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Least energy solutions of the Emden–Fowler equation in hollow thin symmetric domains ☆

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ABSTRACT

In this paper, the Emden–Fowler equation is studied in a hollow thin domain which is invariant under the action of a closed subgroup of the orthogonal group. Then it is proved that if the domain is thin enough, a least energy solution is not group invariant.

1. Introduction

In this paper, we prove the existence of asymmetric positive solutions in hollow thin symmetric domains for the Emden–Fowler equation

$$-\Delta u = u^p, \quad u > 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 2$, $1 < p < \infty$ when $N = 2$ and $1 < p < (N+2)/(N-2)$ when $N \geq 3$. Let G be a closed subgroup of the orthogonal group $O(N)$ such that $G \neq \{I\}$, where I is the unit matrix. We call Ω a *G invariant domain* if $g(\Omega) = \Omega$ for any $g \in G$. We call a solution u a *G invariant solution* if $u(gx) = u(x)$ for any $g \in G$ and $x \in \Omega$. When Ω is G invariant, (1.1) has a G invariant positive solution, which will be proved in Lemma 1.3. However, we are looking for a positive solution without G invariance. Indeed, we shall prove that a least energy solution, which will be defined later on, is not G invariant in a hollow thin domain. Let us give the most typical and simplest example of a hollow thin domain. Consider two regular n polygons with different sizes which have the common center and each side of the small polygon is parallel to a side of the large polygon and the distance between two polygons is small enough. Let Ω be a domain between these two polygons. To set up Ω strictly, we define a scalar multiplication tA by

$$tA := \{tx : x \in A\},$$

for $t > 0$ and a subset A of $\mathbb{R}^N$. Let D be an interior of a regular n polygon in $\mathbb{R}^2$ with center origin. Then $(1+\varepsilon)D$ is a regular polygon too. Remove $\overline{D}$ from $(1+\varepsilon)D$ and define $\Omega := (1+\varepsilon)D \setminus \overline{D}$. Then Ω is a bounded symmetric domain enclosed by boundaries of two polygons $(1+\varepsilon)D$ and D. If $\varepsilon > 0$ is small enough, Ω is a hollow thin domain. We consider the problem below.

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Problem 1.1. Let $\Omega := (1+\varepsilon)D \setminus \overline{D}$ with a regular n polygon D. Does there exists a positive solution which is not invariant under the rotation by angle $2\pi/n$?

This problem is motivated by the works of Coffman [12], Byeon [6] and Li [23]. They consider an annulus $\Omega = A(a, b)$

$$A(a,b) := \{x \in \mathbb{R}^N : a < |x| < b\},$$

with $0 < a < b$. They have proved that (1.1) has a nonradial positive solution if $(b-a)/a$ is small enough, moreover, the number of nonradial positive solutions diverges to infinity as $(b-a)/a \to 0$. This result is due to Coffman [12] for $N = 2$, to Li [23] for $N \geq 4$ and to Byeon [6] for $N = 3$ (see also [10]). On the other hand, Dancer [14] has proved that if $(b-a)/a$ is large enough, then (1.1) has a unique positive solution and it is radially symmetric. For the results related to the annulus, we refer the readers to the Refs. [3,9,8,15,16,18,24–30,34,36]. Moreover, Tanaka and Byeon [7] have studied the expanding annular domain which is not an exact annulus. They have proved the existence of multibump solutions, and hence the number of positive solutions diverges to infinity as the domain is expanding. For the related results to the multibump solutions, we refer the readers to [1,5,17].

In the present paper, we concentrate on the existence of G non-invariant solutions in symmetric domains. The number of positive solutions will be studied in a forthcoming paper. The result mentioned above on the annulus can be rewritten as the existence of a G non-invariant positive solution, where G is taken as $O(N)$. In Problem 1.1, we choose G as

$$G := \{g(2j\pi/n) : j = 0, 1, \dots, n-1\}, \tag{1.2}$$

$$g(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.3}$$

The answer of Problem 1.1 is yes for $\varepsilon > 0$ small enough. This result will be included in Theorem 1.2. A G non-invariant solution will be obtained as a least energy solution. To show it, we define the *Rayleigh quotient* by

$$R(u) := \left(\int_\Omega |\nabla u|^2 dx\right) \left(\int_\Omega |u|^{p+1} dx\right)^{-2/(p+1)}.$$

Moreover, we define the *Nehari manifold* by

$$\mathcal{N} := \left\{u \in H_0^1(\Omega) \setminus \{0\} : \int_\Omega (|\nabla u|^2 - |u|^{p+1}) dx = 0\right\}.$$

Then the *least energy* R_0 is defined by

$$R_0 := \inf\{R(u) : u \in H_0^1(\Omega) \setminus \{0\}\} = \inf\{R(u) : u \in \mathcal{N}\}. \tag{1.4}$$

Because of the Sobolev embedding theorem, the Rayleigh quotient R has a positive lower bound and hence R_0 is well defined and positive. For any $u \in H_0^1(\Omega) \setminus \{0\}$, there is a $\lambda > 0$ such that $\lambda u \in \mathcal{N}$. Moreover, $R(\lambda u) = R(u)$ for any $\lambda > 0$. These facts imply that the second equation in (1.4) is valid. We call u a *least energy solution* if $u \in \mathcal{N}$ and $R(u) = R_0$. Such a solution exists and it solves (1.1) in the distribution sense (see [19] or [37]). Then it belongs to $L^\infty(\Omega) \cap W^{2,q}_{loc}(\Omega)$ for all $q < \infty$ because of the elliptic regularity theorem with the bootstrap argument. Accordingly, a least energy solution belongs to $C^1(\Omega)$. If $\partial\Omega$ is smooth, then any weak solution has a $C^2(\overline{\Omega})$ regularity. A least energy solution is either positive or negative in Ω by the strong maximum principle. We choose a positive solution as a least energy solution.

For a closed subgroup G of $O(N)$, we call u a G invariant least energy solution if it minimizes the Rayleigh quotient R among all G invariant functions in the Nehari manifold. The strict definition will be given after Lemma 1.3. We state some known results related to G invariant least energy solutions in the following. Serra [33] considered the Hénon equation in a ball, which is defined by (1.1) with u^p replaced by $|x|^\alpha u^p$, and proved the existence of a positive non-radial solution if $N \geq 4$, $p = (N+2)/(N-2)$ is the critical exponent and $\alpha > 0$ is large enough. This result was proved by using a $G \times O(N-2)$ invariant least energy solution, where G is defined by (1.2). Badiale and Serra [4] studied the Hénon equation in a ball and proved that an $O(n) \times O(N-n)$ invariant least energy solution is not radial if $N \geq 4$, p is in a certain range including supercritical values and $\alpha > 0$ is large enough. See also our paper [20] for more general nonlinear term in a subcritical case. The author [21] studied the generalized Hénon equation in reflectionally symmetric or point symmetric domains and proved that a least energy solution is neither reflectionally symmetric nor point symmetric. Wang [35] studied the equation with the Neumann problem

$$-\Delta u + \lambda u = u^p, \quad u > 0 \text{ in } B, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } B,$$

where B is a ball and p is the critical exponent. It was proved that for a closed subgroup G of $O(N)$, the minimum number of the cardinal number of the orbit $G(x)$ for $x \in \overline{B} \setminus \{0\}$ determines uniquely a range of λ such that a G invariant least energy solution exists for λ in this range and does not exist for λ out of the range.

Akagi and the author [2] proved that a least energy solution is not G invariant for a suitable G when Ω is an annulus $A(a, b)$, a cylinder $C(a, b, d)$ and a solid of revolution $S_1 \times D$. The annulus $A(a, b)$ has already been defined after Problem 1.1. The cylinder $C(a, b, d)$ is defined by

$$C(a,b,d) := A_{N-1}(a,b) \times (0,d),$$

where $A_{N-1}(a,b) := \{x \in \mathbb{R}^{N-1} : a < |x| < b\}$ is an $N-1$ dimensional annulus. A solid of revolution $S_1 \times D$ is defined by

$$S_1 \times D := \{(r\cos\theta, r\sin\theta, z) : (r,z) \in D, \theta \in [0, 2\pi]\}, \tag{1.5}$$

where S_1 is a circle and D is a bounded domain in $\mathbb{R}^2$ such that the infimum of r for $(r,z) \in D$ is positive. Consider D as an open set in the (x_1, x_3)-plane and revolve it about the x_3-axis. Then we get $S_1 \times D$. In [2], we proved that when Ω is an annulus $A(a,b)$ with $(b-a)/a$ small enough, a least energy solution is nonradial. When $\Omega = C(a,b,d)$, we proved that a least energy solution is not invariant under the rotation around x_3-axis if $b-a$ or d is small enough. When $\Omega = S_1 \times D$, we proved the same result if $\inf_{(r,z)\in D} r > 0$ is large enough. However, these results are based on the perturbation from a G invariant least energy solution and the perturbation function is constructed by using the exact coordinate system. Therefore this method cannot be applied to more general domain Ω. As far as we know, there are no papers which prove the symmetry breaking of a least energy solution in a hollow thin symmetric domain except for the annulus, the cylinder and the solid of revolution.

The purpose of this paper is to prove the symmetry breaking for any hollow thin symmetric domain Ω and for any closed subgroup G of $O(N)$. We shall extend Problem 1.1 to the higher dimensional regular polytopes. For a regular polytope D in $\mathbb{R}^N$, we define the regular polytope group $G(D)$ by the set of rotation matrices which transform D onto itself, i.e.,

$$G(D) := \{g \in SO(N) : g(D) = D\}, \tag{1.6}$$

where $SO(N)$ is a rotation group (special orthogonal group). Let A_n and S_n be the alternating group and the symmetric group, respectively. It is known (see [13, pp. 45–50]) that in $\mathbb{R}^3$, $G(D) \cong A_4$ if D is a tetrahedron, $G(D) \cong S_4$ if D is a cube or an octahedron and $G(D) \cong A_5$ if D is a dodecahedron or an icosahedron.

Theorem 1.2. *Let D be a regular polytope with center origin in $\mathbb{R}^N$ with $N \geq 2$. Put $\Omega := (1+\varepsilon)D \setminus \overline{D}$. If $\varepsilon > 0$ is small enough, then a least energy solution of (1.1) is not invariant under the action of the regular polytope group $G(D)$.*

We define

$$\widetilde{G}(D) := \{g \in O(N) : g(D) = D\}.$$

Even if we employ $\widetilde{G}(D)$ instead of $G(D)$, Theorem 1.2 is still valid. Indeed, by Theorem 1.2, a least energy solution is not $G(D)$ invariant. Then it is not $\widetilde{G}(D)$ invariant because $G(D) \subset \widetilde{G}(D)$.

We denote the usual Lebesgue space by $L^q(\Omega)$ and the Sobolev space by $W^{m,q}(\Omega)$ for $1 \leq q \leq \infty$ and $m \in \mathbb{N}$. We define G invariant function spaces,

$$L^q(\Omega, G) := \{u \in L^q(\Omega) : u \text{ is } G \text{ invariant}\},$$
$$H_0^1(\Omega, G) := \{u \in H_0^1(\Omega) : u \text{ is } G \text{ invariant}\},$$
$$\mathcal{N}(G) := \mathcal{N} \cap H_0^1(\Omega, G).$$

Before going to the main theorems, we show the existence of a G invariant positive solution. The next lemma is valid for any domain Ω even if it has a hole or not.

Lemma 1.3. *Let Ω be a G invariant bounded domain. Then (1.1) has a G invariant positive solution.*

Proof. Put

$$R_G := \inf\{R(u) : u \in H_0^1(\Omega, G) \setminus \{0\}\} = \inf\{R(u) : u \in \mathcal{N}(G)\}. \tag{1.7}$$

Then R_G is achieved at a point $u \in \mathcal{N}(G)$. Then u is a critical point of $R(\cdot)$ in $H_0^1(\Omega, G)$, i.e., $R'(u)v = 0$ for $v \in H_0^1(\Omega, G)$, where R' denotes the Fréchet derivative of R. Then it becomes a critical point in $H_0^1(\Omega)$, i.e., $R'(u)v = 0$ for all $v \in H_0^1(\Omega)$ because of the principle of symmetric criticality by Palais [31]. Since $u \in \mathcal{N}(G)$ and $R'(u) = 0$, u is a solution of (1.1). Consequently, we obtain a G invariant positive solution. □

We sketch our idea. We call u a *G invariant least energy solution* if $u \in \mathcal{N}(G)$ and $R(u) = R_G$ given by (1.7). To avoid confusion, a usual least energy solution is called a *global least energy solution*. Let u be a G invariant least energy solution. We shall construct a function v which satisfies $R(v) < R(u)$. This inequality implies that $R_0 \leq R(v) < R(u) = R_G$, where R_0 (the global least energy) has been defined by (1.4). Therefore a global least energy solution cannot be G invariant.

To accomplish our argument, this paper is organized into five sections. In Section 2, we introduce some notation, which will be needed in the paper. Moreover, we state main results and give examples of Ω. In Section 3, we construct a function v which has a lower energy than the G invariant least energy R_G. In Section 4, we prove the main theorems.

2. Main results

In this section, we state the main results and give several examples of Ω. We first define the *fixed point set* $\mathrm{Fix}(G)$ of G by

$$F = \mathrm{Fix}(G) := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in G\}. \tag{2.1}$$

Then F becomes a linear subspace of $\mathbb{R}^N$. Since $G \neq \{I\}$ with the unit matrix I by assumption, F cannot be the whole space $\mathbb{R}^N$. However, it can occur that $F = \{0\}$. Indeed, when $G = O(N)$ or $\{I, -I\}$, F is equal to $\{0\}$. Thus $0 \leq \dim F \leq N-1$.

Since G is a closed subgroup of $O(N)$, it is compact. Then we can define

$$P(x) := \max_{g\in G} |gx - x| \quad \text{for } x \in \mathbb{R}^N, \tag{2.2}$$

which is continuous. Indeed, we have

Lemma 2.1.

$$|P(x) - P(y)| \le 2|x - y| \quad \textit{for } x, y \in \mathbb{R}^N.$$

Proof. This lemma has been proved in our paper [22], but we give a proof to make the paper self-contained. For $g \in G$, the isometry of g implies that

$$\begin{aligned} |gx - x| &\le |gx - gy| + |gy - y| + |y - x| \\ &= |gy - y| + 2|x - y| \le P(y) + 2|x - y|. \end{aligned}$$

Taking the maximum on $g \in G$, we have $P(x) \le P(y) + 2|x - y|$. Exchanging x with y, we get the conclusion. □

We prepare some definitions and notation.

Definition 2.2. Let D be a bounded open set in $\mathbb{R}^N$.

(i) We define

$$\rho(D) := \min_{\overline{D}} P(x) = \min_{x\in\overline{D}} \max_{g\in G} |gx - x|. \tag{2.3}$$

(ii) For $a > 0$, we define $k(D, a)$ by the smallest positive integer k which satisfies

$$\overline{D} \subset \bigcup_{i=1}^{k} B(x_i, a) \quad \text{with some } x_1, \dots, x_k \in \mathbb{R}^N.$$

Here $B(x, a)$ denotes the open ball in $\mathbb{R}^N$ centered at x with radius a.

(iii) Let $\lambda_1(D)$ denote the first eigenvalue of the problem

$$-\Delta u = \lambda u, \quad \text{in } D,\ u = 0, \text{ on } \partial D.$$

Observe that $P(x) > 0$ if and only if $x \notin F$, where F is defined by (2.1). Hence $\rho(\Omega) > 0$ if and only if $\overline{\Omega} \cap F = \emptyset$. In this case, we consider that Ω has a hole. We state the main result.

Theorem 2.3. *Let G be a closed subgroup of $O(N)$ such that $G \ne \{I\}$. Let Ω be a G invariant bounded domain such that $\overline{\Omega} \cap F = \emptyset$. Put $\rho_0 := \rho(\Omega)/4$ and $k_0 := k(\Omega, \rho_0/2)$. If the inequality*

$$\frac{(2p-1)k_0}{p-1} < \rho_0^2 \lambda_1(\Omega) \tag{2.4}$$

holds, then a least energy solution is not G invariant. Therefore (1.1) has both a G invariant positive solution and a G non-invariant positive solution.

Theorem 2.3 means that if Ω has a hole and the first eigenvalue is large enough, then a least energy solution breaks its symmetry. In particular, if Ω is a hollow thin domain, then $\lambda_1(\Omega)$ is large enough and the theorem is valid. We rewrite (2.4) as

$$\frac{2p-1}{p-1} k(\Omega, \rho_0/2) < \rho_0^2 \lambda_1(\Omega). \tag{2.5}$$

By definition, $k(\Omega, \rho_0/2)$ is nonincreasing with respect to ρ_0. If ρ_0 is large enough, the inequality above holds. For example, in the annulus case $A(a, b)$ with $G = O(N)$, we compute $\rho(A(a, b)) = 2a$. Hence, if a is large and $b = a + 1$, then ρ_0 is large enough and $\lambda_1(\Omega)$ is bounded away from zero. Then (2.5) holds. This means that if Ω has a large hole, then a least energy solution has no symmetry. The assumption that Ω has a large hole is equivalent to the condition that Ω is a hollow thin domain. These are represented as the assumption that $(b - a)/a$ is small enough in the annulus case $A(a, b)$.

Hereafter we denote the Lebesgue measure of Ω by $\mathrm{vol}(\Omega)$. If $\mathrm{vol}(\Omega) \to 0$, then $\lambda_1(\Omega) \to \infty$. This will be proved in Lemma 4.2. Then we have the next corollary.

Corollary 2.4. *Let G be a closed subgroup of $O(N)$ such that $G \ne \{I\}$ and D a G invariant bounded domain such that $\overline{D} \cap F = \emptyset$. Then there is a $\delta > 0$ such that if Ω is a G invariant subdomain of D satisfying* $\mathrm{vol}(\Omega) < \delta$*, then a least energy solution is not G invariant.*

In the corollary above, D is fixed and Ω is a subset of D. Hence it does not seem to be applicable to an expanding annulus $A(a, a + 1)$ as $a \to \infty$. However we use the scaling argument in Example 2.7 so that the corollary works well. We give several examples of Ω having G invariance.

Example 2.5. Let D be a bounded open set in $\mathbb{R}^N$ such that $0 \notin \overline{D}$ and D is point symmetric, i.e., $x \in D$ implies $-x \in D$. If Ω is a point symmetric subdomain of D and $\mathrm{vol}(\Omega)$ is small enough, then a least energy solution of (1.1) is not even. Indeed, choose $G := \{I, -I\}$. Then $\mathrm{Fix}(G) = \{0\}$ and Corollary 2.4 ensures the result above.

Example 2.6. Let D be a bounded open set in $\mathbb{R}^N$ such that $0 \in D$ and D is point symmetric. Put $D_\lambda := \lambda D \setminus \overline{D}$ with $\lambda > 1$. Then D_λ is also point symmetric and $0 \notin \overline{D}_\lambda$. Define $\Omega := D_\lambda \times (0, \varepsilon)$. Then Ω is a bounded domain in $\mathbb{R}^{N+1}$. We denote the variable x in $\mathbb{R}^{N+1}$ by $x = (x', x_{N+1})$ with $x' = (x_1, \dots, x_N)$. If $\lambda > 1$ is large enough, then D_λ is not thin. However when $\varepsilon > 0$ is small enough, Ω has a short height. For any $\lambda > 0$ fixed, if $\varepsilon > 0$ is small enough, then a least energy solution for Ω is not even with respect to x'. Indeed, we choose $G := \{I, I'\}$, where I is the $(N+1) \times (N+1)$ unit matrix and I' is a diagonal matrix whose diagonal elements are $-1, \dots, -1, 1$. Then $\mathrm{Fix}(G)$ is the x_{N+1} axis, which does not intersect $\overline{D}_\lambda$. Even if $\lambda > 0$ is fixed large, we choose $\varepsilon > 0$ so small that $\mathrm{vol}(\Omega)$ is small enough. Hence Corollary 2.4 works well.

Example 2.7. Let Ω be an annulus

$$\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}.$$

If $(b-a)/a$ is small enough, a least energy solution is not even and therefore it is not radially symmetric. Let us show this result. We use a change of variables $v(x) := a^{2/(p-1)} u(ax)$. Then (1.1) is transformed into

$$-\Delta v = v^p, \qquad v > 0 \quad \text{in } \Omega_1, \quad v = 0 \quad \text{on } \partial\Omega_1, \tag{2.6}$$

where

$$\Omega_1 := a^{-1}\Omega = \{x \in \mathbb{R}^N : 1 < |x| < b/a\}.$$

When $0 < a < b$, the convergence $(b-a)/a \to 0$ is equivalent to $b/a \to 1$. Then $\mathrm{vol}(\Omega_1) \to 0$. Example 2.5 ensures the non-evenness of a least energy solution for (2.6), therefore for (1.1) as well.

Example 2.8. Put $U := \{x \in \mathbb{R}^3 : a < |x| < b\}$ with $0 < a < b$ fixed arbitrarily. Let $\Omega := S_1 \times D$ be a solid of revolution defined by (1.5) such that $\Omega \subset U$. Then there is a $\delta > 0$ depending only on a, b such that if $\mathrm{vol}(\Omega) < \delta$, then a least energy solution is not rotationally invariant around the x_3-axis.

Example 2.9. Let U be as in Example 2.8 with $a > 0$ small enough and $b > 0$ large enough. Let Ω be a cylinder

$$\Omega := \{(x_1, x_2, x_3) : \alpha^2 < x_1^2 + x_2^2 < \beta^2, |x_3| < \gamma\},$$

with $\alpha, \beta, \gamma > 0$ and $\alpha < \beta$. There exists a $\delta > 0$ depending only on a and b such that if $\Omega \subset U$ and $(\beta^2 - \alpha^2)\gamma < \delta$, then a least energy solution is not even and not rotationally invariant around the x_3-axis. Indeed, since $\mathrm{vol}(\Omega) = 2\pi(\beta^2 - \alpha^2)\gamma$, Corollary 2.4 shows our claim.

Example 2.10. In $\mathbb{R}^3$, we choose a cube with center origin, whose edges are parallel to the coordinate axes. Denote the union of all edges by E. Let Ω be the ε-neighborhood of E. If $\varepsilon > 0$ is small enough, then a least energy solution is not even and not invariant under the rotation by angle $\pi/2$ around each axis x_i. It is easy to verify the non-evenness. Let us show the non-invariance of the rotation by angle $\pi/2$ around the x_3-axis. The argument below is applicable to any other axis. We choose

$$G := \left\{ \begin{pmatrix} g(j\pi/2) & 0 \\ 0 & 1 \end{pmatrix} : j = 0, 1, 2, 3 \right\},$$

where $g(\theta)$ is defined by (1.3). The fixed point set of G is the x_3-axis, which does not intersect Ω. Therefore a least energy solution is not G invariant if ε is small enough.

3. A function with lower energy

The purpose of this section is to construct a function having an energy lower than the G invariant least energy. To this end, we prepare notation and several lemmas. Hereafter we assume that Ω is G invariant. Let $C_0^\infty(\Omega, G)$ denote the set of G invariant functions in $C_0^\infty(\Omega)$. The next two lemmas have already been proved in our paper [22], however we give their proofs for the reader's convenience.

Lemma 3.1. *$C_0^\infty(\Omega, G)$ is dense in $L^q(\Omega, G)$ for $1 \le q < \infty$.*

Proof. Let $u \in L^q(\Omega, G)$. Put $v(x) := 0$ if $x \in \Omega$ and $\mathrm{dist}(x, \partial\Omega) < \varepsilon$ and $v(x) := u(x)$ if $x \in \Omega$ and $\mathrm{dist}(x, \partial\Omega) \ge \varepsilon$. Here $\mathrm{dist}(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$. Since Ω is G invariant, v lies in $L^q(\Omega, G)$ and $\|v - u\|_q$ converges to 0 as $\varepsilon \to 0$. Hereafter $\|\cdot\|_q$ denotes the $L^q(\Omega)$ norm. Fix $\varepsilon > 0$. A usual mollifier by a radially symmetric function gives us an approximation of v. Indeed, let $J \in C_0^\infty(\mathbb{R}^N)$ be a nonnegative radial function such that the support of J is in $|x| < 1$ and the integral of J on $\mathbb{R}^N$ is one. Put $J_\delta(x) := \delta^{-N} J(x/\delta)$ and define

$$v_\delta(x) := \int_{\mathbb{R}^N} J_\delta(x-y) v(y) dy,$$

which belongs to $C_0^\infty(\Omega, G)$ for $\delta > 0$ small enough and converges to v as $\delta \to 0$. The proof is complete. $\square$

We define the $L^2(\Omega)$ inner product and the $H_0^1(\Omega)$ inner product by

$$(u, v)_{L^2} := \int_\Omega uv dx, \qquad (u, v)_{H_0^1} := \int_\Omega \nabla u \nabla v dx.$$

Denote the orthogonal complement of $L^2(\Omega, G)$ in $L^2(\Omega)$ by $L^2(\Omega, G)^\perp$ and that of $H_0^1(\Omega, G)$ in $H_0^1(\Omega)$ by $H_0^1(\Omega, G)^\perp$, i.e.,

$$L^2(\Omega, G)^\perp := \{u \in L^2(\Omega) : (u, v)_{L^2} = 0 \text{ for all } v \in L^2(\Omega, G)\},$$
$$H_0^1(\Omega, G)^\perp := \{u \in H_0^1(\Omega) : (u, v)_{H_0^1} = 0 \text{ for all } v \in H_0^1(\Omega, G)\}.$$

Lemma 3.2. *The following assertions hold.*

(i) $H_0^1(\Omega, G)^\perp \subset L^2(\Omega, G)^\perp$.

(ii) *Let* $1 \le p, q \le \infty$ *with* $1/p + 1/q = 1$. *For* $u \in L^p(\Omega) \cap L^2(\Omega, G)^\perp$ *and* $v \in L^q(\Omega, G)$, *it holds that*

$$\int_\Omega uv dx = 0.$$

Proof. Let $u \in H_0^1(\Omega, G)^\perp$. Give $f \in L^2(\Omega, G)$ arbitrarily. Choose v in $H_{loc}^2(\Omega, G) \cap H_0^1(\Omega, G)$ satisfying

$$-\Delta v = f \quad \text{in } \Omega, \ v = 0 \text{ on } \partial\Omega.$$

Then we have

$$(u, f)_{L^2} = (u, -\Delta v)_{L^2} = (u, v)_{H_0^1} = 0,$$

which shows that u belongs to $L^2(\Omega, G)^\perp$. Thus (i) is proved.

We shall show (ii). Let p, q, u, v be as in (ii). Assume that $q < \infty$. By Lemma 3.1, we choose a sequence $v_k \in C_0^\infty(\Omega, G)$ converging to v in $L^q(\Omega)$. Since $u \in L^2(\Omega, G)^\perp$ and $v_k \in L^2(\Omega, G)$, it holds that

$$\int_\Omega u v_k dx = 0,$$

which leads to the conclusion as $k \to \infty$. Let $q = \infty$. Since $L^\infty(\Omega, G) \subset L^2(\Omega, G)$, the assertion is trivial. $\square$

The Rayleigh quotient $R(u)$ belongs to $C^2(H_0^1(\Omega))$ in the sense of the Fréchet derivative. The second derivative $R''(u)vw$ is a continuous bilinear form of v and w. We will not need the expression of $R''(u)vw$ but the formula of $R''(u)w^2$ only. It is computed in the next lemma.

Lemma 3.3. *Let u be a positive solution of* (1.1). *For* $w \in H_0^1(\Omega)$, *it holds that*

$$R''(u)w^2 = 2\int_\Omega |\nabla w|^2 dx \left(\int_\Omega |\nabla u|^2 dx\right)^{-2/(p+1)} + 2(p-1)\left(\int_\Omega \nabla u \nabla w dx\right)^2 \left(\int_\Omega |\nabla u|^2 dx\right)^{-(p+3)/(p+1)}$$
$$- 2p \int_\Omega u^{p-1} w^2 dx \left(\int_\Omega |\nabla u|^2 dx\right)^{-2/(p+1)}. \tag{3.1}$$

Proof. For $t \in \mathbb{R}$, we put

$$f(t) := \int_\Omega |\nabla(u + tw)|^2 dx, \qquad g(t) := \int_\Omega |u + tw|^{p+1} dx.$$

Then $R(u + tw) = f(t) g(t)^{-2/(p+1)}$. We compute

$$R''(u + tw)w^2 = f''(t) g(t)^{-2/(p+1)} - \frac{4}{p+1} f'(t) g(t)^{-(p+3)/(p+1)} g'(t)$$
$$+ \frac{2(p+3)}{(p+1)^2} f(t) g(t)^{-2(p+2)/(p+1)} g'(t)^2 - \frac{2}{p+1} f(t) g(t)^{-(p+3)/(p+1)} g''(t). \tag{3.2}$$

Moreover, we have

$$f'(0) = 2\int_\Omega \nabla u \nabla w dx, \qquad f''(0) = 2\int_\Omega |\nabla w|^2 dx,$$
$$g'(0) = (p+1)\int_\Omega u^p w dx, \qquad g''(0) = p(p+1)\int_\Omega u^{p-1} w^2 dx.$$

Substituting $t = 0$ into (3.2), we get

$$\begin{aligned} R''(u)w^2 &= 2\int_\Omega |\nabla w|^2 dx \left(\int_\Omega u^{p+1} dx\right)^{-2/(p+1)} - 8\int_\Omega \nabla u \nabla w dx \left(\int_\Omega u^{p+1} dx\right)^{-(p+3)/(p+1)} \int_\Omega u^p w dx \\ &\quad + 2(p+3)\int_\Omega |\nabla u|^2 dx \left(\int_\Omega u^{p+1} dx\right)^{-2(p+2)/(p+1)} \left(\int_\Omega u^p w dx\right)^2 \\ &\quad - 2p\int_\Omega |\nabla u|^2 dx \left(\int_\Omega u^{p+1} dx\right)^{-(p+3)/(p+1)} \int_\Omega u^{p-1} w^2 dx. \end{aligned} \tag{3.3}$$

Recall that u is a positive solution of (1.1). Multiplying (1.1) by u or w and integrating it over Ω, we have

$$\int_\Omega u^{p+1} dx = \int_\Omega |\nabla u|^2 dx, \qquad \int_\Omega u^p w dx = \int_\Omega \nabla u \nabla w dx.$$

Substituting these identities into (3.3), we obtain (3.1). □

The next proposition plays the most important role to prove the main theorems.

Proposition 3.4. *Let Ω_0 be a G invariant bounded domain in $\mathbb{R}^N$ such that $\Omega \subset \Omega_0$. Let u be a G invariant least energy solution of (1.1) and ϕ be a function in $W^{1,\infty}(\Omega_0) \cap H_0^1(\Omega_0, G)^\perp$. Put $v := (1+\varepsilon\phi)u$. If ϕ satisfies*

$$\int_\Omega |\nabla \phi|^2 u^2 dx < \frac{p-1}{2(2p-1)} \int_\Omega |\nabla u|^2 \phi^2 dx, \tag{3.4}$$

then $R(v) < R(u)$ for $\varepsilon > 0$ small enough. Therefore, a global least energy solution is not G invariant.

Proof. Since $\phi \in W^{1,\infty}(\Omega_0)$, the functions ϕu and v belong to $H_0^1(\Omega)$. Since u is a solution of (1.1), $R'(u)$ vanishes. We apply the Taylor theorem to get

$$R(v) = R(u + \varepsilon\phi u) = R(u) + \frac{\varepsilon^2}{2} R''(u)(\phi u)^2 + o(\varepsilon^2) \tag{3.5}$$

as $\varepsilon \to 0$. Here $o(\varepsilon^2)$ denotes a remainder term satisfying $o(\varepsilon^2)/\varepsilon^2 \to 0$ as $\varepsilon \to 0$. We shall show that $R''(u)(\phi u)^2$ is negative.

Multiplying (1.1) by $\phi^2 u$ and integrating it over Ω, we have

$$\int_\Omega \left(|\nabla u|^2 \phi^2 + 2u\phi \nabla u \nabla \phi\right) dx = \int_\Omega u^{p+1} \phi^2 dx. \tag{3.6}$$

We use the Schwarz inequality to get

$$2|u\phi \nabla u \nabla \phi| \le 2|\nabla \phi|^2 u^2 + \frac{1}{2}|\nabla u|^2 \phi^2.$$

Employing the inequality above with (3.4), we have

$$\int_\Omega |\nabla \phi|^2 u^2 dx - 2(p-1)\int_\Omega u\phi \nabla u \nabla \phi dx < (p-1)\int_\Omega |\nabla u|^2 \phi^2 dx. \tag{3.7}$$

Combining (3.6) with (3.7), we obtain

$$\int_\Omega (|\nabla u|^2 \phi^2 + 2u\phi \nabla u \nabla \phi + |\nabla \phi|^2 u^2) dx < p\int_\Omega (|\nabla u|^2 \phi^2 + 2u\phi \nabla u \nabla \phi) dx = p\int_\Omega u^{p+1} \phi^2 dx,$$

or equivalently

$$\int_\Omega |\nabla(\phi u)|^2 dx < p\int_\Omega u^{p+1} \phi^2 dx. \tag{3.8}$$

Put $u = 0$ outside of Ω. Then $u \in H_0^1(\Omega_0, G)$. Note that $|\nabla u|^2 \in L^1(\Omega_0, G)$ and use Lemma 3.2 with Ω replaced by Ω_0. Then one finds that

$$\int_\Omega |\nabla u|^2 \phi dx = \int_{\Omega_0} |\nabla u|^2 \phi dx = 0.$$

By the usual bootstrap argument with the elliptic regularity theorem, u belongs to $L^\infty(\Omega)$. Since $u \in L^\infty(\Omega) \cap H_0^1(\Omega)$, it holds that $\nabla(u^2) = 2u\nabla u \in L^2(\Omega)$. Hence $u^2 \in H_0^1(\Omega, G)$. Therefore

$$\int_\Omega u \nabla u \nabla \phi dx = \frac{1}{2} \int_{\Omega_0} \nabla(u^2) \nabla \phi dx = 0.$$

Using two identities above, we get

$$\int_\Omega \nabla u \nabla(\phi u) dx = \int_\Omega u \nabla u \nabla \phi dx + \int_\Omega |\nabla u|^2 \phi dx = 0.$$

Substitute $w = \phi u$ into (3.1) and use the equation above. Then we have

$$R''(u)(\phi u)^2 = 2 \int_\Omega |\nabla(\phi u)|^2 dx \left(\int_\Omega |\nabla u|^2 dx \right)^{-2/(p+1)} - 2p \int_\Omega u^{p+1} \phi^2 dx \left(\int_\Omega |\nabla u|^2 dx \right)^{-2/(p+1)}.$$

This inequality with (3.8) implies that $R''(u)(\phi u)^2 < 0$. Then (3.5) ensures that $R(v) < R(u)$ for $\varepsilon > 0$ small enough, and so $R_0 \le R(v) < R(u) = R_G$, where R_0 and R_G have been defined by (1.4) and (1.7), respectively. Therefore a global least energy solution cannot be G invariant. $\square$

4. Proof of the main results

In this section, we shall show the main theorems. We first deal with Theorem 2.3.

Proof of Theorem 2.3. In view of Proposition 3.4, it is enough to construct a function ϕ satisfying the assumption of Proposition 3.4. Let k_0 and ρ_0 be as in Theorem 2.3. Let u be a G invariant least energy solution. Put $u = 0$ outside of Ω. Then $u \in H^1(\mathbb{R}^N, G)$. Choose a point $y_0 \in \mathbb{R}^N$ satisfying

$$\max_{y \in \mathbb{R}^N} \int_{B(y, \rho_0/2)} |\nabla u|^2 dx = \int_{B(y_0, \rho_0/2)} |\nabla u|^2 dx.$$

By the definition of k_0, there exist $y_1, \ldots, y_{k_0} \in \mathbb{R}^N$ such that $\overline{\Omega}$ is covered by the union of $B(y_i, \rho_0/2)$ with $1 \le i \le k_0$. Then we get

$$\int_\Omega |\nabla u|^2 dx \le k_0 \int_{B(y_0, \rho_0/2)} |\nabla u|^2 dx.$$

If $B(y_0, \rho_0/2)$ does not intersect Ω, then the right hand side vanishes. This is impossible. Hence we can choose a point $x_0 \in B(y_0, \rho_0/2) \cap \Omega$. Then it holds that

$$\int_\Omega |\nabla u|^2 dx \le k_0 \int_{B(x_0, \rho_0)} |\nabla u|^2 dx. \tag{4.1}$$

By the definition of ρ_0, there exists a $g_0 \in G$ such that $|g_0 x_0 - x_0| \ge 4\rho_0$. Put $x_1 := g_0 x_0$. Then we have

$$B(x_0, 2\rho_0) \cap B(x_1, 2\rho_0) = \emptyset.$$

Define $\phi_0(|x|)$ by

$$\phi_0(|x|) := \begin{cases} 1 & \text{when } |x| \le \rho_0, \\ (2\rho_0 - |x|)/\rho_0 & \text{when } \rho_0 \le |x| \le 2\rho_0. \end{cases}$$

We define $\phi(x)$ by

$$\phi(x) := \begin{cases} \phi_0(|x - x_0|) & \text{in } B(x_0, 2\rho_0), \\ -\phi_0(|x - x_1|) & \text{in } B(x_1, 2\rho_0), \\ 0 & \text{otherwise}. \end{cases}$$

We put

$$\Omega_0 := \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < 2\rho_0\},$$

where $\mathrm{dist}(x, \Omega)$ is the distance from x to Ω. Then clearly $\Omega \subset \Omega_0$ and $\phi \in W^{1,\infty}(\Omega_0) \cap H_0^1(\Omega_0)$. We shall show that $\phi \in H_0^1(\Omega_0, G)^{\perp}$. Let ψ be any function in $H_0^1(\Omega_0, G)$. Note that $\phi_0(|x|)$ is radial. Using the change of variables $x = g_0 y$, we compute

$$\begin{aligned}\int_{B(x_1,2\rho_0)} \nabla\phi\nabla\psi\, dx &= -\int_{|x-x_1|<2\rho_0} \nabla\phi_0(|x-x_1|)\nabla\psi(x)dx \\ &= -\int_{|g_0y-g_0x_0|<2\rho_0} \nabla\phi_0(|g_0y-g_0x_0|)\nabla\psi(g_0y)dy \\ &= -\int_{B(x_0,2\rho_0)} \nabla\phi(y)\nabla\psi(y)dy.\end{aligned}$$

This implies that

$$\int_{\Omega_0} \nabla\phi\nabla\psi\, dx = 0,$$

that is, $\phi \in H_0^1(\Omega_0, G)^{\perp}$.

Since $\lambda_1 = \lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian, it holds that

$$\int_{\Omega} u^2 dx \le \frac{1}{\lambda_1}\int_{\Omega} |\nabla u|^2 dx.$$

Since $\|\nabla\phi\|_{\infty} = \|\nabla\phi_0\|_{\infty} = 1/\rho_0$ by definition, we have

$$\int_{\Omega} |\nabla\phi|^2 u^2 dx \le \frac{1}{\rho_0^2}\int_{\Omega} u^2 dx \le \frac{1}{\lambda_1\rho_0^2}\int_{\Omega} |\nabla u|^2 dx,$$

which with (4.1) yields

$$\int_{\Omega} |\nabla\phi|^2 u^2 dx \le \frac{k_0}{\lambda_1\rho_0^2}\int_{B(x_0,\rho_0)} |\nabla u|^2 dx.$$

Since $|\phi(x)| = 1$ in $B(x_0, \rho_0) \cup B(x_1, \rho_0)$, we have

$$\int_{B(x_0,\rho_0)} |\nabla u|^2 dx = \int_{B(x_0,\rho_0)} |\nabla u|^2\phi^2 dx \le \frac{1}{2}\int_{\Omega} |\nabla u|^2\phi^2 dx.$$

Combining the two inequalities above and using the assumption (2.4), we obtain

$$\int_{\Omega} |\nabla\phi|^2 u^2 dx < \frac{p-1}{2(2p-1)}\int_{\Omega} |\nabla u|^2\phi^2 dx.$$

Thus (3.4) holds. By Proposition 3.4, a least energy solution cannot be G invariant. The proof is complete. □

By the definitions of $\rho(D)$ and $k(D, a)$, we can prove easily the next lemma.

Lemma 4.1. *Let Ω_1 and Ω_2 be bounded sets in $\mathbb{R}^N$. Then the following assertions hold.*

(i) *$\rho(\Omega_2) \le \rho(\Omega_1)$ if $\Omega_1 \subset \Omega_2$.*
(ii) *$k(\Omega_1, a)$ is nonincreasing with respect to a.*
(iii) *$k(\Omega_1, a) \le k(\Omega_2, a)$ if $\Omega_1 \subset \Omega_2$.*

To prove Corollary 2.4, we need the fact that $\lambda_1(\Omega) \to \infty$ as $\mathrm{vol}(\Omega) \to 0$. This result is well known, however we give a proof for the reader's convenience.

Lemma 4.2. *If $\mathrm{vol}(\Omega) \to 0$, then $\lambda_1(\Omega) \to \infty$.*

Proof. We use the isoperimetric inequality (see [11, p. 87, Theorem 2])

$$\lambda_1(\Omega) \ge \lambda_1(B_r) \quad \text{if } \mathrm{vol}(\Omega) = \mathrm{vol}(B_r) \text{ with a ball } B_r.$$

Here B_r denotes a ball with radius r. Since $\lambda_1(B_r) = r^{-2}\lambda_1(B_1)$ and $\mathrm{vol}(\Omega) = \mathrm{vol}(B_r) = \omega_N r^N$ with ω_N depending only on N, we have

$$\lambda_1(\Omega) \ge r^{-2}\lambda_1(B_1) = \lambda_1(B_1)\omega_N^{2/N}\mathrm{vol}(\Omega)^{-2/N}.$$

This completes the proof. □

Proof of Corollary 2.4. Let D be as in the corollary. Put $\rho_1 := \rho(D)/4$ and $k_1 := k(D, \rho_1/2) = k(D, \rho(D)/8)$. Let Ω be a G invariant subdomain of D. By Lemma 4.2, if $\mathrm{vol}(\Omega)$ is small enough, then

$$\frac{(2p-1)k_1}{(p-1)\rho_1^2} < \lambda_1(\Omega). \tag{4.2}$$

By Lemma 4.1, we have $\rho(D) \leq \rho(\Omega)$ and

$$k(\Omega, \rho(\Omega)/8) \leq k(D, \rho(\Omega)/8) \leq k(D, \rho(D)/8).$$

Then (4.2) implies that

$$\frac{2p-1}{p-1} k(\Omega, \rho(\Omega)/8) < (\rho(\Omega)/4)^2 \lambda_1(\Omega),$$

which is exactly (2.4). Theorem 2.3 gives a conclusion. □

We conclude this paper by proving Theorem 1.2.

Proof of Theorem 1.2. Let D be a regular polytope with center origin in $\mathbb{R}^N$ and put $\Omega := (1+\varepsilon)D \setminus \overline{D}$. Define $G(D)$ by (1.6). It is known that $\mathrm{Fix}(G(D)) = \{0\}$. For the proof, see [32, p. 234, Exercise 6.5] or [22]. By Corollary 2.4, a least energy solution is not $G(D)$ invariant. □

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