

with the constant C independent of u.

Here $\nabla^2 u$ represents the second derivatives of u.

As an application of Propositions 3.2 and 3.3, we have the L^r–L^q estimates for the Stokes semigroup.

Proposition 3.4 ([25,3]). *Let $a \in L^2_\sigma(\mathbb{R}^n_+) \cap L^r(\mathbb{R}^n_+)$ for some $1 \le r < \infty$. Then the estimate*

$$\|e^{-tA}a\|_q \le Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|a\|_r$$

holds with C independent of a and $t > 0$, provided either (i) $1 < r \le q < \infty$; *or* (ii) $1 \le r < q \le \infty$.

4. Slow decay of the Stokes flow

In this section, we study on the decay of the Stokes flow in the whole space $\mathbb{R}^n$ and the half-space $\mathbb{R}^n_+$.

Lemma 4.1 (*The Whole Space*). *Let $n \ge 2$ and put $v(t) = e^{-tA}a$ with the Stokes semigroup e^{-tA} on $L^2_\sigma(\mathbb{R}^n)$. If $a \in L^2_\sigma(\mathbb{R}^n) \cap T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ for some $m \ge 0$ and $\alpha, \gamma, \delta > 0$, then $v(t)$ satisfies*

$$\|v(t)\|_2 \ge Ct^{-\frac{n+2m}{4}} \quad \text{for } t \ge 1, \tag{4.1}$$

where $C = C(n, m, \alpha, \gamma, \delta) > 0$.

Proof. By Plancherel's theorem and Fubini's theorem, we have

$$\begin{aligned}
\|v(t)\|_2^2 &= \|\hat{v}(t)\|_2^2 \ge \int_{|\xi_n|\le\gamma,\ |\xi'|\le\delta} e^{-2t|\xi|^2}|\hat{a}(\xi)|^2\,d\xi \\
&\ge \alpha^2 \int_{|\xi_n|\le\gamma,\ |\xi'|\le\delta} e^{-2t|\xi|^2}|\xi_n|^{2m}\,d\xi \\
&= \alpha^2 \left(\int_{|\xi_n|\le\gamma} e^{-2t\xi_n^2}|\xi_n|^{2m}\,d\xi_n\right)\left(\int_{|\xi'|\le\delta} e^{-2t|\xi'|^2}\,d\xi'\right) \\
&=: \alpha^2 I_1 \cdot I_2,
\end{aligned}$$

for all $t \ge 0$. By changing variables we have

$$\begin{aligned}
I_1 &= 2\int_0^\gamma e^{-2t\xi_n^2}\xi_n^{2m}\,d\xi_n \\
&= 2\int_0^{\sqrt{t}\gamma} e^{-2\rho^2}\left(\frac{\rho}{\sqrt{t}}\right)^{2m}\frac{d\rho}{\sqrt{t}} \\
&\ge 2t^{-\frac{2m+1}{2}}\int_0^\gamma e^{-2\rho^2}\rho^{2m}\,d\rho
\end{aligned}$$

for all $t \ge 1$. Similarly by polar coordinates $\xi' = \varrho\omega \in \mathbb{R}^{n-1}$, we have

$$\begin{aligned}
I_2 &= (n-1)\omega_{n-1}\int_0^\delta e^{-2t\varrho^2}\varrho^{n-2}\,d\varrho \\
&= (n-1)\omega_{n-1}\int_0^{\sqrt{t}\delta} e^{-2\rho^2}\left(\frac{\rho}{\sqrt{t}}\right)^{n-2}\frac{d\rho}{\sqrt{t}} \\
&\ge (n-1)\omega_{n-1}t^{-\frac{n-1}{2}}\int_0^\delta e^{-2\rho^2}\rho^{n-2}\,d\rho,
\end{aligned}$$

for all $t \ge 1$, where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

Therefore, we obtain (4.1) with a constant

$$C^2 = 2\alpha^2(n-1)\omega_{n-1}\left(\int_0^\gamma e^{-2\rho^2}\rho^{2m}\,d\rho\right)\left(\int_0^\delta e^{-2\rho^2}\rho^{n-2}\,d\rho\right).$$

This completes the proof of Lemma 4.1. $\square$

Remark 4.1. We note that Lemma 4.1 still holds, if we replace $a \in L^2_\sigma(\mathbb{R}^n) \cap T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ and e^{-tA} by $a \in T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ and $e^{t\Delta}$ respectively.

Next we consider the decay of the Stokes flow in the half-space $\mathbb{R}^n_+$.

Theorem 4.2 (*The Half-Space*)**.** *Let $n \geq 3$ and put $v(t) = e^{-tA}a$. If $a \in L^2_\sigma(\mathbb{R}^n_+)$ satisfies assumptions* (A1)–(A3), *then the Stokes flow $v(t)$ satisfies*

$$\|v(t)\|_2 \geq Ct^{-\frac{n+2m}{4}} \quad \textit{for } t \geq 1 \tag{4.2}$$

where $C = C(n, m, \alpha, \gamma, \delta) > 0$.

Proof. Since $a^n \equiv 0$ and $\operatorname{div} a = 0$ in the sense of distribution $\mathscr{D}'(\mathbb{R}^n_+)$, we see that $S \cdot a' = 0$. By the Fourier transform in x' we have

$$\widehat{\operatorname{div} a}(\xi', x_n) = \sum_{j=1}^{n-1} \widehat{\partial_j a_j}(\xi', x_n) = \sum_{j=1}^{n-1} i\xi_j \hat{a}_j(\xi', x_n) = 0.$$

Hence we see that

$$\widehat{S \cdot a'}(\xi', x_n) = \sum_{j=1}^{n-1} -\frac{i\xi_j}{|\xi'|}\hat{a}_j(\xi', x_n) = 0. \tag{4.3}$$

Therefore, by Ukai's explicit representation formula of the Stokes flow in the half-space $\mathbb{R}^n_+$, we have $v^n(t) \equiv 0$ and $v'(t) = e^{-tB}a'$. Moreover, we can regard $v(t) = (v'(t), 0)$ as a odd vector field on the whole space $\mathbb{R}^n$ with respect to x_n. Then we have

$$\begin{aligned}
\|v(t)\|^2_{L^2(\mathbb{R}^n_+)} &= \|v'(t)\|^2_{L^2(\mathbb{R}^n_+)} = \frac{1}{2}\|v'(t)\|^2_{L^2(\mathbb{R}^n)} \\
&= \frac{1}{2}\int_{\mathbb{R}^n} |\hat{v}'(\xi, t)|^2\,d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{-2t|\xi|^2}|\widehat{(a')^*}(\xi', \xi_n)|^2\,d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^n} e^{-2t|\xi|^2}|\widehat{a''}(\xi')\widehat{\eta^*}(\xi_n)|^2\,d\xi \\
&= \frac{1}{2}\left(\int_{\mathbb{R}^{n-1}} e^{-2t|\xi'|^2}|\widehat{a''}(\xi')|^2\,d\xi'\right)\left(\int_{\mathbb{R}} e^{-2t\xi_n^2}|\widehat{\eta^*}(\xi_n)|^2\,d\xi_n\right) \\
&=: \frac{1}{2}I_1 \cdot I_2.
\end{aligned} \tag{4.4}$$

By Lemma 4.1 and Remark 4.1, we obtain with some $C = C(n-1, m, \alpha, \gamma, \delta) > 0$

$$I_1 \geq Ct^{-\frac{n-1+2m}{2}} \quad \text{for } t \geq 1. \tag{4.5}$$

On the other hand, since $\widehat{\eta^*}(\xi_n) \geq C$ for $|\xi_n| \leq \tilde{\delta}$, by changing variables we have

$$\begin{aligned}
I_2 &\geq \int_{|\xi_n| \leq \tilde{\delta}} Ce^{-2t\xi_n^2}\,d\xi_n \\
&= 2C\int_0^{\tilde{\delta}} e^{-2t\xi_n^2}\,d\xi_n \\
&= 2C\int_0^{\sqrt{t}\tilde{\delta}} e^{-2\rho^2}\frac{d\rho}{\sqrt{t}} \\
&\geq 2Ct^{-\frac{1}{2}}\int_0^{\tilde{\delta}} e^{-2\rho^2}\,d\rho
\end{aligned} \tag{4.6}$$

for all $t \geq 1$. Hence, by (4.4)–(4.6) we obtain (4.2). $\square$

5. Slow decay of the Navier–Stokes flow

In this section, we study the decay of the Navier–Stokes flow and give the proof of Theorem 2.1. To estimate the nonlinear term, we introduce the following lemma which was proved by [9,3].

Lemma 5.1 (*[9,3]*). *Let $\lambda > 0$. Then*

$$|(u \cdot \nabla v, E_\lambda \phi)| \le C\lambda^{\frac{n+2}{4}} \|u\|_2 \|v\|_2 \|\phi\|_2 \tag{5.1}$$

for all $u,\ v \in H^1_{0,\sigma}(\mathbb{R}^n_+)$ and $\phi \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$.

Proof. Since $\operatorname{div} u = 0$, by integration by parts, the Gagliardo–Nirenberg inequality and Proposition 3.3, we have

$$\begin{aligned} |(u \cdot \nabla v, E_\lambda \phi)| &= |(u \cdot \nabla E_\lambda \phi, v)| \\ &\le \|uv\|_1 \|\nabla E_\lambda \phi\|_\infty \\ &\le C\|u\|_2 \|v\|_2 \|\nabla E_\lambda \phi\|_{2n}^{1/2} \|\nabla^2 E_\lambda \phi\|_{2n}^{1/2} \\ &\le C\|u\|_2 \|v\|_2 \|A^{1/2} E_\lambda \phi\|_{2n}^{1/2} \|A E_\lambda \phi\|_{2n}^{1/2}. \end{aligned} \tag{5.2}$$

Here we note that $E_\lambda \phi \in \bigcap_{m=1}^\infty D(A_2^m) \subset D(A_{2n})$ for any $\lambda > 0$.

Since $1/2n = 1/2 - (n-1)/2n$, by the embedding inequality, we obtain

$$\left\|A^{\frac{1}{2}} E_\lambda \phi\right\|_{2n}^2 \le C \left\|A^{\frac{1}{2}+\frac{n-1}{4}} E_\lambda \phi\right\|_2^2 \le C \int_0^\lambda \rho^{1+\frac{n-1}{2}}\, d\|E_\rho \phi\|_2^2 \le C\lambda^{\frac{n+1}{2}} \|E_\lambda \phi\|_2^2. \tag{5.3}$$

Similarly, we obtain

$$\|A E_\lambda \phi\|_{2n}^2 \le C \left\|A^{1+\frac{n-1}{4}} E_\lambda \phi\right\|_2^2 \le C\lambda^{\frac{n+3}{2}} \|E_\lambda \phi\|_2^2. \tag{5.4}$$

Hence, (5.2)–(5.4) imply (5.1). This completes the proof of Lemma 5.1. □

Next we refer to an upper bound of the energy decay of weak solutions with (1.1). The same results were proved by, for instance, Wiegner [26] in the whole space, and Borchers and Miyakawa [4] in exterior domains. However, we give the complete proof for the reader's convenience.

Lemma 5.2. *Let $1 \le r < 2$ and $a \in L^r(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$. Then every weak solution $u(t)$ of (N–S) with (1.1) satisfies $\|u(t)\|_2 = O(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})})$ as $t \to \infty$.*

Proof. Let u be a weak solution of (N–S) with $u(0) = a$ and satisfy the energy inequality:

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2\, d\tau \le \|u(s)\|_2^2 \tag{5.5}$$

for almost all $s \ge 0$, including $s = 0$, and for all $t \ge s$. Let $\lambda = \lambda(t)$ be a smooth positive function on $(0, \infty)$. From the estimate

$$\begin{aligned} \|\nabla u(t)\|_2^2 &= \|A^{1/2} u(t)\|_2^2 = \int_0^\infty \rho\, d\|E_\rho u(t)\|_2^2 \\ &\ge \int_{\lambda(t)}^\infty \rho\, d\|E_\rho u(t)\|_2^2 \\ &\ge \frac{\lambda(t)}{2} \left(\|u(t)\|_2^2 - \|E_{\lambda(t)} u(t)\|_2^2\right) \end{aligned} \tag{5.6}$$

for all $t > 0$, and the energy inequality (5.5), we obtain

$$\|u(t)\|_2^2 + \int_s^t \lambda(\tau)\|u(\tau)\|_2^2\, d\tau \le \|u(s)\|_2^2 + \int_s^t \lambda(\tau)\|E_{\lambda(\tau)} u(\tau)\|_2^2\, d\tau. \tag{5.7}$$

To estimate $\|E_{\lambda(t)} u(t)\|_2$, for each $\varphi \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ we choose $\phi(\tau) := e^{-(t-\tau)A} E_{\lambda(t)} \varphi$ as a test function of (2.2), which is legitimate since $E_{\lambda(t)}\varphi \in D(A) \cap L^n(\mathbb{R}^n_+)$, and obtain

$$\left(E_{\lambda(t)} u(t), \varphi\right) = \left(E_{\lambda(t)} e^{-(t-s)A} u(s), \varphi\right) - \int_s^t \left(u(\tau) \cdot \nabla u(\tau), E_{\lambda(t)} e^{-(t-\tau)A} \varphi\right) d\tau \tag{5.8}$$

for all $t > s \geq 0$. Therefore, we have with $s = 0$ in (5.8)

$$\|E_{\lambda(t)}u(t)\|_2 \leq \|e^{-tA}a\|_2 + C\lambda(t)^{\frac{n+2}{4}} \int_0^t \|u(\sigma)\|_2^2\, d\sigma. \tag{5.9}$$

Substituting (5.9) into (5.7), we have the following inequality for $y(\tau) := \|u(\tau)\|_2^2$:

$$y(t) - g(t,s) + \int_s^t \lambda(\tau)y(\tau)\, d\tau \leq y(s) \tag{5.10}$$

for a.e. $s \in (0, t)$, where we put

$$g(t,s) := 2\int_s^t \left[\lambda(\tau)\|e^{-\tau A}a\|_2^2 + C\lambda(\tau)^{\frac{n}{2}+2}\left(\int_0^\tau \|u(\sigma)\|_2^2\, d\sigma\right)^2\right] d\tau. \tag{5.11}$$

Now we want to apply Gronwall's lemma to (5.10) with respect to s. We set $h(s) := \int_s^t \lambda(\tau)y(\tau)\, d\tau$, which is almost everywhere differential in $(0, t)$ with $h' \in L^\infty((\varepsilon, t))$ for small $\varepsilon > 0$. From (5.10) we have

$$h'(\tau) = -\lambda(\tau)y(\tau) \leq -\lambda(\tau)[y(t) - g(t,\tau) + h(\tau)]. \tag{5.12}$$

Let $H \geq 0$ be a solution of the equation $H'(\tau) = \lambda(\tau)H(\tau)$. Multiplying (5.12) by H and then integrating over (s, t) yields

$$\big(H(t) - H(s)\big)y(t) \leq H(s)h(s) + \int_s^t H'(\tau)g(t,\tau)\, d\tau, \tag{5.13}$$

since $h(t) = 0$. Applying (5.10) to the second term of the right hand side in (5.13) and integrating by parts, we obtain

$$H(t)y(t) \leq H(s)y(s) - \int_s^t H(\tau)\frac{\partial g}{\partial \tau}(t,\tau)\, d\tau, \tag{5.14}$$

since $g(t,t) = 0$. Now we choose $\lambda(\tau) = m\tau^{-1}$, $m > 0$, so that $H(\tau) = \tau^m$. Since (5.14) holds for almost every $s > 0$ and since $y(\tau)$ is bounded, taking $m > 0$ sufficiently large we can pass to the limit $s \to 0$ in (5.14) to obtain

$$t^m\|u(t)\|_2^2 \leq 2\int_0^t m\tau^{m-1}\|e^{-\tau A}a\|_2^2\, d\tau + C\int_0^t m^{\frac{n}{2}+2}\tau^{m-\frac{n}{2}-2}\left(\int_0^\tau \|u(\sigma)\|_2^2\, d\sigma.\right)^2 d\tau. \tag{5.15}$$

Since $\|u(\sigma)\|_2^2 \leq \|a\|_2^2$ by the energy inequality, the last term of (5.15) is $O(t^{1-n/2})$. Hence we see that $\|u(t)\|_2^2 \leq Ct^{-\alpha}$, where $\alpha = \min\{n(1/r - 1/2), n/2 - 1\}$. By the same argument, we finally obtain $\|u(t)\|_2^2 \leq Ct^{-n(1/r-1/2)}$.

This completes the proof of Lemma 5.2. $\square$

Lemma 5.3. *Let $1 \leq r < 2$ and $a \in L^r(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$. We put $v(t) = e^{-tA}a$. Then every weak solution $u(t)$ of (N–S) with (1.1) satisfies*

$$\|u(t) - v(t)\|_2 = \begin{cases} O\left(t^{-\left(\frac{n}{r}-\frac{n}{4}-\frac{1}{2}\right)}\right), & n\left(\dfrac{1}{r} - \dfrac{1}{2}\right) < 1, \\ O\left(t^{-\left(\frac{n+2}{4}\right)}\log t\right), & n\left(\dfrac{1}{r} - \dfrac{1}{2}\right) = 1, \\ O\left(t^{-\left(\frac{n+2}{4}\right)}\right), & n\left(\dfrac{1}{r} - \dfrac{1}{2}\right) > 1, \end{cases} \tag{5.16}$$

as $t \to \infty$.

Proof. Let $w(t) := u(t) - v(t)$. Since $u(t)$ and $v(t)$ satisfy strong energy inequality (1.1), we obtain

$$\begin{aligned} &\|w(t)\|_2^2 + 2\int_s^t \|\nabla w(\tau)\|_2^2\, d\tau \\ &\quad = \|u(t)\|_2^2 + \|v(t)\|_2^2 - 2\big(u(t), v(t)\big) + 2\int_s^t \left[\|\nabla u(\tau)\|_2^2 + \|\nabla v(\tau)\|_2^2 - 2\big(\nabla u(\tau), \nabla v(\tau)\big)\right] d\tau \\ &\quad \leq \|u(s)\|_2^2 + \|v(s)\|_2^2 - 2\big(u(t), v(t)\big) - 4\int_s^t \big(\nabla u(\tau), \nabla v(\tau)\big)\, d\tau, \end{aligned} \tag{5.17}$$

for almost every $s > 0$, and for all $t \geq s$. We substitute $\phi(\tau) = v(\tau)$ for the test function in (2.2) and obtain

$$\big(u(t), v(t)\big) = \big(u(s), v(s)\big) - 2\int_s^t \big(\nabla u(\tau), \nabla v(\tau)\big)\, d\tau - \int_s^t \big(u(\tau)\cdot\nabla u(\tau), v(\tau)\big)\, d\tau \tag{5.18}$$

for all $t \geq s > 0$, since $dv/dt = -Av$. Hence (5.17) and (5.18) yield

$$\|w(t)\|_2^2 + 2\int_s^t \|\nabla w(\tau)\|_2^2\, d\tau \leq \|w(s)\|_2^2 + 2\int_s^t \big(u(\tau)\cdot\nabla u(\tau), v(\tau)\big)\, d\tau \tag{5.19}$$

for almost every $s > 0$ and all $t \geq s$. We estimate the last term in (5.19), Since $(u\cdot\nabla v, v) = 0$, by Proposition 3.4 and the Hölder and the Young inequalities we have

$$\begin{aligned}\big|\big(u(\tau)\cdot\nabla u(\tau), v(\tau)\big)\big| &= \big|\big(u(\tau)\cdot\nabla w(\tau), v(\tau)\big)\big| \\ &\leq \|u(\tau)\|_2\|\nabla w(\tau)\|_2\|v(\tau)\|_\infty \\ &\leq C\|u(\tau)\|_2\|\nabla w(\tau)\|_2\, \tau^{-\frac{n}{2r}}\|a\|_r \\ &\leq \frac{1}{2}\|\nabla w(\tau)\|_2^2 + C\tau^{-\frac{n}{r}}\|a\|_r^2\|u(\tau)\|_2^2 \end{aligned}\tag{5.20}$$

for all $\tau > 0$. The remaining argument is nearly the same as in the proof of Lemma 5.2. Let $\lambda(t)$ be the same one in the proof of Lemma 5.2. Similarly, choosing $\phi(\tau) = e^{-(t-\tau)A}E_{\lambda(t)}\varphi$, $\varphi \in C_{0,\sigma}^\infty(\mathbb{R}_+^n)$ as the test function in (2.2), we obtain

$$\|E_{\lambda(t)}w(t)\|_2^2 \leq C\lambda(t)^{\frac{n}{2}+1}\left(\int_0^t \|u(\sigma)\|_2^2\, d\sigma\right)^2 \tag{5.21}$$

for all $t > 0$. Putting $y(t) := \|w(t)\|_2^2$ and

$$g(t,s) := C\int_s^t \lambda(\tau)^{\frac{n}{2}+2}\left(\int_0^\tau \|u(\sigma)\|_2^2\, d\sigma\right)^2 d\tau + C\|a\|_r^2\int_s^t \tau^{-\frac{n}{r}}\|u(\tau)\|_2^2\, d\tau,$$

finally we obtain with $\lambda(t) = mt^{-1}$ for sufficient large $m > 0$,

$$t^m\|w(t)\|_2^2 \leq Cm^{\frac{n}{2}+2}\int_0^t \tau^{m-\frac{n}{2}-2}\left(\int_0^\tau \|u(\sigma)\|_2^2\, d\sigma\right)^2 d\tau + C\|a\|_r^2\int_0^t \tau^{m-\frac{n}{r}}\|u(\tau)\|_2^2\, d\tau \tag{5.22}$$

for $t > 0$.

Now we note that by Lemma 5.2 every weak solution of (N–S) with (1.1) satisfies

$$\|u(t)\|_2^2 \leq Ct^{-n\left(\frac{1}{r}-\frac{1}{2}\right)},$$

if $a \in L^r(\mathbb{R}_+^n) \cap L_\sigma^2(\mathbb{R}_+^n)$, $1 \leq r < 2$.

First we consider the case $n(1/r - 1/2) < 1$. Since $\|u(t)\|_2^2 \leq Ct^{-n(1/r-1/2)}$, (5.22) yields

$$\|w(t)\|_2^2 \leq Ct^{1+\frac{n}{2}-\frac{2n}{r}}.$$

We next consider the case $n(1/r - 1/2) = 1$. Since $\|u(t)\|_2^2 \leq C(1+t)^{-1}$ we have

$$\begin{aligned}\|w(t)\|_2^2 &\leq C\left[t^{-\frac{n}{2}-1}\big(\log(1+t)\big)^2 + t^{-\frac{n}{r}}\log(1+t)\right] \\ &\leq Ct^{-\frac{n}{2}-1}\big(\log(1+t)\big)^2\end{aligned}$$

for large $t > 0$.

Finally we consider the case $n(1/r-1/2) > 1$. Since $\|u(t)\|_2^2 \leq C(1+t)^{-1-\beta}$ for some $\beta > 0$, and so $\int_0^\infty \|u(\tau)\|_2^2\, d\tau < \infty$, (5.22) gives

$$\|w(t)\|_2^2 \leq C\left[t^{-1-\frac{n}{2}} + t^{-\frac{n}{r}}\right].$$

Since $1 + n/2 < n/r$, we obtain the desired result. The proof of Lemma 5.3 is complete. □

5.1. *Completion of the proof of Theorem 2.1*

Let $v(t) = e^{-tA}a$. By Theorem 4.2 and Lemma 5.3, it suffices to show that

$$\lim_{t\to\infty} \frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2} = 0. \tag{5.23}$$

Indeed, under the condition (5.23), there exists $T \geq 1$ such that

$$\frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2} \leq \frac{1}{2}$$

for all $t \geq T$. Hence, by the triangle inequality and Theorem 4.2 we have

$$\begin{aligned}\|u(t)\|_2 &\geq \|v(t)\|_2 - \|u(t) - v(t)\|_2 \\ &= \|v(t)\|_2 \left(1 - \frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2}\right) \\ &\geq \frac{1}{2}\|v(t)\|_2 \\ &\geq Ct^{-\frac{n+2m}{4}}\end{aligned}$$

for all $t \geq T$.

Now it remains to prove (5.23). First we consider the case $1 < r < 2n/(n+2)$. The assumption (i) implies $n(1/r - 1/2) > 1$ and $(n + 2m)/4 < (n + 2)/4$. Hence, from Theorem 4.2 and Lemma 5.3 it follows that

$$\frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2} \leq C\frac{t^{-\frac{n+2}{4}}}{t^{-\frac{n+2m}{4}}} \to 0,$$

as $t \to \infty$.

Next we consider the case $r = 2n/(n + 2)$. Since $m < 1$ and $n(1/r - 1/2) = 1$ by assumption (i), we have

$$\begin{aligned}\frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2} &\leq C\frac{t^{-\frac{n+2}{4}}\log t}{t^{-\frac{n+2m}{4}}} \\ &\leq Ct^{-\varepsilon}\log t \to 0,\end{aligned}$$

as $t \to \infty$, where $\varepsilon := (1 - m)/2 > 0$.

Finally we consider the case $2n/(n + 2) < r < 2n/(n + 1)$. Assumption (ii) implies $n(1/r - 1/2) < 1$ and $(n + 2m)/4 < n/r - n/4 - 1/2$. Hence, we obtain

$$\frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2} \leq C\frac{t^{-\left(\frac{n}{r}-\frac{n}{4}-\frac{1}{2}\right)}}{t^{-\frac{n+2m}{4}}} \to 0,$$

as $t \to \infty$. Since all constants which appear in the proof of Lemmas 5.2 and 5.3 are independent of u, we can choose $T > 0$ independently of u. The proof of Theorem 2.1 is complete. □

6. Proof of Theorem 2.2

In this section we consider the concentration phenomenon of the energy of weak solutions of (N–S). For this purpose, the following lemmas play an important role in the proof.

Lemma 6.1. *Let $n \geq 3$ and let $1 < r \leq n/(n - 1)$. If $a \in L^r(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$, Then every weak solution $u(t)$ of (N–S) lies in $L^r(\mathbb{R}^n_+)$ for all $t \geq 0$.*

Proof. For each $\varphi \in C^\infty_{0,\sigma}(\mathbb{R}^n_+)$ and $t > 0$, we put $\phi(\tau) := e^{-(t-\tau)A}\varphi$. We substitute ϕ for the test function in (2.2) with $s = 0$ and obtain

$$\big(u(t), \varphi\big) = \big(a, e^{-tA}\varphi\big) + \int_0^t \big(u(\tau)\cdot\nabla u(\tau), \phi(\tau)\big)\,d\tau. \tag{6.1}$$

Since $2 < 2r'/(r' + 2) < 2n/(n + 2)$, By the Hölder and the Sobolev inequalities we have

$$\begin{aligned}\left|\big(u(\tau)\cdot\nabla u(\tau), e^{-(t-\tau)A}\varphi\big)\right| &\leq \|e^{-(t-\tau)A}\varphi\|_{r'}\|u(\tau)\|_{2r'/(r'+2)}\|\nabla u(\tau)\|_2 \\ &\leq C\|\varphi\|_{r'}\|u(\tau)\|_2^{1-n/r'}\|u(\tau)\|_{2n/(n+2)}^{n/r'}\|\nabla u(\tau)\|_2 \\ &\leq C\|\varphi\|_{r'}\|u(\tau)\|_2^{1-n/r'}\|\nabla u(\tau)\|_2^{1+n/r'}.\end{aligned} \tag{6.2}$$

From energy inequality (1.1) with $s = 0$, we have the uniform estimate $\|u(t)\|_2 \le \|a\|_2$ for all $t \ge 0$. Also by the Young inequality we have

$$\int_0^t \|\nabla u(\tau)\|_2^{1+n/r'}\, d\tau \le t^{1/2-n/2r'} \left(\int_0^t \|\nabla u(\tau)\|_2^2\, d\tau\right)^{1/2+n/2r'}$$
$$\le t^{1/2-n/2r'} \|a\|_2^{1+n/r'}. \tag{6.3}$$

Hence we obtain by (6.1)–(6.3)

$$\|u(t)\|_r \le C\left[\|a\|_r + t^{1/2-n/2r'}\|a\|_2^2\right] < \infty \tag{6.4}$$

for all $t \ge 0$. The proof of Lemma 6.1 is complete. □

Lemma 6.2. *Let $n = 3, 4$ and let r and m satisfy*

$$1 < r < \frac{n}{n-1}, \qquad 0 \le m < \frac{n}{r} - n + 1.$$

If $a \in L^r(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^2_+)$ satisfies the assumptions (A1)–(A3) for some $\alpha, \gamma, \delta > 0$, then for every weak solution $u(t)$ of (N–S) with (1.1) satisfies

$$\frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_2^2} = O\left(t^{-\left(\frac{n}{r}-n+1-m\right)}\right) \tag{6.5}$$

as $t \to \infty$.

Proof. By the well-known Leray's structure theorem, every weak solution of (N–S) with (1.1) becomes a strong solution after some definite time T which depends on u. Furthermore, it is shown by Kozono [10] that the strong solution of (N–S) decays in the same way as the Stokes flow $e^{-tA}a$ as $t \to \infty$. With the aid of Lemma 6.1, since $a \in L^r(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$, we have $\|\nabla u(t)\|_2 = Ct^{-n(1/r-1/2)/2-1/2}$ for sufficiently large t. Here we should note that C depends on u and T. Hence by Theorem 2.1, we obtain (6.5). □

6.1. Completion of the proof of Theorem 2.2

For each $\lambda > 0$, we have the estimate from below as

$$\|\nabla u(t)\|_2^2 = \|A^{1/2}u(t)\|_2^2 = \int_0^\infty \rho\, d\|E_\rho u(t)\|_2^2$$
$$\ge \int_\lambda^\infty \rho\, d\|E_\rho u(t)\|_2^2$$
$$\ge \lambda \int_\lambda^\infty d\|E_\rho u(t)\|_2^2 = \lambda\left(\|u(t)\|_2^2 - \|E_\lambda u(t)\|_2^2\right). \tag{6.6}$$

Dividing both sides of (6.6) by $\lambda\|u(t)\|_2^2$, we obtain

$$1 - \frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2} \le \frac{1}{\lambda}\frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_2^2}.$$

Hence by Lemma 6.2, there exist $T > 0$ and $C > 0$ such that

$$\left|\frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2} - 1\right| \le \frac{C}{\lambda} t^{-\left(\frac{n}{r}-n+1-m\right)}$$

for all $t \ge T$, where C is independent of $\lambda > 0$. This completes the proof of Theorem 2.2. □

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Appendix. Example of initial data

We construct an example of the initial data which belongs to $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ for all dimension $n \geq 2$. In this paper, we are interested in the exponent m of $T^m_{\alpha,\gamma,\delta}(\mathbb{R}^n)$ as in Theorems 2.1 and 2.2.

We first show the example of initial data by induction. Consider $a' \in T^m_{\alpha,\gamma,\delta}(\mathbb{R}^{n-1})$ for some $\alpha, \gamma, \delta > 0$ and take $\eta \in L^2(\mathbb{R})$ with $|\hat{\eta}(\xi)| \geq C$ for all $|\xi| \leq \tilde{\delta}$ with some $C > 0$ and $\tilde{\delta} > 0$. Setting $a(x', x_n) := a'(x')\eta(x_n)$, we have

$$|\hat{a}(\xi)| = |\widehat{a'}(\xi')\hat{\eta}(\xi_n)| \geq C\alpha|\xi_{n-1}|^m$$

for all $|\xi''| \leq \delta$, $|\xi_{n-1}| \leq \gamma$ and $|\xi_n| \leq \tilde{\delta}$, where $\xi'' = (\xi_1, \ldots, \xi_{n-1})$. Therefore, we see that $a \in T^m_{\alpha',\gamma',\delta'}(\mathbb{R}^n)$ for some $\alpha', \gamma', \delta' > 0$.

Now it remains to show for $n = 2$. Moreover, we get the initial data when the dimension n is even. Let $n = 2k$ for some $k \in \mathbb{N}$. For each $1 \leq j \leq k$, we put

$$M_j := \begin{pmatrix} 0 & R_{2j} \\ -R_{2j-1} & 0 \end{pmatrix}$$

where $\widehat{R_j u}(\xi) = -i\xi_j\hat{u}(\xi)/|\xi|, j = 1, \ldots, n$ denotes the Riesz transform on $\mathbb{R}^n$. Putting

$$a = \begin{pmatrix} M_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_k \end{pmatrix} \begin{pmatrix} e^{-|x|^2} \\ \vdots \\ e^{-|x|^2} \end{pmatrix},$$

we see that

$$\hat{a}(\xi) = Ce^{-|\xi|^2/4}\left(-i\frac{\xi_2}{|\xi|}, i\frac{\xi_1}{|\xi|}, \ldots, -i\frac{\xi_{2k}}{|\xi|}, i\frac{\xi_{2k-1}}{|\xi|}\right). \tag{A.1}$$

Since $i\xi \cdot \hat{a}(\xi) = 0$, we obtain div $a = 0$. On the other hand, for any fixed $\delta > 0$, (A.1) yields

$$|\hat{a}(\xi)| \geq Ce^{-\delta^2/4}$$

for all $|\xi| \leq \delta$. So it follows that $a \in T^0_{\alpha,\gamma,\delta}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n_+)$ for some $\alpha, \gamma, \delta > 0$. Moreover, we see that $a \in L^r(\mathbb{R}^n)$ for $1 < r < \infty$.

Next we construct an example of η as in (A2). Take a Schwartz function $\psi \in \mathscr{S}(\mathbb{R})$ such that it is an even function and $|\hat{\psi}(\xi)| \geq C$ near $\xi = 0$. Set $\eta = \mathcal{H}\psi|_{\mathbb{R}_+}$, where $\widehat{\mathcal{H}\psi}(\xi) = -i\,\mathrm{sgn}\,\xi\,\hat{\psi}(\xi)$ is the Hilbert transform. Since $\mathcal{H}\psi$ is an odd function on $\mathbb{R}$, we see that $\eta^*(x) = \mathcal{H}\psi(x)$ and $|\widehat{\eta^*}(\xi)| \geq C$ for $|\xi| \leq \tilde{\delta}$ with some $\tilde{\delta} > 0$. Furthermore, we have $\eta \in L^r(\mathbb{R})$ for $1 < r < \infty$.

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