J. Math. Anal. Appl. 402 (2013) 702–709

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On very weak solutions to a class of double obstacle problems ☆

Gejun Bao, Tingting Wang *, Guanfeng Li

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, PR China

ARTICLE INFO

Article history:
Received 4 December 2012
Available online 5 February 2013
Submitted by Junping Shi

Keywords:
Uniqueness and existence
Very weak solutions
Double obstacle problems
Hodge decomposition

ABSTRACT

We prove the uniqueness of very weak solutions by Hodge decomposition and obtain the existence by an approximation argument. Moreover, we prove that the solution quasiminimizes the r-Dirichlet integral in $\mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, and finally the stability of very weak solutions is established for varying obstacle functions.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain [10]. This paper deals with quasilinear elliptic equations of the form

$$-\operatorname{div}A(x, \nabla u) = -\operatorname{div}F(x) \tag{1.1}$$

where $F(x) \in L^{\frac{r}{p-1}}(\Omega, \mathbb{R}^n)$ with $\max\{1, p-1\} < r \le p$, $1 < p < \infty$, and $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory mapping satisfying the following assumptions for fixed $0 < \alpha \le \beta < \infty$:

(i) the Lipschitz continuity

$$|A(x, \xi) - A(x, \zeta)| \le \beta|\xi - \zeta|^{p-1}$$

(ii) the monotonicity inequality

$$\langle A(x, \xi) - A(x, \zeta), \xi - \zeta\rangle \ge \alpha|\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2}$$

(iii) the homogeneity condition

$$A(x, \lambda\xi) = \lambda|\lambda|^{p-2}A(x, \xi)$$

for almost every $x \in \Omega$ and all $\xi, \zeta \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

The obstacle problem is a classic topic in the theory of variational inequalities and free boundary problems. The problem is to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle. The mathematical formulation of the problem is to seek minimizers of the Dirichlet energy functional,

$$I[u] = \int_\Omega |\nabla u(x)|^p \mathrm{d}x \tag{1.2}$$

☆ This work was supported by the National Natural Science Foundation of China (Grant No. 11071048).

* Corresponding author.
E-mail addresses: baogj@hit.edu.cn (G. Bao), ttwanghit@gmail.com, hnjzwtt@163.com (T. Wang), liguanfeng1984@126.com (G. Li).

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2013.01.065

in addition to satisfying Dirichlet boundary conditions corresponding to the fixed boundary of the membrane, the functions u are in addition constrained to be greater than some given obstacle function, see [1,5,17,21]. As its generalization, the theory of the obstacle problem is extended to other divergence form uniformly elliptic operators or degenerate elliptic operators A and their associated energy functionals, see [7]. The double obstacle problem is a problem in the field of partial differential equations with constraints limiting the solution from attaining too high and low values. It belongs to the class of free boundary problems, which comprises the branch of partial differential equations with an unknown boundary-the free boundary, is of great interest and has obtained many important results, see [13,14,16].

The theory of very weak solutions of (1.1) or minimizers of a variational integral (1.2) has been initiated by Iwaniec and Sbordone [10]. Recall that a local minimizer of I is a function $u \in W^{1,p}(\Omega)$ such that $I[u+\phi] \geq I[u]$ for each $\phi \in W_0^{1,p}(\Omega)$, that is:

$$\int_\Omega \left(|\nabla u + \nabla\phi|^p - |\nabla u|^p\right)\mathrm{d}x \geq 0 \tag{1.3}$$

for all $\phi \in C_0^\infty(\Omega)$. Note that $|\,|\xi|^p - |\zeta|^p\,| \leq p|\xi-\zeta|(|\xi|+|\zeta|)^{p-1}$, it is clear that it is only necessary to assume $u \in W^{1,r}(\Omega)$ with $r \geq \max\{1, p-1\}$ to ensure (1.3) is meaningful, see [10,8,19]. They pointed out the necessity to build up estimates below the natural exponent p and gave the definition of very weak solutions of (1.1):

Definition 1.1 (*[10]*)**.** Let $F(x) \in L^{\frac{r}{p-1}}(\Omega)$ with $\max\{1, p-1\} < r \leq p$. A very weak solution u of (1.1) is an element of the Sobolev space $W^{1,r}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \langle A(x, \nabla u), \nabla\phi\rangle \mathrm{d}x = \int_\Omega \langle F, \nabla\phi\rangle \mathrm{d}x \tag{1.4}$$

for all $\phi \in W^{1,\frac{r}{r-p+1}}(\Omega)$ with compact support.

The main tool they used is the Hodge decomposition and for equations of the form (1.1) they proved that if r is sufficiently close to p, then a very weak solution is a weak solution. Later other authors used the same technique to obtain many results under various conditions, see [2,4,12,18,20,22], and the key point was to use the Hodge decomposition in construction of a test function. Especially, in the case when the right hand-side of (1.1) is not in divergence form, other methods, based on the techniques of harmonic analysis, were established to construct test functions to deal with the regularity of very weak solutions, see [11,3,6]. Based on the close relationship between A-harmonic equation and the obstacle problem, see [7], we extend our study of very weak solutions to the double obstacle problems.

Suppose that $\theta \in W^{1,r}(\Omega)$, and that $\varphi, \psi : \Omega \to [-\infty, +\infty]$. Let

$$\mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega) = \left\{u \in W^{1,r}(\Omega) : \varphi \leq u \leq \psi \text{ a.e. } \Omega,\, u-\theta \in W_0^{1,r}(\Omega)\right\}.$$

Here the function θ is called a boundary datum, and φ, ψ lower obstacle and upper obstacle, respectively.

Dealing with very weak solutions requires some estimates below the natural exponent. The key tool of the paper is the following Hodge decomposition.

Lemma 1.2 (*[10, Hodge Decomposition]*)**.** *Let $\Omega \subset \mathbb{R}^n$ be a regular domain and $w \in W_0^{1,r}(\Omega)$, $r > 1$, and let $-1 < \varepsilon < r-1$. Then there exist $\phi \in W_0^{1,\frac{r}{1+\varepsilon}}(\Omega)$ and a (divergence free) function $H \in L_0^{\frac{r}{1+\varepsilon}}(\Omega)$ such that*

$$|\nabla w|^\varepsilon \nabla w = \nabla\phi + H.$$

Moreover

$$\|H\|_{\frac{r}{1+\varepsilon}} \leq C|\varepsilon|\,\|\nabla w\|_r^{1+\varepsilon}$$

where $C = C(r, \Omega)$.

For all $u, v \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, we apply the Hodge decomposition with $w = u - v$ and $\varepsilon = r - p$ yield that

$$|\nabla(v-u)|^{r-p}\nabla(v-u) = \nabla\phi_{v,u} + H_{v,u} \tag{1.5}$$

where $\phi_{v,u} \in W_0^{1,\frac{r}{r-p+1}}(\Omega)$ and $H_{v,u} \in L^{\frac{r}{r-p+1}}(\Omega)$ is divergence free, and satisfy the following estimates

$$\begin{aligned} &\|H_{v,u}\|_{\frac{r}{r-p+1}} \leq C(p-r)\|\nabla(v-u)\|_r^{r-p+1} \\ &\|\nabla\phi_{v,u}\|_{\frac{r}{r-p+1}} \leq C\|\nabla(v-u)\|_r^{r-p+1}. \end{aligned} \tag{1.6}$$

Notice that

$$\phi_{v,u} = -\phi_{u,v}, \qquad H_{v,u} = -H_{u,v} \tag{1.7}$$

by the uniqueness of the Hodge decomposition.

Now we give the definition of very weak solutions to the $\mathcal{K}^{r,\theta}_{\varphi,\psi}$-obstacle problem.

Definition 1.3. A function $u \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$ is called a very weak solution to the $\mathcal{K}^{r,\theta}_{\varphi,\psi}$-obstacle problem associated with A-harmonic Eq. (1.1) if

$$\int_\Omega \langle A(x, \nabla u), |\nabla(v-u)|^{r-p}\nabla(v-u)\rangle \mathrm{d}x \geq \int_\Omega \langle A(x,\nabla u), H_{v,u}\rangle \mathrm{d}x + \int_\Omega \langle F, \nabla\phi_{v,u}\rangle \mathrm{d}x \tag{1.8}$$

whenever $v \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$ and $H_{v,u}$, $\nabla\phi_{v,u}$ come from (1.5).

Remark. (a) The term "very weak solution" means the Sobolev integrability exponent r of u can be below the natural exponent p.

(b) If $r = p$, then by the uniqueness of the Hodge decomposition we have $H_{v,u} = 0$. So then the above definition about very weak solution agrees with the usual weak solution, i.e., (1.8) is simplified into

$$\int_\Omega \langle A(x, \nabla u) - F, \nabla(v-u)\rangle \mathrm{d}x \geq 0. \tag{1.9}$$

Proceeding the proof similarly as [15] we know that the problem (1.9) admits a unique solution.

The study of very weak solutions of double obstacle problems has aroused deep concern recently. There have been many results abut the regularity of very weak solutions of double obstacle problems, such as, higher integrability [22] and local boundedness [20]. In this paper, we focus on the existence and uniqueness of very weak solutions, and then we derive its property as the quasiminimizer of the r-Dirichlet integral and finally a convergence is established for varying obstacle functions.

2. Main results

2.1. Uniqueness and existence

In this section, we prove the uniqueness and the existence of very weak solutions to the obstacle problem (1.8) if r is sufficiently close to p. We prove the following

Theorem 2.1. *There exists $r_0 = r_0(r, p, \alpha, \beta, \Omega) > 0$ such that for $|p - r| < r_0$ and $F, G \in L^{\frac{r}{p-1}}(\Omega, \mathbb{R}^n)$, each of the two double obstacle problems*

$$\int_\Omega \langle A(x, \nabla u_1), |\nabla(v-u_1)|^{r-p}\nabla(v-u_1)\rangle \mathrm{d}x \geq \int_\Omega \langle A(x,\nabla u_1), H_{v,u_1}\rangle \mathrm{d}x + \int_\Omega \langle F, \nabla\phi_{v,u_1}\rangle \mathrm{d}x \tag{2.1}$$

and

$$\int_\Omega \langle A(x, \nabla u_2), |\nabla(v-u_2)|^{r-p}\nabla(v-u_2)\rangle \mathrm{d}x \geq \int_\Omega \langle A(x,\nabla u_2), H_{v,u_2}\rangle \mathrm{d}x + \int_\Omega \langle G, \nabla\phi_{v,u_2}\rangle \mathrm{d}x \tag{2.2}$$

has a unique solution $u_1, u_2 \in \mathcal{K}^{r,\theta}_{\varphi,\psi}$ respectively provided $\theta \in W^{1,p}(\Omega)$, and it holds

$$\|u_1 - u_2\|_{W_0^{1,r}(\Omega)} \leq C\|F - G\|^{\frac{1}{p-1}}_{L^{\frac{r}{p-1}}(\Omega,\mathbb{R}^n)} \tag{2.3}$$

where $C = C(n, r, p, \alpha, \beta, \Omega)$.

Remark. The assumption $\theta \in W^{1,p}(\Omega)$ is only needed to derive the result of the existence.

Proof. For $u_1, u_2 \in \mathcal{K}^{r,\theta}_{\varphi,\psi}$, we have by applying (1.5)–(1.7) that

$$\begin{aligned} |\nabla(u_1-u_2)|^{r-p}\nabla(u_1-u_2) &= \nabla\phi_{u_1,u_2} + H_{u_1,u_2} \\ |\nabla(u_2-u_1)|^{r-p}\nabla(u_2-u_1) &= \nabla\phi_{u_2,u_1} + H_{u_2,u_1} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \|H_{u_1,u_2}\|_{\frac{r}{r-p+1}} = \|H_{u_2,u_1}\|_{\frac{r}{r-p+1}} &\leq C(p-r)\|\nabla(u_1-u_2)\|_r^{r-p+1} \\ \|\nabla\phi_{u_1,u_2}\|_{\frac{r}{r-p+1}} = \|\nabla\phi_{u_2,u_1}\|_{\frac{r}{r-p+1}} &\leq C\|\nabla(u_1-u_2)\|_r^{r-p+1}. \end{aligned} \tag{2.5}$$

Taking $v = u_2$ and $v = u_1$ in (2.1) and (2.2), respectively, implies that

$$\int_\Omega \langle A(x, \nabla u_1) - A(x, \nabla u_2), |\nabla(u_1 - u_2)|^{r-p}\nabla(u_1 - u_2)\rangle dx \\ \le \int_\Omega \langle A(x, \nabla u_1) - A(x, \nabla u_2), H_{u_1,u_2}\rangle dx + \int_\Omega \langle F - G, \nabla\phi_{u_1,u_2}\rangle dx. \tag{2.6}$$

On the one hand, it follows from the monotonicity inequality (ii) that

$$\int_\Omega \langle A(x, \nabla u_1) - A(x, \nabla u_2), |\nabla(u_1 - u_2)|^{r-p}\nabla(u_1 - u_2)\rangle dx \ge \alpha \int_\Omega |\nabla(u_1 - u_2)|^r dx = \alpha\|\nabla(u_1 - u_2)\|_r^r. \tag{2.7}$$

On the other hand, we have by the Lipschitz continuity (i), Hölder inequality, (2.4) and (2.5) that

$$\begin{aligned}\int_\Omega \langle A(x, \nabla u_1) - A(x, \nabla u_2), H_{u_1,u_2}\rangle dx &\le \beta \int_\Omega |\nabla(u_1 - u_2)|^{p-1}|H_{u_1,u_2}|dx \\ &\le \beta\|\nabla(u_1 - u_2)\|_r^{p-1}\|H_{u_1,u_2}\|_{\frac{r}{r-p+1}} \\ &\le \beta C(p - r)\|\nabla(u_1 - u_2)\|_r^r \end{aligned}\tag{2.8}$$

and

$$\begin{aligned}&\int_\Omega \langle F - G, \nabla\phi_{u_1,u_2}\rangle dx \\ &= \int_\Omega \langle F - G, |\nabla(u_1 - u_2)|^{r-p}\nabla(u_1 - u_2) - H_{u_1,u_2}\rangle dx \\ &\le \int_\Omega |F - G|\,|H_{u_1,u_2}|dx + \int_\Omega |F - G|\,|\nabla(u_1 - u_2)|^{r-p+1}dx \\ &\le \|F - G\|_{\frac{r}{p-1}}\|H_{u_1,u_2}\|_{\frac{r}{r-p+1}} + \|F - G\|_{\frac{r}{p-1}}\|\nabla(u_1 - u_2)\|_r^{r-p+1} \\ &\le C(p - r)\|F - G\|_{\frac{r}{p-1}}\|\nabla(u_1 - u_2)\|_r^{r-p+1} + \|F - G\|_{\frac{r}{p-1}}\|\nabla(u_1 - u_2)\|_r^{r-p+1}.\end{aligned}\tag{2.9}$$

Inserting (2.7)–(2.9) into (2.6), we obtain

$$\big(\alpha - \beta C(p - r)\big)\|\nabla(u_1 - u_2)\|_r^{p-1} \le \big(1 + C(p - r)\big)\|F - G\|_{\frac{r}{p-1}}.$$

Choosing r_0 such that for $|p - r| < r_0$ we have $\alpha - \beta C(p - r) > 0$, and therefore we have

$$\|\nabla(u_1 - u_2)\|_r^{p-1} \le C\|F - G\|_{\frac{r}{p-1}}$$

for $|p - r| < r_0$. Since $u_1 - \theta \in W_0^{1,r}(\Omega)$ and $u_2 - \theta \in W_0^{1,r}(\Omega)$, we get $u_1 - u_2 \in W_0^{1,r}(\Omega)$. Thus the Poincaré inequality yields

$$\|u_1 - u_2\|_{L^r(\Omega)} \le C(n, r, \Omega)\|\nabla(u_1 - u_2)\|_{L^r(\Omega)}.$$

Combined the above two inequalities, (2.3) holds. Inequality (2.3) implies the uniqueness in Theorem 2.1.

It remains to prove existence. We prove it by an approximation argument. For $F \in L^{\frac{r}{p-1}}(\Omega, \mathbb{R}^n)$, let $F_j \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$ be mappings converging to F in $L^{\frac{r}{p-1}}(\Omega, \mathbb{R}^n)$ and let $u_j \in \mathcal{K}_{\varphi,\psi}^{p,\theta} \subset \mathcal{K}_{\varphi,\psi}^{r,\theta}$ be the unique solutions of the problem

$$\int_\Omega \langle A(x, \nabla u_j) - F_j, \nabla\phi_{v,u_j}\rangle dx \ge 0 \tag{2.10}$$

whenever $v \in \mathcal{K}_{\varphi,\psi}^{p,\theta}(\Omega)$, by Remark (b). We use inequality (2.3) to get

$$\|u_j - u_k\|_{W_0^{1,r}(\Omega)} \le C\|F_j - F_k\|_{L^{\frac{r}{p-1}}(\Omega,\mathbb{R}^n)}^{\frac{1}{p-1}} \tag{2.11}$$

which implies that u_j is a Cauchy sequence in $W_0^{1,r}(\Omega)$. Let $u \in W_0^{1,r}(\Omega)$ be the limit of u_j. Let k tends to ∞ in (2.11) yields

$$\|u_j - u\|_{W_0^{1,r}(\Omega)} \le C\|F_j - F\|_{L^{\frac{r}{p-1}}(\Omega,\mathbb{R}^n)}^{\frac{1}{p-1}}. \tag{2.12}$$

Then, by (2.12) and the uniqueness of the Hodge decomposition, (2.10) yields that

$$\int_\Omega \langle A(x, \nabla u) - F, \nabla \phi_{v,u} \rangle \mathrm{d}x \geq 0 \tag{2.13}$$

whenever $v \in \mathcal{K}^{p,\theta}_{\varphi,\psi}(\Omega)$.

Moreover, for all $\widetilde{v} \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, since $\widetilde{v} - \theta \in W^{1,r}_0(\Omega)$, then there exists $\phi_n \in C^\infty_0(\Omega)$ such that $\phi_n \to \widetilde{v} - \theta$ in $W^{1,r}(\Omega)$. Recall that $\theta \in W^{1,p}(\Omega)$, we then have

$$\begin{aligned}
&\phi_n + \theta \to \widetilde{v} \quad \text{in } W^{1,r}(\Omega)\\
&\phi_n + \theta \in W^{1,p}(\Omega)\\
&(\phi_n + \theta) - \theta \in W^{1,p}_0(\Omega)\\
&\varphi \leq (\phi_n + \theta) \leq \psi \quad \text{a.e. } \Omega \text{ when } n \text{ sufficiently large.}
\end{aligned}$$

So then taking $v = (\phi_n + \theta) \in \mathcal{K}^{p,\theta}_{\varphi,\psi}(\Omega)$ in (2.13) and taking $n \to \infty$ yield

$$\int_\Omega \langle A(x, \nabla u) - F, \nabla \phi_{\widetilde{v},u} \rangle \mathrm{d}x \geq 0$$

whenever $\widetilde{v} \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, which means that u is a very weak solution. The theorem follows. □

2.2. Quasiminimizers

In this paper, we study some properties of very weak solutions to the $\mathcal{K}^{r,\theta}_{\varphi,\psi}$-obstacle problem associated with (1.1). As a generalization of the weak solution to the single obstacle problem, we firstly prove that very weak solutions keep the property as quasiminimizers to the r-Dirichlet integral in $\mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, that is:

Theorem 2.2. *Suppose that u is a very weak solution to the $\mathcal{K}^{r,\theta}_{\varphi,\psi}$-obstacle problem, then there exists $r_0 = r_0(r, p, \alpha, \beta) > 0$ such that for $|p - r| < r_0$ and for all $v \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, it holds*

$$\int_\Omega |\nabla u|^r \mathrm{d}x \leq C \left(\int_\Omega |\nabla v|^r \mathrm{d}x + \int_\Omega |F|^{\frac{r}{p-1}} \mathrm{d}x \right)$$

where $C = C(r, p, \alpha, \beta)$.

Proof. For u a very weak solution and all $v \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, write

$$E(v, u) = |\nabla(v - u)|^{r-p} \nabla(v - u) + |\nabla u|^{r-p} \nabla u.$$

According to the elementary inequality [9],

$$\big| |X|^{-\varepsilon} X - |Y|^{-\varepsilon} Y \big| \leq 2^\varepsilon \frac{1 + \varepsilon}{1 - \varepsilon} |X - Y|^{1-\varepsilon}, \quad X, Y \in \mathbb{R}^n,\ 0 \leq \varepsilon < 1 \tag{2.14}$$

and by applying (2.14) with

$$\varepsilon = p - r, \qquad X = \nabla(v - u), \qquad Y = -\nabla u$$

we can derive that

$$|E(v, u)| \leq 2^{p-r} \frac{1 + p - r}{1 - p + r} |\nabla v|^{1-p+r}. \tag{2.15}$$

We then have by the definition (1.8) that

$$\begin{aligned}
\int_\Omega \langle A(x, \nabla u), |\nabla u|^{r-p} \nabla u \rangle \mathrm{d}x &= \int_\Omega \langle A(x, \nabla u), E(v, u) \rangle \mathrm{d}x - \int_\Omega \langle A(x, \nabla u), |\nabla(v - u)|^{r-p} \nabla(v - u) \rangle \mathrm{d}x\\
&\leq \int_\Omega \langle A(x, \nabla u), E(v, u) \rangle \mathrm{d}x - \int_\Omega \langle A(x, \nabla u), H_{v,u} \rangle \mathrm{d}x - \int_\Omega \langle F, \nabla \phi_{v,u} \rangle \mathrm{d}x.
\end{aligned} \tag{2.16}$$

On the one hand, the monotonicity inequality (ii) yields that

$$\int_\Omega \langle A(x, \nabla u), |\nabla u|^{r-p} \nabla u \rangle \mathrm{d}x \geq \alpha \int_\Omega |\nabla u|^r \mathrm{d}x. \tag{2.17}$$

On the other hand, by using the Hölder inequality, we have from (2.15), (2.16) and (1.6) that

$$\begin{aligned}\int_\Omega \langle A(x,\nabla u), E(v,u)\rangle \mathrm{d}x &\le C(p,r,\beta)\int_\Omega |\nabla u|^{p-1}|\nabla v|^{1-p+r}\mathrm{d}x\\ &\le C\|\nabla u\|_r^{p-1}\|\nabla v\|_r^{r-p+1}\\ \int_\Omega \langle A(x,\nabla u), H_{v,u}\rangle \mathrm{d}x &\le \beta\int_\Omega |\nabla u|^{p-1}|H_{v,u}|\mathrm{d}x\\ &\le \beta\|\nabla u\|_r^{p-1}\|H_{v,u}\|_{\frac{r}{r-p+1}}\\ &\le C(p-r)\|\nabla u\|_r^{p-1}\|\nabla(v-u)\|_r^{r-p+1}\\ \int_\Omega \langle F, \nabla\phi_{v,u}\rangle \mathrm{d}x &\le \int_\Omega |F|\,|\nabla\phi_{v,u}|\mathrm{d}x\\ &\le \|F\|_{\frac{r}{p-1}}\|\nabla\phi_{v,u}\|_{\frac{r}{r-p+1}}\\ &\le C\|F\|_{\frac{r}{p-1}}\|\nabla(v-u)\|_r^{r-p+1}.\end{aligned}$$

Inserting the above inequalities and (2.17) into (2.16), we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^r \mathrm{d}x \le{}& C\|\nabla u\|_r^{p-1}\|\nabla v\|_r^{r-p+1} + C(p-r)\|\nabla u\|_r^{p-1}\|\nabla v\|_r^{r-p+1}\\ &+ C(p-r)\|\nabla u\|_r^{r} + C\|F\|_{\frac{r}{p-1}}\|\nabla v\|_r^{r-p+1} + C\|F\|_{\frac{r}{p-1}}\|\nabla u\|_r^{r-p+1}\end{aligned}$$

where $C = C(r,p,\alpha,\beta)$. Next we apply Young's inequality

$$ab \le \varepsilon a^t + \varepsilon^{-1/(t-1)} b^{t/(t-1)}, \quad \varepsilon > 0 \text{ and } t > 1$$

yields that

$$\begin{aligned}\|\nabla u\|_r^r \le{}& \varepsilon\|\nabla u\|_r^r + CC_1(\varepsilon,r,p)\|\nabla v\|_r^r + \varepsilon\|\nabla u\|_r^r + (p-r)CC_1(\varepsilon,r,p)\|\nabla v\|_r^r\\ &+ C(p-r)\|\nabla u\|_r^r + \varepsilon\|F\|_{\frac{r}{p-1}}^{\frac{r}{p-1}} + CC_1(\varepsilon,r,p)\|\nabla v\|_r^r + \varepsilon\|\nabla u\|_r^r + CC_1(\varepsilon,r,p)\|F\|_{\frac{r}{p-1}}^{\frac{r}{p-1}}\\ ={}& (3\varepsilon + C(p-r))\|\nabla u\|_r^r + (\varepsilon + CC_1(\varepsilon,r,p))\|F\|_{\frac{r}{p-1}}^{\frac{r}{p-1}} + (p-r+2)CC_1(\varepsilon,r,p)\|\nabla v\|_r^r \end{aligned}\tag{2.18}$$

where $C = C(r,p,\alpha,\beta)$. Let ε small enough and choosing r_0 such that for $|p-r| < r_0$, we have $(3\varepsilon + C(p-r)) < 1/2$. Therefore we obtain from (2.18) that

$$\|\nabla u\|_r^r \le C\left(\|\nabla v\|_r^r + \|F\|_{\frac{r}{p-1}}^{\frac{r}{p-1}}\right)$$

where $C = C(r,p,\alpha,\beta)$. The theorem follows. □

Remark. In other words, the theorem expresses that among all functions v having the same boundary values as u and which is bounded by φ and ψ as their obstacles, the solution u has the least r-Dirichlet integral. When $r = p$, the conclusion agrees with the classic result, see [7].

2.3. Stability of very weak solutions

The last section presents the stability of very weak solutions about varying obstacle functions. Suppose that obstacle sequences $\{\varphi_j\}$ and $\{\psi_j\}$, converging to φ and ψ, respectively. We consider the sequence of sets

$$\mathcal{K}^{r,\theta}_{\varphi_j,\psi_j}(\Omega) = \left\{u \in W^{1,r}(\Omega) : \varphi_j \le u \le \psi_j \text{ a.e. } \Omega,\, u-\theta \in W^{1,r}_0(\Omega)\right\}.$$

For such $\{\varphi_j\}$, $\{\psi_j\}$ and $F(x) \in L^{\frac{r}{p-1}}(\Omega)$, we apply Theorem 2.1 and obtain that there exists unique solution $u_j \in \mathcal{K}^{r,\theta}_{\varphi_j,\psi_j}$ to the $\mathcal{K}^{r,\theta}_{\varphi_j,\psi_j}$-obstacle problem, i.e. it holds

$$\int_\Omega \langle A(x,\nabla u_j), |\nabla(v-u_j)|^{r-p}\nabla(v-u_j)\rangle \mathrm{d}x \ge \int_\Omega \langle A(x,\nabla u_j), H_{v,u_j}\rangle \mathrm{d}x + \int_\Omega \langle F, \nabla\phi_{v,u_j}\rangle \mathrm{d}x$$

whenever $v \in \mathcal{K}^{r,\theta}_{\varphi_j,\psi_j}$.

To establish the convergence of solutions u_j, we assume that the sequence φ_j converges to φ from below while ψ_j converges to ψ from above.

Theorem 2.3. *Under the hypotheses above, we obtain that $u_j \to u$ in $W^{1,r}(\Omega)$ and the limit function u is a very weak solution to the $\mathcal{K}^{r,\theta}_{\varphi,\psi}$-obstacle problem.*

Proof. For all $v \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, since $\varphi_j \le \varphi \le \psi \le \psi_j$ and $v-\theta \in W^{1,r}_0(\Omega)$, we get $v \in \mathcal{K}^{r,\theta}_{\varphi_j,\psi_j}(\Omega)$ for all $j \in \mathbb{N}$. Fix $v_0 \in \mathcal{K}^{r,\theta}_{\varphi_j,\psi_j}(\Omega)$ and Theorem 2.2 implies that

$$\int_\Omega |\nabla u_j|^r \mathrm{d}x \le C\left(\int_\Omega |\nabla v_0|^r \mathrm{d}x + \int_\Omega |F|^{\frac{r}{p-1}} \mathrm{d}x\right) < \infty. \tag{2.19}$$

Note that $u_j, \theta \in W^{1,r}(\Omega)$ and $u_j - \theta \in W^{1,r}_0(\Omega)$, then it follows from the Poincaré inequality and (2.19) that

$$\begin{aligned}\int_\Omega |u_j|^r \mathrm{d}x &\le 2^r \int_\Omega \left(|u_j-\theta|^r + |\theta|^r\right)\mathrm{d}x \\ &\le C(n,r,\Omega)\int_\Omega |\nabla(u_j-\theta)|^r \mathrm{d}x + 2^r\int_\Omega |\theta|^r \mathrm{d}x \\ &< C\left(\int_\Omega |\nabla u_j|^r \mathrm{d}x + \int_\Omega \left(|\theta|^r + |\nabla\theta|^r\right)\mathrm{d}x\right) < \infty. \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20), we obtain that $\{u_j\}$ is bounded in $W^{1,r}(\Omega)$. Then there exists a subsequence, still denoted by $\{u_j\}$ and $u \in W^{1,r}(\Omega)$ such that

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } W^{1,r}(\Omega) \\ u_j \to u & \text{in } L^r(\Omega) \\ u_j \to u & \text{a.e. } \Omega. \end{cases} \tag{2.21}$$

Moreover, since $u_j - \theta \in W^{1,r}_0(\Omega)$, then $u-\theta \in W^{1,r}_0(\Omega)$ and $\varphi_j \le u_j \le \psi_j$ a.e. Ω yields $\varphi \le u \le \psi$ a.e. Ω, so we have $u \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$.

Next we are to extract a further subsequence such that

$$\nabla u_n \to \nabla u \quad \text{a.e. } \Omega \tag{2.22}$$

then this immediately implies that

$$u_j \to u \quad \text{in } W^{1,r}(\Omega). \tag{2.23}$$

To prove (2.22), firstly we observe from the Lipschitz condition (i) and (2.19), that

$$\begin{aligned}\int_\Omega \left|\left(A(x,\nabla u_j) - A(x,\nabla u)\right)\cdot|\nabla(u_j-u)\,|^{r-p}\right|^{\frac{r}{r-1}}\mathrm{d}x &\le \beta^{\frac{r}{r-1}}\int_\Omega |\,|\nabla(u_j-u)\,|^{r-1}\,|^{\frac{r}{r-1}}\mathrm{d}x \\ &= \beta^{\frac{r}{r-1}}\int_\Omega |\nabla(u_j-u)|^r \mathrm{d}x \\ &\le C\left(\int_\Omega |\nabla u_j|^r \mathrm{d}x + \int_\Omega |\nabla u|^r \mathrm{d}x\right) < \infty\end{aligned}$$

which implies that

$$\left(A(x,\nabla u_j) - A(x,\nabla u)\right)\cdot|\nabla(u_j-u)|^{r-p} \in L^{\frac{r}{r-1}}(\Omega).$$

Then since $\nabla u_j \rightharpoonup \nabla u$ weakly in $L^r(\Omega)$, we obtain

$$\begin{aligned}&\int_\Omega \langle A(x,\nabla u_j) - A(x,\nabla u), |\nabla(u_j-u)|^{r-p}(\nabla u_j - \nabla u)\rangle \mathrm{d}x \\ &\quad= \int_\Omega \langle \left(A(x,\nabla u_j) - A(x,\nabla u)\right)\cdot|\nabla(u_j-u)|^{r-p}, \nabla u_n - \nabla u\rangle \mathrm{d}x \longrightarrow 0\end{aligned} \tag{2.24}$$

as $j \to \infty$. On the other hand, the monotonicity assumption (ii) yields

$$\begin{aligned}
&\int_\Omega \langle A(x, \nabla u_j) - A(x, \nabla u), |\nabla(u_j - u)|^{r-p}(\nabla u_j - \nabla u)\rangle \mathrm{d}x \\
&\quad \geq \alpha \int_\Omega |\nabla(u_j - u)|^{r-p} \cdot |\nabla u_j - \nabla u|^2 (|\nabla u_n| + |\nabla u|)^{p-2} \mathrm{d}x \\
&\quad \geq \alpha \int_\Omega |\nabla(u_j - u)|^r \mathrm{d}x
\end{aligned}$$

then we have that

$$\int_\Omega |\nabla(u_j - u)|^r \mathrm{d}x \to 0.$$

Thus there exists a subsequence of $\{\nabla u_j\}$, still denoted by $\{\nabla u_j\}$, such that $\nabla u_j \to \nabla u$ a.e. Ω which implies that (2.23) holds.

Now we show that u is a very weak solution to the obstacle problem in $\mathcal{K}^{r,\theta}_{\varphi,\psi}$. For all $v \in \mathcal{K}^{r,\theta}_{\varphi,\psi}(\Omega)$, since u_j is a solution to the $\mathcal{K}^{r,\theta}_{\varphi_j,\psi_j}$-obstacle problem, we have

$$\int_\Omega \langle A(x, \nabla u_j) - F, \nabla \phi_{v,u_j}\rangle \mathrm{d}x \geq 0 \tag{2.25}$$

then by (2.23) and the uniqueness of the Hodge decomposition, (2.25) yields that

$$\int_\Omega \langle A(x, \nabla u) - F, \nabla \phi_{v,u}\rangle \mathrm{d}x \geq 0.$$

We then obtain that u is a very weak solution by Definition 1.3. The theorem follows. □

References

[1] L.A. Caffarelli, The obstacle problem revisited, J. Fourier Anal. Appl. 4 (1998) 383–402.
[2] M. Carozza, G. Moscariello, A. Passarelli di Napoli, Nonlinear equations with growth coefficients in BMO, Houston J. Math. 28 (2002) 917–929.
[3] M. Carozza, A. Passarelli di Napoli, On very weak solutions of a class of nonlinear elliptic systems, Comment. Math. Univ. Carolin. 41 (2000) 493–508.
[4] A. Firoenza, C. Sbordone, Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1, Studia Math. 127 (1998) 223–231.
[5] A. Friedman, Variational Principles and Free Boundary Problems, Dover Publications, Dover, 2010.
[6] F. Giannetti, A. Passarelli di Napoli, On very weak solutions of degenerate p-harmonic equations, NoDEA 14 (2007) 739–751.
[7] J. Heinonen, T. Kilpelainen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford University Press, Oxford, 1993.
[8] T. Iwaniec, p-Harmonic tensors and quasiregular mappings, Ann. of Math. 136 (1992) 589–624.
[9] T. Iwaniec, L. Migiaccio, L. Nania, C. Sbordone, Integrability and removability results for quasiregular mappings in high dimensions, Math. Scand. 75 (1994) 263–279.
[10] T. Iwaniec, C. Sbordone, Weak minima of variational integrals, J. Reine Angew. Math. 454 (1994) 143–161.
[11] J. Lewis, On very weak solutions of certain elliptic systems, Comm. Partial Differential Equations 18 (1993) 1515–1537.
[12] J. Li, H.Y. Gao, Local regularity result for very weak solutions of obstacle problems, Rad. Mat. 12 (2003) 19–26.
[13] G.B. Li, O. Martio, Stability and higher integrability of derivatives of solutions in double obstacle problems, J. Math. Anal. Appl. 272 (2002) 19–29.
[14] G.B. Li, O. Martio, Stability of solutions of varying degenerate elliptic equations, Indiana Univ. Math. J. 47 (1998) 873–891.
[15] G. Li, Y. Wang, G. Bao, The existence of solutions to the nonhomogeneous A-harmonic equation, J. Inequal. Appl. 2011 (2011) 80. http://dx.doi.org/10.1186/1029-242X-2011-80.
[16] M. Reppen, P. Moosavi, A review of the double obstacle problem, SA104X Degree Project, in Engineering Physics, First Level Department of Mathematics, KTH, Stockholm 2011.
[17] J.F. Rodrigues, Obstacle Problems in Mathematical Physics, in: North Holland Mathematics Studies, North-Holland, 1987.
[18] C. Sbordone, Very weak minimizers of integrals of the calculus of variations, Rend. Semin. Mat. Univ. Politec. Torino 52 (1994) 65–69.
[19] B. Stroffolini, On weakly A-harmonic tensors, Studia Math. 114 (1995) 289–301.
[20] Y. Tong, J. Li, J. Gu, Local boundedness results for very weak solutions of double obstacle problems, J. Inequal. Appl. 2012 (2012) 43. http://dx.doi.org/10.1186/1029-242X-2012-43.
[21] G.M. Troianiello, Elliptic Differential Equations and Obstacle Problems, Plenum Press, New York and London, 1987.
[22] X. Xu, X. Lu, Y. Tong, Regularity of the very weak solutions for double obstacle problems, Adv. Mater. Res. 143–144 (2011) 1396–1400.