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The Riemann problem for one dimensional generalized Chaplygin gas dynamics☆

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ABSTRACT

The Riemann problem for one dimensional generalized Chaplygin gas dynamics is considered. Its two characteristic fields are genuinely nonlinear, but the nonclassical solutions appear. The formation of mechanism for δ-shock is analyzed, that is the one-shock curve and the two-shock curve do not intersect each other in the phase plane. The Riemann solutions are constructed, and the generalized Rankine–Hugoniot conditions and the δ-entropy condition are clarified. By the interaction of the delta-shock wave with the elementary waves, the generalized Riemann problem for this system is presented. Furthermore, by studying the limits of the solutions as perturbed parameter ε approaches zero, one can observe that the Riemann solutions are stable for such perturbations of the initial data. Some numerical simulations are given to illustrate our analysis.

1. Introduction

In this paper, we consider the Riemann problem for the Euler equations modeling isentropic compressible fluids

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \end{cases} \tag{1.1}$$

where the unknown variable ρ denotes the density of the mass, u the velocity. The system (1.1) with the equation of state

$$p = -s\rho^{-\alpha}, \quad 0 < \alpha \le 1,\ s > 0 \tag{1.2}$$

is called the generalized Chaplygin gas dynamics.

The existence of entropy solutions for the Cauchy problem associated with (1.1) was established in the case of polytropic perfect gases

$$p = s\rho^{\gamma}, \quad s > 0,\ \gamma > 1, \tag{1.3}$$

first by Diperna [16], Ding, Chen and Luo [15], and Chen [4] based on the compensated compactness method, and then motivated by a kinetic formulation of hyperbolic conservation laws, by Lions, Perthame, and Tadmor [31]. General pressure laws $p(\rho)$ were covered first by Chen and Lefloch [5,6]. Lu [34] used the theory of compensated compactness coupled with some basic ideas of the kinetic formulation to establish an existence theorem. Recently, Chen and Perepelitsa [8] established the vanishing viscosity limit of the Navier–Stokes equations to the isentropic Euler equations for one-dimensional compressible fluid flow. For $\gamma = 1$, the system (1.1) is called the isothermal gas dynamics and the existence of

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a weak entropy solution to Cauchy problems of the system was obtained by Huang and Wang [21], among others [36,32,12, 33,45,30].

The purpose of this paper is to deal with the Riemann problem and the generalized Riemann problem for system (1.1) and (1.2), which has been advertised as a possible model for dark energy [17,10,41]. When $s = 1$, $\alpha = 1$ in (1.2), (1.1) is the isentropic Chaplygin gas, which were introduced by Chaplygin [3], Tsien [49] and von Karman [50] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. Y. Brenier [1] studied the one dimensional Riemann problems and obtained the solutions with concentration. Cheng and Yang [9] studied the Riemann problem for the relativistic Chaplygin Euler equations. In addition, D. Serre [40] studied the interaction of pressure waves for the 2-D isentropic irrotational Chaplygin gas. He constructively proved the existence of transonic solutions for two cases: saddle and vortex of 2-D Riemann problem. Guo, Sheng and Zhang [18] considered the 2-D Riemann problem for the Chaplygin gas. Lai, Sheng and Zheng [26] discussed simple waves for two-dimensional self-similar flow for the Chaplygin gas and found a new type of discontinuity which is a discontinuity supported by a pressure delta function in course of constructing the global solutions, also see [23] for related results.

In this paper, we pay more attention to the case $0 < \alpha < 1$ in (1.2), which substantial difference with Chaplygin gas $\alpha = 1$ lies in that its two characteristic fields are genuinely nonlinear, but nonclassical solutions also appear: delta-shock wave type solution. In addition, the formation of mechanism for the singular solution may different from the case of the pressureless fluids, see the analysis and numerical simulations presented by Chen and Liu in [7].

Delta-shock wave is a kind of nonclassical nonlinear waves on which at least one of the state variables becomes a singular measure. Korchinski [24] introduced the concept of the Dirac function into the classical weak solution when he studied the Riemann problem for the following system

$$\begin{cases} u_t + \left(\dfrac{1}{2}u^2\right)_x = 0, \\ v_t + \left(\dfrac{1}{2}uv\right)_x = 0, \end{cases} \tag{1.4}$$

in his unpublished Ph.D. Thesis in 1977. Tan, Zhang and Zheng [48] considered the system

$$\begin{cases} u_t + (u^2)_x = 0, \\ v_t + (uv)_x = 0 \end{cases} \tag{1.5}$$

and discovered that the form of Dirac delta functions supported on shocks was used as parts in their Riemann solutions for certain initial data. There is another well-known example, i.e. the one-dimensional system of pressureless Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = 0, \end{cases} \tag{1.6}$$

which has been analyzed extensively since 1994; for example, see [7,20,27–29,44] and the references cited therein. Recently, the weak asymptotic method was widely used to study the δ-shock wave type solution by Danilov and Shelkovich et al. [14,37,42], and also see papers [38,47,22,13,19,52,51,35,43] for the related equations and results.

The organization of this paper is as follows. In Section 2, by characteristic analysis, the Riemann solutions to the generalized Chaplygin gas dynamics are constructed and the existence of the nonclassical solutions: δ-shock waves, is analyzed. The generalized Rankine–Hugoniot conditions and δ-entropy condition are clarified. In Section 3, we consider the initial value problem with three constant states. With the help of the interaction of the δ-shock and elementary waves, the global solutions are constructed. Moreover, we prove that the solutions of the perturbed initial value problem converge to the corresponding Riemann solutions as ε approaches zero, which shows the stability of the Riemann solutions for the small perturbation, and analyze the large time-asymptotic behaviors of the solutions. In Section 4, we present some representative numerical results, produced by semidiscrete central-upwind schemes in [25], to investigate the interaction of δ-shock wave and rarefaction waves.

2. Solutions to the Riemann problem

2.1. Elementary waves and some Riemann solutions

In this section, we will discuss Riemann solutions to Eqs. (1.1). Consider the Riemann problem of the generalized Chaplygin gas equations (1.1) with Riemann initial data

$$(\rho, u)(x, 0) = (\rho_\pm, u_\pm), \quad \pm x > 0, \tag{2.1}$$

where $\rho_\pm > 0$ and $u_\pm$ are constants.

The Eqs. (1.1) can be written in matrix form

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ p'(\rho)/\rho & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0. \tag{2.2}$$

It is easy to see that the eigenvalues are

$$\lambda_1 = u - \sqrt{s\alpha}\rho^{-\frac{1+\alpha}{2}} \quad \text{and} \quad \lambda_2 = u + \sqrt{s\alpha}\rho^{-\frac{1+\alpha}{2}}, \tag{2.3}$$

so that the system is strictly hyperbolic. The corresponding right-eigenvectors are

$$\overrightarrow{r_1} = \left(1, -\frac{c}{\rho}\right)^T \quad \text{and} \quad \overrightarrow{r_2} = \left(1, \frac{c}{\rho}\right)^T, \tag{2.4}$$

where $c = \sqrt{s\alpha}\rho^{-\frac{1+\alpha}{2}}$. The first and second characteristic fields are genuinely nonlinear with $\nabla\lambda_1 \cdot \overrightarrow{r_1} \neq 0$ and $\nabla\lambda_2 \cdot \overrightarrow{r_2} \neq 0$ for $\alpha \neq 1$, in which ∇ denotes the gradient with respect to (ρ, u). Therefore, in classical sense, the associated waves are rarefaction waves or shocks.

The Riemann invariants along the characteristic fields are

$$\omega = u - \frac{2\sqrt{s\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} \quad \text{and} \quad z = u + \frac{2\sqrt{s\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}. \tag{2.5}$$

Since Eqs. (1.1) and the Riemann data are invariant under uniform stretching of coordinates

$$(x, t) \longrightarrow (\kappa x, \kappa t), \quad \kappa \text{ is constant},$$

we consider the self-similar solutions of (1.1) and (2.1)

$$(\rho, u)(x, t) = (\rho, u)(\xi), \quad \xi = x/t.$$

Then the Riemann problem is reduced to a boundary value problem of ordinary differential equations:

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ [6pt] - \xi(\rho u)_\xi + (\rho u^2 - s\rho^{-\alpha})_\xi = 0, \end{cases} \tag{2.6}$$

with

$$(\rho, u)(\pm\infty) = (\rho_\pm, u_\pm).$$

For smooth solutions, Eqs. (2.6) can be rewritten as

$$\begin{pmatrix} -\xi + u & \rho \\ p'(\rho)/\rho & -\xi + u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_\xi = 0. \tag{2.7}$$

It follows from (2.7) that besides the constant solution ($\rho > 0$), it provides a rarefaction wave which is a continuous solution of (2.7) in the form $(\rho, u)(\xi)$. Given a state (ρ_-, u_-), the possible states (ρ, u) that can be connected to state (ρ_-, u_-) on the right by a centered rarefaction wave in the 1-family are as follows:

$$R_1(\rho_-, u_-): \begin{cases} \xi = u - \sqrt{s\alpha}\rho^{-\frac{1+\alpha}{2}}, \\ u - \dfrac{2\sqrt{s\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} = u_- - \dfrac{2\sqrt{s\alpha}}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}}, \quad \rho < \rho_-. \end{cases} \tag{2.8}$$

Similarly, for a given right state (ρ_+, u_+), the rarefaction wave curve which are the sets of states that can be connected on the left in the 2-family are as follows:

$$R_2(\rho_+, u_+): \begin{cases} \xi = u + \sqrt{s\alpha}\rho^{-\frac{1+\alpha}{2}}, \\ u + \dfrac{2\sqrt{s\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} = u_+ + \dfrac{2\sqrt{s\alpha}}{1+\alpha}\rho_+^{-\frac{1+\alpha}{2}}, \quad \rho < \rho_+. \end{cases} \tag{2.9}$$

Given a state (ρ_-, u_-), we consider possible state (ρ, u) that can be connected to state (ρ, u) on the right by a shock wave. For a bounded discontinuous solutions, the Rankine–Hugoniot condition holds

$$\begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma[\rho u] + [\rho u^2 - s\rho^{-\alpha}] = 0, \end{cases} \tag{2.10}$$

where and in what follows we use the notation $[h] = h_r - h_\ell$ with h_ℓ and h_r the values of function h on the left and right-hand sides of the discontinuity curve, respectively, and σ is the velocity of the discontinuity.

By direct calculation, the Lax shock inequalities implies $\rho > \rho_-$. So for a given left state (ρ_-, u_-), the possible state can be connected to (ρ_-, u_-) on the right by one-shock wave are as follows:

$$S_1(\rho_-, u_-): u = u_- - \left(\frac{1}{\rho\rho_-}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_-), \quad \rho > \rho_-. \tag{2.11}$$

Similarly, for a given right state (ρ_+, u_+), it is easy to get two-shock wave as follows:

$$S_2(\rho_+, u_+) : u = u_+ + \left(\frac{1}{\rho\rho_+}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_+), \quad \rho > \rho_+. \tag{2.12}$$

Now, we consider the asymptote of the shock wave curves.

Lemma 2.1. *The* 1*-shock wave curve* $S_1(\rho_-, u_-)$ *has a straight line* $u = u^-$ *as its asymptote, and the* 2*-shock wave curve* $S_2(\rho_+, u_+)$ *has a straight line* $u = u^+$ *as its asymptote, where* $u^- = u_- - \sqrt{s}\rho_-^{-\frac{1+\alpha}{2}}$ *and* $u^+ = u_+ + \sqrt{s}\rho_+^{-\frac{1+\alpha}{2}}$.

Proof. It is clear from (2.11) that

$$u(\rho) = u_- - \left(\frac{1}{\rho_-}\left(1 - \frac{\rho_-}{\rho}\right)\right)^{1/2}(s\rho_-^{-\alpha} - s\rho^{-\alpha})^{1/2}, \quad \rho > \rho_-,$$

which implies

$$\lim_{\rho\to+\infty} u(\rho) = u_- - \sqrt{s}\rho_-^{-\frac{1+\alpha}{2}}.$$

Similarly, for 2-shock wave curve, it is easy to obtain that

$$\lim_{\rho\to+\infty} u(\rho) = u_+ + \sqrt{s}\rho_+^{-\frac{1+\alpha}{2}}.$$

We complete the proof. □

Let (ρ, u) be the intermediate state in the sense that (ρ_-, u_-) and (ρ, u) are connected by one-shock S_1 and that (ρ, u) and (ρ_+, u_+) are connected by two-shock S_2, then it is clear from (2.11) and (2.12) that

$$u_- - u_+ = \left(\frac{1}{\rho\rho_-}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_-) + \left(\frac{1}{\rho\rho_+}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_+). \tag{2.13}$$

If $u^- \geq u^+$, then it follows from (2.13) that

$$\left(\frac{1}{\rho\rho_-}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_-) + \left(\frac{1}{\rho\rho_+}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_+) \geq \sqrt{s}\rho_-^{-\frac{1+\alpha}{2}} + \sqrt{s}\rho_+^{-\frac{1+\alpha}{2}}. \tag{2.14}$$

Next, we proof that unless $\rho \to \infty$, the inequality (2.14) does not hold.

In fact, we only need to obtain

$$\left(\frac{1}{\rho\rho_-}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_-) < \sqrt{s}\rho_-^{-\frac{1+\alpha}{2}}, \tag{2.15}$$

$$\left(\frac{1}{\rho\rho_+}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_+) < \sqrt{s}\rho_+^{-\frac{1+\alpha}{2}} \tag{2.16}$$

for $\rho \nrightarrow \infty$ and the sign of equality holds in (2.14) as $\rho \to \infty$.

For (2.15), it suffices to show

$$\left((\rho_-^{-\alpha} - \rho^{-\alpha})\left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)\right)^{\frac{1}{2}} < \rho_-^{-\frac{1+\alpha}{2}}, \tag{2.17}$$

which yields

$$\rho_-\rho^{-\alpha} - \rho^{1-\alpha} - \rho_-^{1-\alpha} < 0. \tag{2.18}$$

Setting $\beta = \frac{\rho}{\rho_-} > 1$, for $\rho > \rho_-$, we deduce from (2.18)

$$1 - \beta - \beta^{\alpha} < 0,$$

which is true for $\beta > 1$. A similar argument gives the same results for (2.16). In addition, it is easy to obtain from (2.17) that the sign of equality holds in (2.14) as $\rho \to \infty$.

Through the above analysis, it is clear from $u^+ \leq u^-$ that 1-shock curve $S_1(\rho_-, u_-)$ does not intersect 2-shock curve $S_2(\rho_+, u_+)$ and we must construct the solutions in nonclassical sense. Now, we summarize that the sets of states consist of the rarefaction wave curve R and the shock wave curve S for a given left state (ρ_-, u_-). Starting from point A, which is

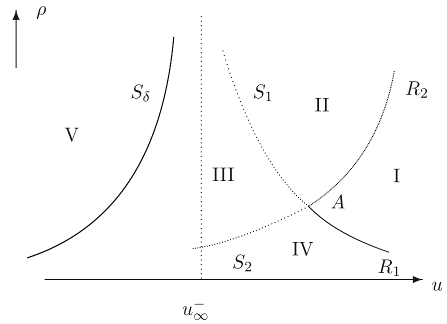

Fig. 2.1. Curves of elementary waves.

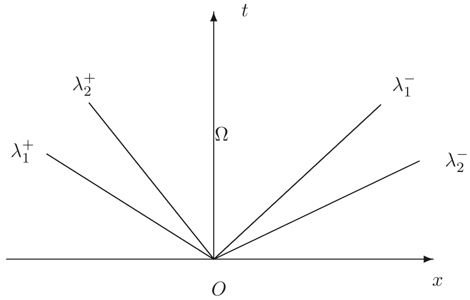

Fig. 2.2. Analysis of characteristics for delta-shock waves.

(ρ_-, u_-) in the (u, ρ) phase plane, we draw four curves R_1, R_2, S_1 and S_2, where the R_2 curve consists of right states (ρ, u), which can be connected with $A(\rho_-, u_-)$ by rarefaction waves defined as follows:

$$u + \frac{2\sqrt{s\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} = u_- + \frac{2\sqrt{s\alpha}}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}}, \quad \rho > \rho_-, \tag{2.19}$$

and the S_2 curve consists of right states (ρ, u), connected with point A by shock wave, which is defined by

$$u = u_- + \left(\frac{1}{\rho\rho_-}\frac{[p]}{[\rho]}\right)^{1/2}(\rho - \rho_-), \quad \rho < \rho_-. \tag{2.20}$$

In addition, we draw a S_δ curve, which is determined as follows

$$u^+ = u + \sqrt{s}\rho^{-\frac{1+\alpha}{2}} = u^- = u_- - \sqrt{s}\rho_-^{-\frac{1+\alpha}{2}}, \quad \rho > 0. \tag{2.21}$$

These curves divide the phase plane $(\rho > 0)$ into five regions, as shown in Fig. 2.1.

Using these elementary waves: shocks and rarefaction waves, one can construct the solutions of (1.1) and (2.1) by the analysis method in phase plane. According to the right state (ρ_+, u_+) in the different region, One can construct the unique global Riemann solution connecting two constant states $(\rho_\pm, u_\pm)$ as follows:

1. $(\rho_+, u_+) \in \mathrm{I}: \ R_1 + R_2$, 2. $(\rho_+, u_+) \in \mathrm{II}: \ S_1 + R_2$
3. $(\rho_+, u_+) \in \mathrm{III}: \ S_1 + S_2$, 4. $(\rho_+, u_+) \in \mathrm{IV}: \ R_1 + S_2$.

In addition, when $(\rho_+, u_+) \in \mathrm{V}$, we need to seek a nonclassical solution.

2.2. *Delta-shock wave type solution*

It is obvious to construct the unique global Riemann solution that we need to consider the existence and uniqueness of the nonclassical solutions. From the above discussion, we can obtain that the nonclassical solution may occur under the condition $u^+ \le u^-$. In this case, we have

$$\lambda_1^+ = u_+ - c_+ < \lambda_2^+ = u_+ + c_+ < \lambda_1^- = u_- - c_- < \lambda_2^- = u_- + c_-, \tag{2.22}$$

which means that the characteristic lines from initial data will overlap in the domain Ω, shown in Fig. 2.2. So singularity must happen in Ω. It is well known that the singularity is impossible to be a jump with finite amplitude, which implies that there is no solution that is piecewise smooth and bounded, Hence, the Riemann solutions with weighted δ-measure should be constructed.

In what follows, we introduce a definition of a generalized solution [35,37,42] for system (1.1) with (1.2).

Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a graph in the closed upper half-plane $\{(x,t) : x \in \mathbb{R}, t \in [0,+\infty)\} \in \mathbb{R}^2$ containing smooth arc $\gamma_i = \{(x,t) : S_i(x,t) = 0\}$, $i \in I$, and I is a finite set. Let I_0 be subset of I such that an arc γ_i for $i \in I_0$ starts from the points of the x-axis; $\Gamma_0 = \{x_k^0; i \in I_0\}$ is the set of initial points of arc γ_i, $i \in I_0$.

Consider δ-shock wave type initial data $(\rho^0(x), u^0(x))$, where

$$\rho^0(x) = \rho_0(x) + e^0\delta(\Gamma_0),$$

$u^0,\ \rho_0 \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$, $e^0\delta(\Gamma_0) = \sum_{i\in I} e_i^0\delta(x - x_i^0)$ and e_i^0 are constants for $i \in I_0$.

In addition, according to [1] the pressure $p = -s\rho^{-\alpha}$ in (1.1), which is a non-linear terms with respect to ρ, is defined by

$$p^0(x,t) = -s\rho_0^{-\alpha}, \tag{2.23}$$

where the pressure should be noticed that the delta measure does not contribute.

Definition 2.2. A pair of distributions $(\rho(x,t), u(x,t))$ and a graph Γ, where $\rho(x,t)$ has the form of the sum

$$\rho(x,t) = \widehat{\rho}(x,t) + e(x,t)\delta(\Gamma) \quad \text{and} \quad p(x,t) = -s(\widehat{\rho}(x,t))^{-\alpha},$$

$u,\ \widehat{\rho} \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ and $e(x,t)\delta(\Gamma) = \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C(\Gamma)$ for $i \in I$, is called a generalized δ-shock wave type solution of system (1.1) with the δ-shock wave type initial data $(\rho^0(x), u^0(x))$ if the integral identities

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \big(\widehat{\rho}\phi_t + (\widehat{\rho}u)\phi_x\big)dxdt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\phi(x,t)}{\partial\ell}d\ell + \int_{\mathbb{R}} \rho_0(x)\phi(x,0)dx + \sum_{i\in I_0} e_i^0\phi(x_i^0,0) = 0,\\
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} \big(\widehat{\rho}u\phi_t + (\widehat{\rho}u^2 + p)\phi_x\big)dxdt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)u_\delta(x,t)\frac{\partial\phi(x,t)}{\partial\ell}d\ell\\
&\quad + \int_{\mathbb{R}} \rho_0(x)u^0(x)\phi(x,0)dx + \sum_{i\in I_0} e_i^0 u_\delta^0(x_i^0)\phi(x_i^0,0) = 0,
\end{aligned} \tag{2.24}$$

hold for all test functions $\phi(x,t) \in \mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$, where $\frac{\partial\phi(x,t)}{\partial\ell}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i} d\ell$ is a line integral over the arc γ_i, $u_\delta(x,t)$ is the velocity of the δ-shock wave and

$$u_\delta^0(x_i^0) = u_\delta(x_i^0,0) = -\frac{(S_i)_t}{(S_i)_x}\bigg|_{(x_i^0,0)}, \quad i \in I_0.$$

Theorem 2.3. *For the Cauchy problem (1.1) and (2.1), when $u^+ \le u^-$ (i.e., $(\rho_+, u_+) \in V$), (1.1) has a δ-shock wave type solution*

$$\begin{aligned}
&u(x,t) = u_- + [u]H(x - x(t))\\
&\rho(x,t) = \rho_- + [\rho]H(x - x(t)) + e(t)\delta(x - x(t)),
\end{aligned}$$

which satisfies the integral identities (2.24) in the sense of Definition 2.2, where $\Gamma = \{(x,t) : x = x(t) = \sigma_\delta t,\ t \ge 0\}$, $\widehat{\rho}(x,t) = \rho_- + [\rho]H(x - x(t))$,

$$\int_\Gamma e(x,t)\frac{\partial\phi(x,t)}{\partial\ell} = \int_0^\infty e(x,t)\frac{\mathrm{d}\phi(x,t)}{dt}dt,$$

and $H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 0, & \text{for } x < 0,\\ 1, & \text{for } x > 0.\end{cases}$$

In addition

$$e(t) = \sqrt{\rho_-\rho_+[u]^2 - [\rho][p]}\,t, \qquad \sigma_\delta = \frac{[\rho u] + \frac{de}{dt}}{[\rho]}, \tag{2.25}$$

as $\rho_- \ne \rho_+$ and

$$e(t) = (\rho_- u_- - \rho_+ u_+)t, \qquad \sigma_\delta = \frac{1}{2}(u_- + u_+), \tag{2.26}$$

as $\rho_- = \rho_+$.

Proof. We need to check that the constructed δ-measure solution satisfies the Definition 2.2 in the sense of distributions, that is

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}\big(\widehat{\rho}\phi_t+(\widehat{\rho}u)\phi_x\big)dxdt+\int_0^\infty e(t)\frac{\mathrm{d}\phi(x(t),t)}{dt}dt+\int_{\mathbb{R}}\rho_0(x)\phi(x,0)dx=0, \tag{2.27}$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}\big(\widehat{\rho}u\phi_t+(\widehat{\rho}u^2+p)\phi_x\big)dxdt+\int_0^\infty e(t)u_\delta\frac{\mathrm{d}\phi(x(t),t)}{dt}dt+\int_{\mathbb{R}}\rho_0(x)u^0(x)\phi(x,0)dx=0 \tag{2.28}$$

for all the test functions $\phi(x,t)\in\mathbb{D}(R\times R_+)$ and $u_\delta=\sigma_\delta$, where $\rho_0(x)=\rho_-+[\rho]H(x)$ and $u^0(x)=u_-+[u]H(x)$.

Denote by $\mathcal{A}$ the left-hand side of (2.27), we have

$$\begin{aligned}\mathcal{A}=&\int_0^\infty\int_{-\infty}^{x(t)}(\rho_-\phi_t+\rho_-u_-\phi_x)dxdt+\int_0^\infty\int_{x(t)}^{+\infty}(\rho_+\phi_t+\rho_+u_+\phi_x)dxdt\\&+\int_0^\infty e(t)\frac{\mathrm{d}\phi(x(t),t)}{dt}dt+\int_{-\infty}^0\rho_-\phi(x,0)dx+\int_0^\infty\rho_+\phi(x,0)dx.\end{aligned} \tag{2.29}$$

Without loss of generality, we assume $\sigma_\delta>0$, then the first term on the right-hand side of (2.29) equals

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^0\rho_-\phi_tdxdt+\int_0^\infty\int_0^{x(t)}\rho_-\phi_tdxdt+\int_0^\infty\int_{-\infty}^{x(t)}\rho_-u_-\phi_xdxdt\\&=-\int_{-\infty}^0\rho_-\phi(x,0)dx+\int_0^\infty\int_0^{x(t)}\rho_-\phi_tdxdt+\int_0^\infty\rho_-u_-\phi(x(t),t)dt\\&=-\int_{-\infty}^0\rho_-\phi(x,0)dx+\int_0^\infty dx\int_{t(x)}^\infty\rho_-\phi_tdt+\int_0^\infty\frac{\rho_-u_-}{\sigma_\delta}\phi(x,t(x))dx\\&=-\int_{-\infty}^0\rho_-\phi(x,0)dx+\int_0^\infty\left(\frac{u_-}{\sigma_\delta}-1\right)\rho_-\phi(x,t(x))dx.\end{aligned} \tag{2.30}$$

The second term on the right-hand side of (2.29) equals

$$\int_0^\infty dx\int_0^{t(x)}\rho_+\phi_tdt-\int_0^\infty\rho_+u_+\phi(x(t),t)dt=-\int_0^\infty\rho_+\phi(x,0)dx+\int_0^\infty\left(1-\frac{u_+}{\sigma_\delta}\right)\rho_+\phi(x,t(x))dx. \tag{2.31}$$

The third term on the right-hand side of (2.29) equals

$$(\sigma_\delta[\rho]-[\rho u])\,t\phi(x(t),t)\,|_0^{+\infty}-(\sigma_\delta[\rho]-[\rho u])\int_0^{+\infty}\phi(x(t),t)dt=-\frac{(\sigma_\delta[\rho]-[\rho u])}{\sigma_\delta}\int_0^\infty\phi(x,t(x))dx. \tag{2.32}$$

From (2.29)–(2.32), we obtain

$\mathcal{A}=0.$

Similarly, we easily obtain (2.28). So, we complete the proof. □

Using Definition 2.2 and repeating the proof of Theorem 2.3 almost word-for-word, one can derive the generalized Rankine–Hugoniot conditions for δ-shock wave type solutions of the system (1.1).

Theorem 2.4. *Suppose that $\Omega\subset\mathbb{R}\times\mathbb{R}_+$ is some region cut by a smooth curve $\Gamma=\{(x,t):x=x(t)\}$ into a left- and right-hand parts $\Omega_\pm=\{(x,t):\pm(x-x(t))>0\}$, $(\rho(x,t),u(x,t))$, Γ is a generalized δ-shock wave type solution of system (1.1), functions $\widehat{\rho}(x,t),u(x,t)$ are smooth in $\Omega_\pm$, and have one-side limits $\widehat{\rho}_\pm,u_\pm$ on the curve Γ. Then the generalized Rankine–Hugoniot conditions for δ-shocks*

$$\begin{cases}\dfrac{de(t)}{dt}=\big(\dot{x}(t)[\rho]-[\rho u]\big)|_{x=x(t)},\\[2mm]\dfrac{d(e(t)\dot{x}(t))}{dt}=\big(\dot{x}(t)[\rho u]-[\rho u^2+p]\big)|_{x=x(t)},\end{cases} \tag{2.33}$$

where $e(t)\doteq e(x(t),t)$ and $\dot{(\cdot)}=\frac{d}{dt}(\cdot)$.

In addition to the generalized Rankine–Hugoniot conditions (2.33), to guarantee uniqueness, the discontinuity must satisfy

$$\lambda_1^+=u_+-c_+<\lambda_2^+=u_++c_+<\sigma_\delta<\lambda_1^-=u_--c_-<\lambda_2^-=u_-+c_-, \tag{2.34}$$

where $\rho_\pm$ and $u_\pm$ are the respective left- and right-hand limit values of $\rho(x,t),\ u(x,t)$ on the discontinuity curve. Condition (2.34) is called as δ-*entropy condition*. It is overcompressive and means that all the characteristic lines on both sides of the discontinuity are in-coming. A discontinuity satisfying (2.33) and (2.34) will be called a δ-shock wave to system (1.1).

So, we complete the construction of the Riemann solutions to system (1.1).

3. The interaction of δ-shock and elementary waves

To start off we consider the initial value problem with three pieces constant states

$$(\rho,u)(x,0)=\begin{cases}(\rho_-,u_-), & -\infty<x<0,\\ (\rho_m,u_m), & 0<x<\varepsilon,\\ (\rho_+,u_+), & x>\varepsilon,\end{cases} \tag{3.1}$$

where $\varepsilon>0$ is arbitrarily small. The data (3.1) is a perturbation of the Riemann initial data (2.1). Our interest is to investigate whether the Riemann solutions of (1.1) and (2.1) are the limits of the solutions of (1.1) and (3.1) as $\varepsilon\to 0$. In this section, we only consider the interaction of the δ-shock and elementary waves. For the interactions of elementary waves, we refer the readers to the book of Smoller [46] and the monograph of Chang and Hsiao [2]. For a comprehensive survey, we can see the books written by Dafermos [11] and Serre [39]. The above problem can be divided into four cases as follows:

$$\delta\text{-shock}\ \oplus R_1,\qquad \delta\text{-shock}\ \oplus R_2,\qquad \delta\text{-shock}\ \oplus S_1,\qquad \delta\text{-shock}\ \oplus S_2.$$

Before discussing the problem, we will consider some important properties about the shock curves and the rarefaction wave curves in the following lemmas.

Lemma 3.1. *If point B is the interaction point of one-shock curve S_1 with the left state (ρ_A,u_A) and two-shock curve S_2 with the right state (ρ_C,u_C), i.e., $B=S_1\bigcap S_2$, in addition $\rho_A<\rho_B$ and $\rho_C<\rho_B$, then*

$$\lambda_1(A)>\sigma_1(AB)>\lambda_1(B), \tag{3.2}$$

$$\sigma_2(CB)>\lambda_1(C) \tag{3.3}$$

where $\lambda_1(A)$, $\lambda_1(B)$ and $\lambda_1(C)$ stand for the speed of characteristic of the one-rarefaction wave at points A, B and C, respectively; $\sigma_1(AB)$ is the speed of one-shock wave with the left and the right states (ρ_A,u_A) and (ρ_B,u_B) and $\sigma_2(CB)$ is the speed of two-shock wave with the left and the right states (ρ_B,u_B) and (ρ_C,u_C), see Fig. 3.1.

Proof. First, we will proof the result

$$\sigma_1(AB)>\lambda_1(B). \tag{3.4}$$

From (2.11), we have

$$\sigma_1(AB)=\frac{\rho_Bu_B-\rho_Au_A}{\rho_B-\rho_A}=u_B+\frac{\rho_A}{\rho_B-\rho_A}(u_B-u_A). \tag{3.5}$$

Combining (3.5) with (2.11) gives

$$\sigma_1(AB)=u_B-\left(\frac{\rho_A(p_B-p_A)}{\rho_B(\rho_B-\rho_A)}\right)^{\frac{1}{2}}. \tag{3.6}$$

It is clear from (2.8) that

$$\lambda_1(B)=u_B-\sqrt{s\alpha}\,\rho_B^{-\frac{1+\alpha}{2}}. \tag{3.7}$$

Based on (3.6) and (3.7), the inequality (3.4) is equivalent to

$$-\left(\frac{\rho_A(p_B-p_A)}{\rho_B(\rho_B-\rho_A)}\right)^{-\frac{1}{2}}>-\sqrt{s\alpha}\,\rho_B^{\frac{1+\alpha}{2}},$$

i.e.,

$$\frac{s\rho_A(\rho_A^{-\alpha}-\rho_B^{-\alpha})}{\rho_B(\rho_B-\rho_A)}<s\alpha\rho_B^{-(1+\alpha)},$$

which becomes

$$\rho_A\left(\left(\frac{\rho_B}{\rho_A}\right)^{\alpha}-1\right)<\alpha(\rho_B-\rho_A).$$

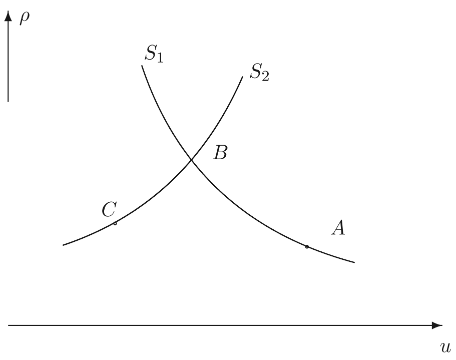

Fig. 3.1. The graphs of shocks S_1 and S_2.

It is enough to obtain

$$\left(\frac{\rho_B}{\rho_A}\right)^{\alpha} - 1 < \alpha\left(\frac{\rho_B}{\rho_A} - 1\right). \tag{3.8}$$

Setting $\frac{\rho_B}{\rho_A} = \beta > 1$ for $\rho_B > \rho_A$, then (3.8) reduces

$$\beta^{\alpha} - 1 < \alpha(\beta - 1). \tag{3.9}$$

Let $f(\beta) = \beta^{\alpha} - \alpha(\beta - 1) - 1$ and we obtain

$$f(1) = 0, \qquad f'(\beta) = \alpha(\beta^{\alpha-1} - 1) < 0, \quad \text{for } 0 < \alpha < 1,$$

which implies

$$f(\beta) < 0, \quad \text{for } \beta > 1.$$

So, we obtain the inequality (3.4). A similar argument gives $\lambda_1(A) > \sigma_1(AB)$.

Now, we will proof (3.3). By (2.12), we have

$$\sigma_2(CB) = \frac{\rho_C u_C - \rho_B u_B}{\rho_C - \rho_B} = u_C + \frac{\rho_B}{\rho_C - \rho_B}(u_C - u_B). \tag{3.10}$$

Combining (2.12) with (3.10) gives

$$\begin{aligned}\sigma_2(CB) &= u_C - \frac{\rho_B}{\rho_C - \rho_B}\left(\frac{p_B - p_C}{\rho_B\rho_C(\rho_B - \rho_C)}\right)^{\frac{1}{2}}(\rho_B - \rho_C)\\ &= u_C + \left(\frac{\rho_B}{\rho_C}\frac{p_B - p_C}{\rho_B - \rho_C}\right)^{\frac{1}{2}}.\end{aligned} \tag{3.11}$$

From (2.8), one can obtain

$$\lambda_1(C) = u_C - \sqrt{s\alpha}\,\rho_C^{-\frac{1+\alpha}{2}}. \tag{3.12}$$

It is easy to obtain (3.3) from (3.11) and (3.12). Therefore, we complete the proof of the lemma. □

Lemma 3.2. *For a given constant state* $\oplus =: (\rho_+, u_+)$, *let* R_2 *be two-rarefaction wave curve with* (ρ_+, u_+) *being the right state,* S_2 *two-shock curve with* (ρ_+, u_+) *being the right state and* S_δ *be a curve which passes through the state* (ρ_+, u_+), *then (see Fig. 3.2)*

(1) *the curve* R_2 *lies above the curve* S_2 *for* $\rho > \rho_+$.
(2) *the curve* S_2 *lies above the curve* S_δ *for* $\rho > \rho_+$.

Proof. From (2.9) and (2.12), we have for $\rho > \rho_+$

$$R_2 : \; u_R = u_+ + \frac{2\sqrt{s\alpha}}{1+\alpha}\rho_+^{-\frac{1+\alpha}{2}} - \frac{2\sqrt{s\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}},$$

$$S_2 : \; u_S = u_+ + \left(\frac{1}{\rho\rho_+}(p - p_+)(\rho - \rho_+)\right)^{\frac{1}{2}}.$$

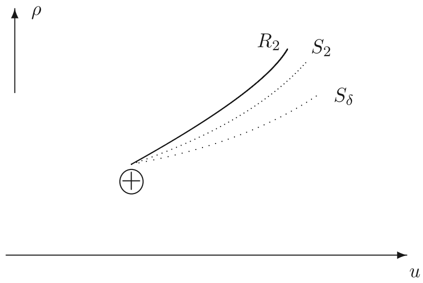

Fig. 3.2. The locations of curves R_2, S_2 and S_δ.

We only need to prove $u_S - u_R > 0$ for $\rho > \rho_+$, i.e.,

$$\left(\frac{1}{\rho\rho_+}(p-p_+)(\rho-\rho_+)\right)^{\frac{1}{2}} > \frac{2\sqrt{s\alpha}}{1+\alpha}\left(\rho_+^{-\frac{1+\alpha}{2}} - \rho^{-\frac{1+\alpha}{2}}\right),$$

which is equivalent to

$$\frac{1}{\rho}\left(\rho_+^{-\alpha} - \rho^{-\alpha}\right)\left(\frac{\rho}{\rho_+} - 1\right) > \frac{4\alpha}{(1+\alpha)^2}\left(\rho_+^{-(1+\alpha)} + \rho^{-(1+\alpha)} - 2\rho_+^{-\frac{1+\alpha}{2}}\rho^{-\frac{1+\alpha}{2}}\right). \tag{3.13}$$

Multiplying by $\rho^{1+\alpha}$ on both sides of (3.13) gives

$$\left(\left(\frac{\rho}{\rho_+}\right)^{\alpha} - 1\right)\left(\frac{\rho}{\rho_+} - 1\right) > \frac{4\alpha}{(1+\alpha)^2}\left(\left(\frac{\rho}{\rho_+}\right)^{1+\alpha} + 1 - 2\left(\frac{\rho}{\rho_+}\right)^{\frac{1+\alpha}{2}}\right). \tag{3.14}$$

Setting $\frac{\rho}{\rho_+} = \beta > 1$ for $\rho > \rho_+$, then (3.14) reduces

$$\left(\beta^{\alpha} - 1\right)\left(\beta - 1\right) > \frac{4\alpha}{(1+\alpha)^2}\left(\beta^{1+\alpha} + 1 - 2\beta^{\frac{1+\alpha}{2}}\right).$$

Denote

$$f(\beta) = \left(\beta^{\alpha} - 1\right)\left(\beta - 1\right) - \frac{4\alpha}{(1+\alpha)^2}\left(\beta^{1+\alpha} + 1 - 2\beta^{\frac{1+\alpha}{2}}\right),$$

then

$$f'(\beta) = \frac{(1-\alpha)^2}{1+\alpha}\beta^{\alpha} - \alpha\beta^{\alpha-1} - 1 + \frac{4\alpha}{1+\alpha}\beta^{\frac{\alpha-1}{2}}, \tag{3.15}$$

$$f''(\beta) = \alpha(1-\alpha)\beta^{\alpha-2}\left(\frac{1-\alpha}{1+\alpha}\beta + 1 - \frac{2}{1+\alpha}\beta^{\frac{1-\alpha}{2}}\right). \tag{3.16}$$

Setting $g(\beta) = \left(\frac{1-\alpha}{1+\alpha}\beta + 1 - \frac{2}{1+\alpha}\beta^{\frac{1-\alpha}{2}}\right)$ gives

$$g(1) = 0, \qquad g'(\beta) = \frac{1-\alpha}{1+\alpha}\left(1 - \beta^{-\frac{1+\alpha}{2}}\right) > 0, \quad \text{for } 0 < \alpha < 1,$$

which reduces $g(\beta) > 0$ for $\beta > 1$. Then one can obtain from (3.16) $f''(\beta) > 0$ for $\beta > 1$. In addition, it is clear from (3.15) that $f'(1) = 0$. So $f'(\beta) > 0$ for $\beta > 1$, which implies $f(\beta) > 0$ with the fact $f(1) = 0$. So, the first statement (1) is true.

From (2.21), we have

$$S_\delta : u_\delta = u_+ + \sqrt{s}\rho_+^{-\frac{1+\alpha}{2}} - \sqrt{s}\rho^{-\frac{1+\alpha}{2}}, \quad \rho > 0. \tag{3.17}$$

It suffices to prove $u_\delta - u_S > 0$ for $\rho > \rho_+$, i.e.,

$$\left(\frac{1}{\rho\rho_+}(p-p_+)(\rho-\rho_+)\right)^{\frac{1}{2}} < \sqrt{s}\left(\rho_+^{-\frac{1+\alpha}{2}} - \rho^{-\frac{1+\alpha}{2}}\right),$$

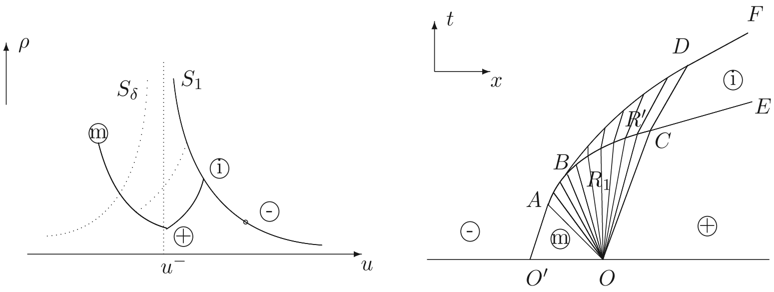

Fig. 3.3. Case 1.1, δ-shock $\bigoplus R_1$.

which is equivalent to

$$\frac{1}{\rho}\left(\rho_+^{-\alpha}-\rho^{-\alpha}\right)\left(\frac{\rho}{\rho_+}-1\right)<\left(\rho_+^{-(1+\alpha)}+\rho^{-(1+\alpha)}-2\rho_+^{-\frac{1+\alpha}{2}}\rho^{-\frac{1+\alpha}{2}}\right). \tag{3.18}$$

Multiplying by $\rho^{1+\alpha}$ on both sides of (3.18) gives

$$\left(\left(\frac{\rho}{\rho_+}\right)^{\alpha}-1\right)\left(\frac{\rho}{\rho_+}-1\right)<\left(\left(\frac{\rho}{\rho_+}\right)^{1+\alpha}+1-2\left(\frac{\rho}{\rho_+}\right)^{\frac{1+\alpha}{2}}\right). \tag{3.19}$$

Setting $\beta=\frac{\rho}{\rho_+}>1$ for $\rho>\rho_+$, (3.19) gives

$$\left(\beta^{\alpha}-1\right)\left(\beta-1\right)<\left(\beta^{1+\alpha}+1-2\beta^{\frac{1+\alpha}{2}}\right),$$

which reduces

$$\beta^{\alpha}+\beta-2\beta^{\frac{1+\alpha}{2}}>0. \tag{3.20}$$

Obviously, the above inequality is true for $\beta>1$ and $0<\alpha<1$, which gives the second statement (2). Then, the proof of the lemma is completed. □

Now, we begin to discuss the interaction of δ-shock with elementary waves.

Case 1. δ-shock $\bigoplus R_1$. Based on the fact whether the two-shock S_2 with the right state (ρ_+, u_+) interacts the one-shock S_1 with the left state (ρ_-, u_-) or not, the case can be divided into two subcases: case 1.1: $S_1\bigcap S_2\neq\varnothing$ and case 1.2: $S_1\bigcap S_2=\varnothing$, that is to say $R_1(\ominus)\bigcap S_2\neq\varnothing$, where $R_1(\ominus)$ is one-rarefaction wave curve with the left state (ρ_-, u_-).

Case 1.1, $S_1\bigcap S_2\neq\varnothing$. Denote by ⓘ $:=(\rho_i, u_i)$ the intersect state of S_1 and S_2, see Fig. 3.3.

The δ-shock $O'A$ intersects with the rarefaction wave R_1 at point A. When the rarefaction wave curve R_1 intersects with the curve S_δ and the intersection is denoted by the state (ρ_B, u_B), it is easy to see from points A to B that the δ-entropy condition is satisfied. The values of (ρ_B, u_B) are determined by

$$\begin{cases} u_B-\dfrac{2\sqrt{s\alpha}}{1+\alpha}\rho_B^{-\frac{1+\alpha}{2}}=u_m-\dfrac{2\sqrt{s\alpha}}{1+\alpha}\rho_m^{-\frac{1+\alpha}{2}},\\ u_B+\sqrt{s}\rho_B^{-\frac{1+\alpha}{2}}=u_--\sqrt{s}\rho_-^{-\frac{1+\alpha}{2}}.\end{cases}$$

Meanwhile, two shocks $\widehat{BDF}$ and $\widehat{BCE}$ are formed at the point B and a new one-rarefaction wave R' occurs, which state (ρ, u) is determined by

$$\begin{cases} u=u_--\left(\dfrac{1}{\rho\rho_-}(\rho-\rho_-)(p-p_-)\right)^{\frac{1}{2}},\\ u=\overline{u}+\left(\dfrac{1}{\rho\overline{\rho}}(\rho-\overline{\rho})(p-\overline{p})\right)^{\frac{1}{2}},\\ \overline{u}-\dfrac{2\sqrt{s\alpha}}{1+\alpha}\overline{\rho}^{-\frac{1+\alpha}{2}}=u_m-\dfrac{2\sqrt{s\alpha}}{1+\alpha}\rho_m^{-\frac{1+\alpha}{2}},\\ \rho_+\leq\overline{\rho}\leq\rho_B.\end{cases} \tag{3.21}$$

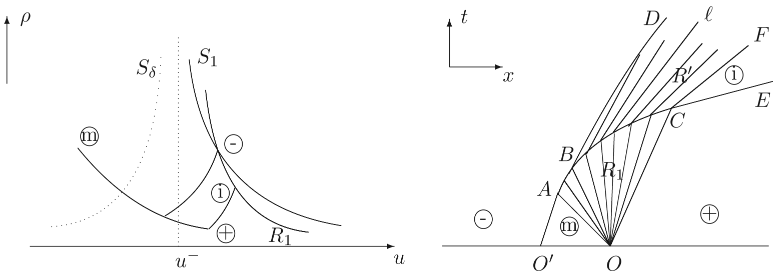

Fig. 3.4. Case 1.2, δ-shock $\bigoplus R_1$.

Based on (2.11), the one-shock curve $\widehat{BD}$ is calculated by

$$\begin{cases} \dfrac{dx}{dt} = \sigma_{BD}, \\ x|_{t=t_B} = x_B, \end{cases}$$

where

$$\sigma_{BD} = \frac{\rho u - \rho_- u_-}{\rho - \rho_-}$$

and (ρ, u) is given by (3.21).

Similarly, the two-shock curve $\widehat{BC}$ is determined by

$$\begin{cases} \dfrac{dx}{dt} = \sigma_{BC}, \\ x|_{t=t_B} = x_B, \end{cases}$$

where

$$\sigma_{BC} = \frac{\rho u - \overline{\rho}\overline{u}}{\rho - \overline{\rho}},$$

and (ρ, u) and $(\overline{\rho}, \overline{u})$ are given by (3.21).

With the help of Lemma 3.1, it is easy to obtain that the shock $\widehat{BDF}$ can penetrate the rarefaction wave R' and the shock $\widehat{BDE}$ can penetrate the rarefaction wave R_1.

Thus, as $t \to \infty$, the solution can be described as

$$(\rho_-, u_-) + S_{DF} + (\rho_i, u_i) + S_{CE} + (\rho_+, u_+),$$

and as $\varepsilon \to 0$, the limit of the solution of (1.1) and (3.1) is the corresponding Riemann solution of (1.1) and (2.1).

Case 1.2 $R_1(\ominus) \bigcap S_2 \neq \varnothing$. Denote by ① $:= (\rho_i, u_i)$ the intersect state of R_1 and S_2, see Fig. 3.4. Unlike the above case 1.1, the shock $\widehat{BD}$ cannot penetrate the R'. The shock $\widehat{BCE}$ can penetrate the R_1 and can be determined by the above method and omit it here.

Now, we consider the property of one-shock curve $\widehat{BD}$, which is determined by

$$\begin{cases} \dfrac{dx}{dt} = \sigma_{BD}, \\ x|_{t=t_B} = x_B, \end{cases}$$

where

$$\sigma_{BD} = \frac{\rho u - \rho_- u_-}{\rho - \rho_-}$$

and (ρ, u) is given by (3.21).

From (2.11), we have

$$\sigma_{BD}(\rho) = u_- + \frac{\rho}{\rho - \rho_-}(u - u_-) = u_- - \left(\frac{\rho}{\rho_-}\frac{p - p_-}{\rho - \rho_-}\right)^{\frac{1}{2}},$$

which implies

$$\lim_{\rho \to \rho_-} \sigma_{BD}(\rho) = u_- - \sqrt{p'(\rho_-)} = u_- - \sqrt{s\alpha}\,\rho_-^{-\frac{1+\alpha}{2}}. \tag{3.22}$$

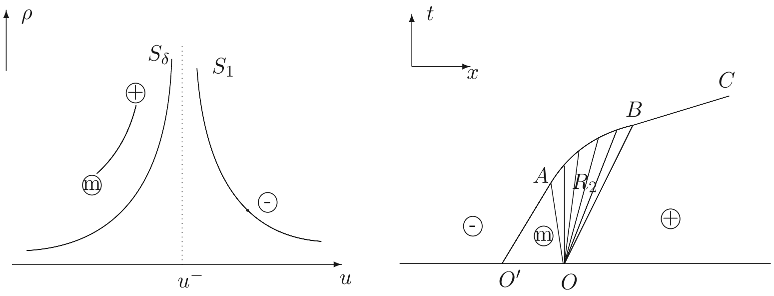

Fig. 3.5. Case 2, δ-shock $\bigoplus R_2$.

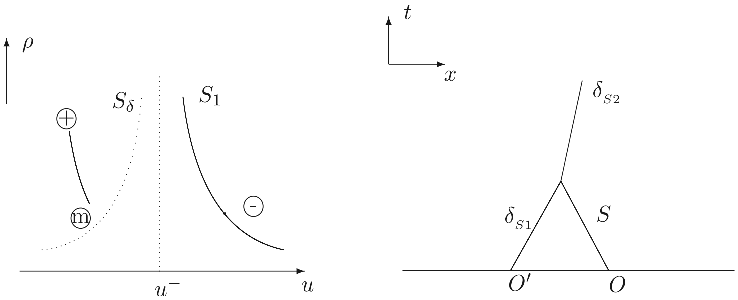

Fig. 3.6. Case 3, δ-shock $\bigoplus S_1$.

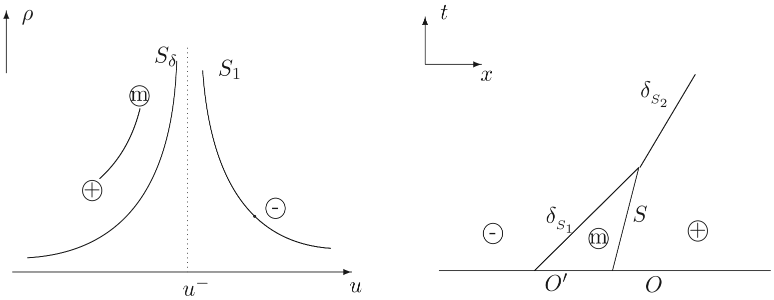

Fig. 3.7. Case 4, δ-shock $\bigoplus S_2$.

From the first identity of (2.8) and (3.22), we can conclude that the shock $\widehat{BD}$ has the straight line $\ell : \frac{dx}{dt} = \xi_\ell = u_- - \sqrt{s\alpha}\rho_-^{-\frac{1+\alpha}{2}}$ as its asymptote, which gives that the shock $\widehat{BD}$ cannot penetrate the rarefaction wave R'. For large time, i.e., as $t \to \infty$, the solution can be expressed as

$$(\rho_-, u_-) + R' + (\rho_i, u_i) + S_{cE} + (\rho_+, u_+).$$

Case 2, δ-shock $\bigoplus R_2$. By Lemma 3.2, the two-rarefaction wave curve R_2 from the state (ρ_m, u_m) to the state (ρ_+, u_+) is on the left-hand side of the curve S_δ, which implies that in the curse of interaction of the δ-shock and the R_2, the δ-entropy condition (2.34) is satisfied and the δ-shock can penetrate the R_2, see Fig. 3.5.

Case 3, δ-shock $\bigoplus S_1$. In this case, the δ-shock overtakes the one-shock at some time t, and after the time t, the δ-entropy condition is satisfied, then a new δ-shock is formed, see Fig. 3.6.

Case 4, δ-shock $\bigoplus S_2$. In this case, the δ-shock overtakes the two-shock at some time t, and after the time t, the δ-entropy condition is satisfied, then a new δ-shock is formed, see Fig. 3.7.

So far, we have finished the discussion for the interactions of the δ-shock and the elementary waves and the global solutions for the perturbed initial value problem (1.1) and (3.1) have been constructed. We summarize our results in the following.

Theorem 3.3. *There exists a unique generalized solution to the perturbed initial value problem (1.1) and (3.1). The limits of the perturbed Riemann solution of (1.1) and (3.1) are exactly the corresponding Riemann solutions of (1.1) and (2.1). The Riemann solutions of (1.1) and (2.1) are stable with respect to such small perturbations of the initial data.*

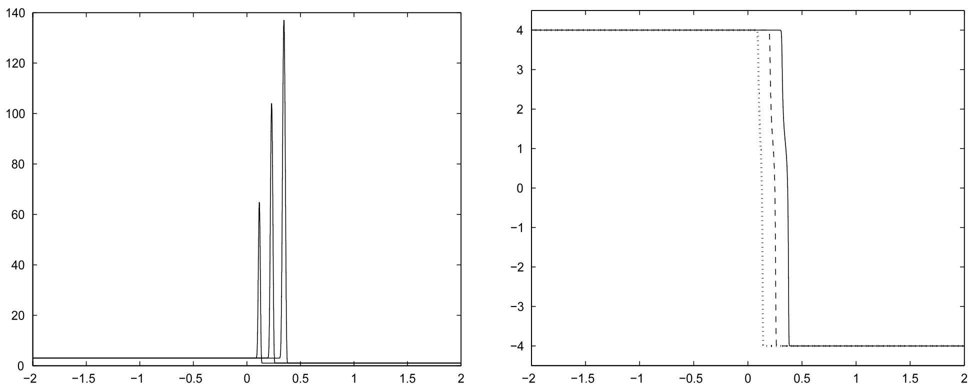

Fig. 4.1. Example 1. Left: the plots of density at times 0.05, 0.1 and 0.2, respectively. Right: the plots of velocity at corresponding times, respectively.

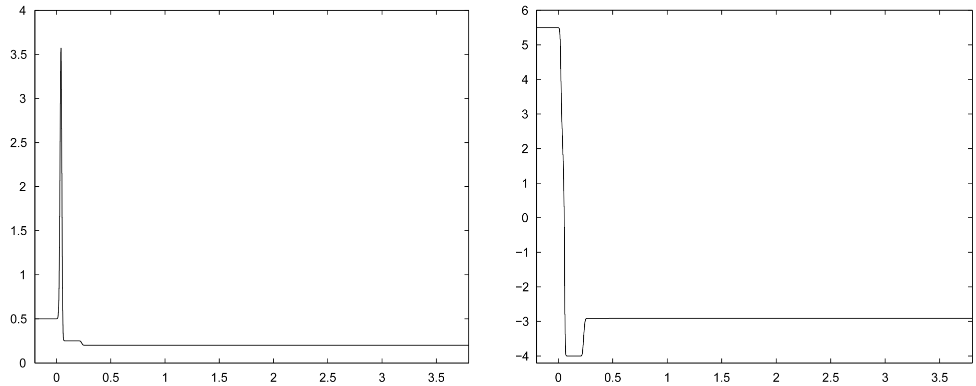

Fig. 4.2. Example 2. Left: the plot of density at time 0.02. Right: the plot of velocity at time 0.02.

4. Numerical simulations

In this section, we will provide some corresponding numerical results, which are consistence with the above discussions. In the following examples, we take $\alpha = 0.5$ and $s = 5.0$ in (1.2). To investigate the interaction of δ-shock and elementary waves, we use the semidiscrete central-upwind scheme [25] and the space step $dx = 1/1000$ and mesh ration $\lambda = 0.10$.

Example 1, we solve the Riemann problem for (1.1) and (1.2) and the Riemann initial data is

$$(\rho, u)(x, 0) = \begin{cases} (3.0, 4.0) & \text{for } x < 0, \\ (1.0, -4.0) & \text{for } x > 0. \end{cases}$$

The numerical results are shown in Fig. 4.1, which indicate the formation of a δ-shock and that the strength of δ-shock increases dramatically as time process.

Example 2, to investigate the interaction of the δ-shock and one-family rarefaction wave, we compute the initial data

$$(\rho, u)(x, 0) = \begin{cases} (0.5, 5.5), & \text{for } x < 0, \\ (0.25, -4.0), & \text{for } 0 < x < 0.4, \\ (0.20, -2.9137), & \text{for } x > 0.4. \end{cases} \tag{4.1}$$

Fig. 4.2 shows that at time 0.02 (less than t_A), the solution consists of a δ-shock and a one-family rarefaction wave, also see Fig. 3.3. The numerical result at time 0.26 are shown in Fig. 4.3, which indicates that when $t_C < t < t_D$, the solution may expressed as

$$(\rho_-, u_-) + S_{BD} + R' + (\rho_i, u_i) + S_{CE} + (\rho_+, u_+).$$

At $t = 0.9 > t_D$, the solution is changed into

$$(\rho_-, u_-) + S_{DF} + (\rho_i, u_i) + S_{CE} + (\rho_+, u_+)$$

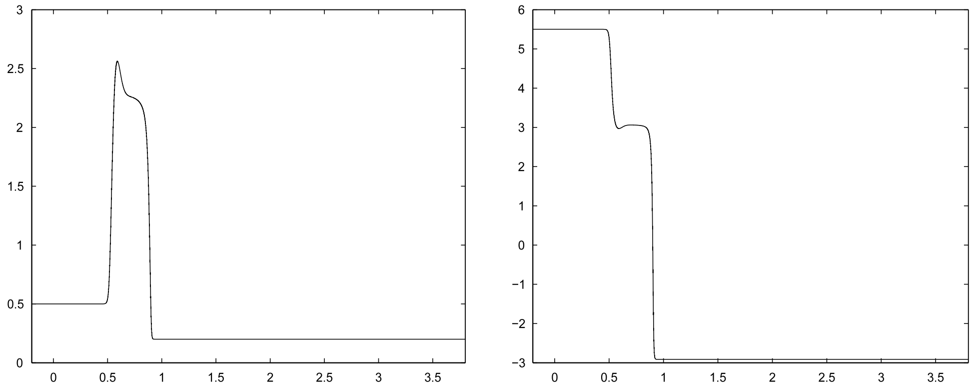

Fig. 4.3. Example 2. Left: the plot of density at time 0.26. Right: the plot of velocity at time 0.26.

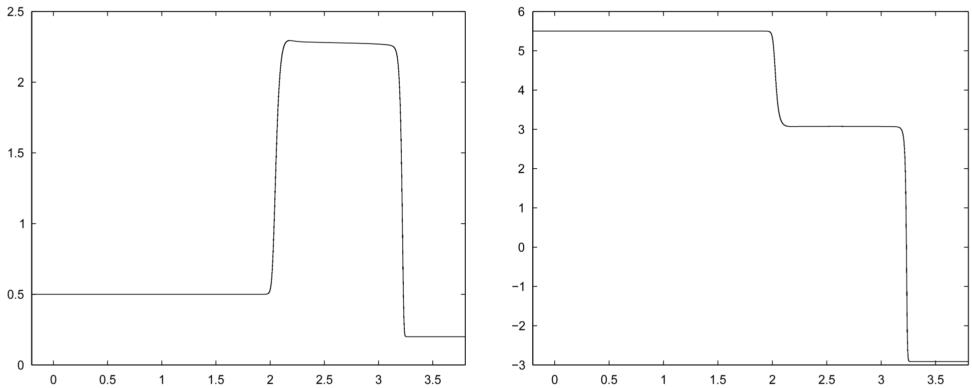

Fig. 4.4. Example 2. Left: the plot of density at time 0.90. Right: the plot of velocity at time 0.90.

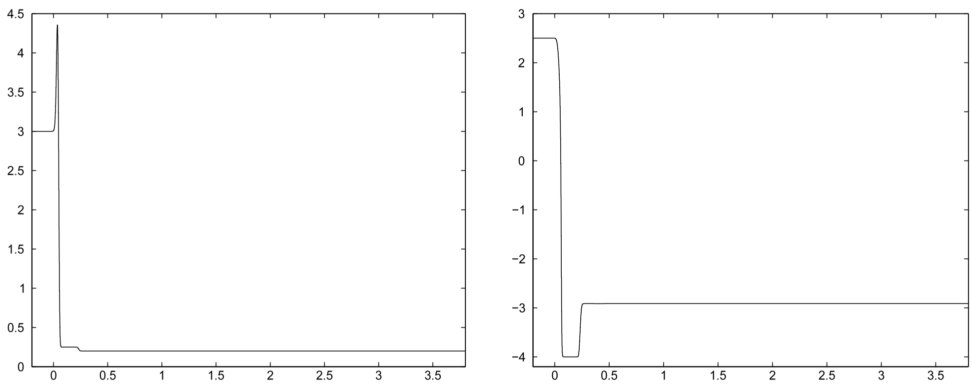

Fig. 4.5. Example 3. Left: the plot of density at time 0.02. Right: the plot of velocity at time 0.02.

which is displayed in Fig. 4.4. Example 3, we consider the initial data

$$(\rho, u)(x, 0) = \begin{cases} (3.0, 2.5), & \text{for } x < 0, \\ (0.25, -4.0), & \text{for } 0 < x < 0.4, \\ (0.20, -2.9137), & \text{for } x > 0.4 \end{cases} \tag{4.2}$$

as further discussion about the interaction of a δ-shock and a one-family rarefaction wave, see Fig. 3.4. The numerical results are shown in Figs. 4.5–4.6. Fig. 4.5 shows that at time $t = 0.02 < t_A$, the solution is a δ-shock and a one-family rarefaction wave. When $t = 0.4 > t_C$, the solution is a shock, a rarefaction wave and a shock, which is shown in Fig. 4.6.

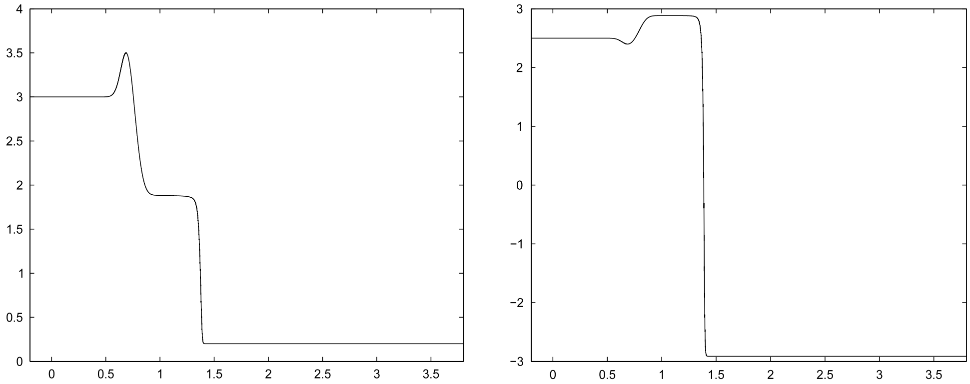

Fig. 4.6. Example 3. Left: the plot of density at time 0.40. Right: the plot of velocity at time 0.40.

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