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Spectrality of one dimensional self-similar measures with consecutive digits

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ABSTRACT

Assume $0 <| \rho |< 1$ and m is a prime, let μ_ρ be the self-similar measure defined by $\mu_\rho(A) = \frac{1}{m}\sum_{j=0}^{m-1}\mu_\rho(\rho^{-1}A - j), \forall A \in \mathcal{B}$. We prove that $L^2(\mu_\rho)$ contains an orthonormal basis of exponential functions if and only if $\rho = \pm 1/mk$ for some $k \in \mathbb{N}$.

1. Introduction

Let μ be a Borel probability measure on $\mathbb{R}$. We say that μ is a spectral measure if there exists a discrete set Λ such that $E_\Lambda := \{e^{2\pi\lambda xi} : \lambda \in \Lambda\}$ forms an orthonormal basis of $L^2(\mu)$. In this case, we call Λ a spectrum of μ and (μ, Λ) a spectral pair, respectively.

Jorgenson and Pederson [4]studied the spectral property of general Cantor measures. They proved that the $1/k$-Cantor measure $\mu_{1/k}$ on $\mathbb{R}$ is a spectral measure if k is even (Strichartz provided a simplified proof in [9]). This result was investigated by Laba and Wang in more details in [5] and for the general Borel measures in [6].

Hu and Lau [3] further studied the spectral property of Bernoulli convolutions. They proved that the necessary and sufficient condition that the Bernoulli convolution has an infinite orthonormal set E_Λ of exponential functions is that the contraction ratio ρ is the n-th root of a fraction p/q, where p is odd and q is even. Recently, Dai [1] proved that the Bernoulli convolution has an orthonormal basis E_Λ of exponential functions if and only if the contraction ratio ρ is the reciprocal of an even integer.

Motivated by the above results, we study the spectral property of one dimensional self-similar measures with consecutive digits.

Let ρ be a real number such that $0 < |\rho| < 1$, it is well known that for any positive integer $m \geq 2$, there exists a unique probability measure, denoted by μ_ρ, such that

$$\mu_\rho(A) = \frac{1}{m}\sum_{j=0}^{m-1}\mu_\rho(\rho^{-1}(A) - j) \tag{1.1}$$

for all Borel set $A \in \mathcal{B}$. μ_ρ is called a self-similar measure.

For the self-similar measure μ_ρ defined in (1.1), our main theorem is as follows.

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Theorem 1.1. *If m is a prime, then $L^2(\mu_\rho)$ contains an orthonormal basis of exponential functions only if $\rho = \pm\frac{1}{mk}$ for some $k \in \mathbb{N}$.*

This is an extension of the result of [1].

On the other hand, Dai etc. proved that $L^2(\mu_{1/mk})$ contains an orthonormal basis of exponential functions for any $k \in \mathbb{N}$ in [2]. It is easy to see that $L^2(\mu_{-1/mk})$ also contains an orthonormal basis of exponential functions for any $k \in \mathbb{N}$. Hence we have the following.

Theorem A. *If m is a prime, then $L^2(\mu_\rho)$ contains an orthonormal basis of exponential functions if and only if $\rho = \pm\frac{1}{mk}$ for some $k \in \mathbb{N}$.*

Remarks. Theorem 1.1 indicates that the main theorems of [3,1] also hold for $-1 < \rho < 0$. Our proof of Theorem 1.1 strongly depends on the structure of the zeros of $\hat{\mu}_\rho$. The set of zeros of $\hat{\mu}_\rho$ will be very complicated if the digit set is replaced by a non-consecutive digit set. So far, we do not know how to deal with the case of non-consecutive digits. Also, some of our proofs do not work when m is not a prime. For integral self-affine measures, Li studied the spectrality of a class of planar self-affine measures with decomposable digit sets in [7] and with three non-consecutive digit set in [8].

If we only consider the existence of infinite orthonormal set of exponential functions, we have the following theorem which is an extension of the result of [3].

Theorem 1.2. *Assume m is a prime, then $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions if and only if $\rho = \pm(q/p)^{1/r}$ for some $p, q, r \in \mathbb{N}$ with the properties: p, q are co-prime and $m|p$.*

Since E_Λ forms an orthonormal set in $L^2(\mu_\rho)$ if and only if $E_{t+\Lambda}$ forms an orthonormal set in $L^2(\mu_\rho)$ for any fixed $t \in \mathbb{R}^d$. For simplicity we assume that $0 \in \Lambda$ throughout this paper.

Notations. We will use the following notations. Let $\mathbb{Z}$ be the set of all integers, let $\mathbb{N}$ be the set of all positive integers. For any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}$ and $r \in \mathbb{N}$, we use $\mathbf{x} \equiv \mathbf{y} (\text{mod } r)$ to denote $\mathbf{x} - \mathbf{y} \in r\mathbb{Z}$.

For the iterated function system $\{S_j\}_{j=0}^{m-1}$ with $S_j(x) = \rho(x + j)$ and the associated μ_ρ defined in (1.1), let $\hat{\mu}_\rho(t) = \int e^{2\pi xti} d\mu_\rho(x)$ be the Fourier transform of μ_ρ. Define

$$\mathcal{Z}_\rho = \{t \in \mathbb{R} : \hat{\mu}_\rho(t) = 0\}$$

to be the set of zeros of $\hat{\mu}_\rho(t)$. Let

$$\mathbb{O} = \{\pm(q/p)^{1/r} : p, q, r \in \mathbb{N}\}.$$

It is clear that $\beta \in \mathbb{O}$ if and only if $|\beta|$ is an algebraic rational with a minimal polynomial $px^r - q$ for some $p, q, r \in \mathbb{N}$.

Throughout this paper, we always use

$$E_\Lambda = \{e^{2\pi\lambda xi} : \lambda \in \Lambda\},$$

to denote an orthonormal set of exponential functions in $L^2(\mu_\rho)$, where Λ is a subset of $\mathbb{R}$ containing 0. For this Λ, we define

$$Q_\Lambda(t) = \sum_{\lambda\in\Lambda} |\hat{\mu}_\rho(\lambda - t)|^2.$$

We organize the paper as follows. Some preliminary lemmas are given in Section 2. Section 3 is devoted to prove Theorem 1.2. While Theorem 1.1 is proven in Section 4.

2. Some preliminary lemmas

We first give some preliminary results associated with the self-similar measure μ_ρ. Then we will use them to prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

It is easy to prove the following.

Lemma 2.1. *Let $\hat{\mu}_\rho(t)$ be the Fourier transform of the self-similar measure μ_ρ defined in (1.1), then*

$$\hat{\mu}_\rho(t) = \prod_{k=1}^{n} \left[\frac{1}{m}\sum_{j=0}^{m-1} (e^{2\pi\rho^k ti})^j\right] \hat{\mu}_\rho(\rho^n t) \tag{2.1}$$

for all positive integers $n > 0$.

Lemma 2.2. $\mathcal{Z}_\rho = \{\frac{\ell}{m\rho^k} : k \in \mathbb{N}, \ell \in \mathbb{Z} \setminus m\mathbb{Z}\}$.

Proof. Use (2.1), since $\hat{\mu}_\rho(\rho^n t) \to 1$ as $n \to +\infty$, $\hat{\mu}_\rho(\lambda) = 0$ if and only if there exists a positive integer $k > 0$ so that

$$\sum_{j=0}^{m-1} (e^{2\pi\rho^k\lambda i})^j = 0.$$

Hence $e^{2\pi\rho^k\lambda i} \neq 1$, multiplying both sides by $1 - e^{2\pi\rho^k\lambda i}$, we see that the above equation is equivalent to

$$1 - (e^{2\pi\rho^k\lambda i})^m = 0, \quad e^{2\pi\rho^k\lambda i} \neq 1.$$

Hence $\hat{\mu}_\rho(\lambda) = 0$ if and only if there exist integers $\ell \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $2m\pi\rho^k\lambda = 2\ell\pi$ and $\rho^k\lambda$ is not an integer, i.e. $\lambda = \frac{\ell}{m\rho^k}$ with $\ell \in \mathbb{Z} \setminus m\mathbb{Z}$ and $k \in \mathbb{N}$, the conclusion follows. $\square$

Remark. It is possible that $\lambda \in \mathcal{Z}_\rho$ has another representation different from the one in Lemma 2.2. For example, if $p = m$ and $\rho = \frac{m-1}{m}$, then $\frac{1}{\rho} \in \mathcal{Z}_\rho$, since $\frac{1}{\rho} = \frac{m-1}{m\rho^2}$.

Lemma 2.3. *Let Λ be a subset of $\mathbb{R}$ containing 0, then E_Λ is an orthonormal set of $L^2(\mu_\rho)$ if and only if $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}_\rho$. Equivalently, the following two conditions are satisfied:*

(i) *$\Lambda = \{0\} \cup \{\frac{\ell_j}{m\rho^{k_j}} : 1 \le j < N\}$ with $k_j \in \mathbb{N}$, $\ell_j \in \mathbb{Z} \setminus m\mathbb{Z}$, $0 < j < N$, where N is a finite positive integer or the infinity.*

(ii) *There exist $\frac{v_{s,t}}{m\rho^{r_{s,t}}} \in \mathcal{Z}_\rho$ with $v_{s,t} \in \mathbb{Z} \setminus m\mathbb{Z}$ such that*

$$\frac{\ell_s}{m\rho^{k_s}} - \frac{\ell_t}{m\rho^{k_t}} = \frac{v_{s,t}}{m\rho^{r_{s,t}}}, \quad 1 \le s \neq t < N. \tag{2.2}$$

Proof. It is clear that E_Λ is an orthonormal set in $L^2(\mu_\rho)$ if and only if $\hat{\mu}_\rho(\lambda_1 - \lambda_2) = 0$ for any distinct $\lambda_1, \lambda_2 \in \Lambda$. Hence (ii) follows from Lemma 2.2, and (i) follows from the assumption $0 \in \Lambda$. $\square$

Lemma 2.4. *If $L^2(\mu_\rho)$ has an orthonormal set of exponential functions with at least $m + 1$ elements, then ρ^{-1} is a zero of an integral polynomial.*

Proof. Let E_Λ be an orthonormal set with at least $m + 1$ elements. Then the N defined in Lemma 2.3 is at least $m + 1$. Hence (2.2) holds for all $1 \le s < t \le m$.

If $k_s = k_t = r_{s,t}$ for all $1 \le s < t \le m$, then $\ell_s - \ell_t = v_{s,t}$ for all $1 \le s < t \le m$. It is clear that there exist s and t such that $0 \le s < t \le m$ and $\ell_s - \ell_t \in m\mathbb{Z}$. Hence $v_{s,t} \in m\mathbb{Z}$, it contradicts Lemma 2.3(ii). Therefore, there is a pair (s, t) so that at least two of $k_s, k_t, r_{s,t}$ are distinct, so ρ^{-1} is a zero of an integral polynomial by using (2.2). $\square$

Lemma 2.5. *Assume $\beta \in \mathbb{O}$ admits a minimal polynomial $p\beta^r - (\pm 1)^r q = 0$ and satisfies $a_1\beta^k + a_2\beta^j = a_3\beta^u$, where $k, j, u \ge 0$ are nonnegative integers and $a_1, a_2, a_3 \in \mathbb{Z} \setminus \{0\}$. Then $k \equiv j \equiv u (\mathrm{mod}\ r)$.*

Proof. Let $k = k_1 r + s, j = j_1 r + t, u = u_1 r + v$ with $0 \le s, t, v < r$. Since $p\beta^r = (\pm 1)^r q$, so β satisfies

$$b_1\beta^s + b_2\beta^t = b_3\beta^v$$

for some integers $b_1, b_2, b_3 \neq 0$. Since $px^r - (\pm 1)^r q$ is the minimal polynomial of β with order r, in view of $0 \le s, t, v < r$, we see that $s = t = v$, the conclusion follows. $\square$

3. Proof of Theorem 1.2

We first prove Theorem 1.2 for the case $\rho \in \mathbb{O}$.

Proposition 3.1. *Let $px^r - q$ be the minimal polynomial of $|\rho|$ for some $p, q, r \in \mathbb{N}$, where p, q are co-prime. Then $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions if and only if p, m have a common divisor larger than one.*

Proof. Consider the necessity. Let E_Λ be an infinite orthonormal set with $0 \in \Lambda$, then $\Lambda \setminus \{0\} \subseteq \mathcal{Z}_\rho$.

For any $x \in \mathcal{Z}_\rho$, there exist $u \in \mathbb{N}$ and $v \in \mathbb{Z} \setminus m\mathbb{Z}$ such that $x = \frac{v}{m|\rho|^u}$ by Lemma 2.2. Furthermore, there exist integers $\ell \ge 0$ and $v_0 \in \mathbb{Z}$ so that $v = p^l v_0$ with $p \nmid v_0$. Hence $\frac{v}{m|\rho|^u} = \frac{q^l v_0}{m|\rho|^{u+rl}}$, $p \nmid v_0$ and $m \nmid p^l v_0$. Since $\Lambda - \Lambda \subseteq \{0\} \cup \mathcal{Z}_\rho$. Let k_1 be the smallest positive integer such that $\frac{\ell_1}{m|\rho|^{k_1}} \in \Lambda - \Lambda \subseteq \{0\} \cup \mathcal{Z}_\rho$ for some $\ell_1 \in (\mathbb{Z} \setminus p\mathbb{Z})$.

Let $\lambda_1, \lambda_0 \in \Lambda$ be such that $\lambda_1 - \lambda_0 = \frac{\ell_1}{m|\rho|^{k_1}}$.

For any $\lambda \in \Lambda$, Lemma 2.3 implies that there exists a $\lambda' \in \mathcal{Z}_\rho$ such that $\frac{\ell_1}{m\rho^{k_1}} - (\lambda - \lambda_0) = \lambda_1 - \lambda = \lambda'$. Write $\lambda = \lambda_0 + \frac{q^s \ell}{m|\rho|^{k+rs}}$ and $\lambda' = \frac{q^t v}{m|\rho|^{u+rt}}$ with the properties: $\ell, v \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z}$; $k, u \in \mathbb{N}$; $s, t \geq 0$. Then we have

$$\frac{\ell_1}{m|\rho|^{k_1}} - \frac{q^s \ell}{m|\rho|^{k+rs}} = \frac{q^t v}{m|\rho|^{u+rt}}.$$

Therefore, Lemma 2.5 implies that $k - k_1 \equiv u - k_1 \equiv 0 \pmod r$. Let $k + rs = nr + k_1, u + rt = u'r + k_1$, then the definition of k_1 implies $n, u' \geq 0$. The above equality becomes

$$\ell_1 - q^{s-n} p^n \ell = q^{t-u'} v p^{u'}.$$

If $n = 0$, then $\lambda = \lambda_0 + \frac{q^s \ell}{m|\rho|^{k_1}}$. If $n > 0$, note that p, q are co-prime, if $u' > 0$, then the above equality implies $p|\ell_1$, a contradiction. Hence $u' = 0$, so the above equality implies $q^n | q^s \ell$. Hence λ can be written as the form $\lambda = \lambda_0 + \frac{q^n \ell'}{m|\rho|^{k_1+rn}} = \frac{p^n \ell'}{m|\rho|^{k_1}}$ with $\ell' \in \mathbb{Z}$. Therefore, there exist non-zero integers z_j such that

$$\Lambda = \{\lambda_0\} \cup \{\lambda_j\}_{j=1}^{+\infty} \quad \text{with } \lambda_j = \lambda_0 + \frac{z_j}{m|\rho|^{k_1}} \ (j > 0). \tag{3.1}$$

Lemma 2.3(ii) implies that $\frac{z_s - z_t}{m|\rho|^{k_1}} \in \mathcal{Z}_\rho$ for all distinct $s, t > 0$. Choose $s > t > 0$ so that $m|(z_s - z_t)$, let $z_s - z_t = mz$. Then Lemma 2.2 implies that there exist $u_{s,t} \in \mathbb{Z} \setminus m\mathbb{Z}$, $v_{s,t} \in \mathbb{N}$ such that

$$\frac{z}{|\rho|^{k_1}} = \lambda_s - \lambda_t = \frac{u_{s,t}}{m|\rho|^{v_{s,t}}}.$$

Lemma 2.5 implies that $v_{s,t} - k_1 = r\xi_{s,t}$ for some integer $\xi_{s,t}$. Hence

$$\left(\frac{q}{p}\right)^{\xi_{s,t}} = \frac{u_{s,t}}{mz}. \tag{3.2}$$

The definition of k_1 implies $\xi_{s,t} \geq 0$. Note that $m \nmid u_{s,t}$, we see that $\xi_{s,t} > 0$ and p, m have a common divisor larger than one. The necessity follows.

We now prove the sufficiency. Suppose that p, m have a common divisor larger than one. Let $m_0 > 1$ be the greatest common divisor of m and p. Since p, q are co-prime, so m_0, q are co-prime.

Let

$$\hat{\Lambda} = \{0\} \cup \left\{ \frac{q^n}{m|\rho|^{nr}} : n \in \mathbb{N} \right\}.$$

Since m_0, q are co-prime and $m_0 > 1$ is the greatest common divisor of m and p, so neither $\frac{q^n}{m}$ nor $\frac{q^n(p^k-1)}{m}$ is an integer for all $n, k \in \mathbb{N}$. Therefore, both $\frac{q^n}{m|\rho|^{nr}}$ and

$$\frac{q^{n+k}}{m|\rho|^{nr+kr}} - \frac{q^n}{m|\rho|^{nr}} = \frac{q^n(p^k - 1)}{m|\rho|^{nr}}$$

are zeros of $\hat{\mu}_\rho(t)$ for all $n, k \in \mathbb{N}$ by using Lemma 2.2. Hence $(\hat{\Lambda} - \hat{\Lambda}) \setminus \{0\} \subset \mathcal{Z}_\rho$, so $E_{\hat{\Lambda}}$ is an infinite orthonormal set in $L^2(\mu_\rho)$ by Lemma 2.3. The sufficiency follows. □

To prove Theorem 1.2 for the case $\rho \notin \mathbb{O}$. We will suppose on the contrary that $L^2(\mu_\rho)$ has an infinite orthonormal set of exponential functions for some $\rho \notin \mathbb{O}$, then obtain a contradiction.

Let E_Λ be an infinite orthonormal set in $L^2(\mu_\rho)$, then Lemma 2.4 implies that ρ^{-1} is a zero of an integral polynomial. Let

$$g(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{3.3}$$

be the minimal integral polynomial of ρ^{-1}, where $a_0 > 0, a_1, \ldots, a_n$ are relative prime and $a_n \neq 0$.

It is easy to see that the following lemma holds.

Lemma 3.2. *If ρ^{-1} is a zero of an integral polynomial $f(x) = d_0 + d_1 x + \cdots + d_k x^k$, then $a_0 | d_0$ and $a_n | d_k$.*

Lemma 3.3. *Let E_Λ be an orthonormal set in $L^2(\mu_\rho)$ and $\rho \notin \mathbb{O}$, let $\Lambda_k = \frac{\mathbb{Z} \setminus m\mathbb{Z}}{m\rho^k} \cap \Lambda$, then $\{\Lambda_k\}_{k>0}$ are disjoint and each one has cardinality at most $m - 1$.*

Proof. For the disjointness of $\{\Lambda_k\}_{k>0}$, suppose on the contrary, $\Lambda_k \cap \Lambda_r \neq \emptyset$ for some $0 < k < r$. Then there exist $\frac{s}{m\rho^k} \in \Lambda_k$ and $\frac{t}{m\rho^r} \in \Lambda_r$ so that $\frac{s}{m\rho^k} = \frac{t}{m\rho^r}$. Hence

$$\rho = \pm(|t/s|)^{\frac{1}{r-k}} \in \mathbb{O},$$

a contradiction.

For the cardinality of Λ_k. If Λ_k has at least m elements, let $\frac{\ell_j}{m\rho^k} \in \Lambda_k, j = 1, 2, \ldots, m$. Then there exist $1 \le j_1 < j_2 \le m$ so that $\ell_{j_1} - \ell_{j_2} = rm$ for a non-zero integer r. Hence Lemma 2.3 implies that

$$\frac{r}{\rho^k} = \frac{u}{m\rho^v}$$

for some $u \in \mathbb{Z} \setminus m\mathbb{Z}$ and $v \in \mathbb{N}$. This means that $v \neq k$ and

$$\rho = \left(\frac{u}{mr}\right)^{\frac{1}{v-k}} \quad (v > k) \quad \text{or} \quad \rho = \left(\frac{mr}{u}\right)^{\frac{1}{k-v}} \quad (v < k),$$

a contradiction to the assumption $\rho \notin \mathbb{O}$. □

Lemma 3.4. *Assume $\rho \notin \mathbb{O}$. If $L^2(\mu_\rho)$ has an infinite orthonormal set of exponential functions, then ρ^{-1} has a minimal polynomial $g(x)$ as (3.3) with $a_0 = 1$.*

Proof. Lemma 2.4 implies that ρ^{-1} is a zero of an integral polynomial. Hence ρ^{-1} has a minimal polynomial $g(x)$ as given by (3.3). Since $\rho \notin \mathbb{O}$, it is easy to see that $z\rho^{-k} \notin \mathcal{Z}_\rho$ for all $z \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Since $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions E_Λ, Lemma 3.3 implies that there exist infinitely many $\lambda_j = \frac{\ell_j}{m\rho^{k_j}} \in \Lambda$ with $0 < k_j < k_{j+1}$. Without loss of generality, we assume

$$\Lambda = \{0\} \cup \left\{\frac{\ell_j}{m\rho^{k_j}} : 0 < k_1 < k_2 < \cdots\right\}.$$

By Lemma 2.2, we can assume that $\frac{\ell_j}{m} \notin \mathbb{Z}$. Lemma 2.3(ii) then implies that there exist $v_{s,t} \in \mathbb{Z} \setminus m\mathbb{Z}$, $r_{s,t} \in \mathbb{N}$ such that

$$\frac{\ell_s}{m\rho^{k_s}} - \frac{\ell_t}{m\rho^{k_t}} = \frac{v_{s,t}}{m\rho^{r_{s,t}}}, \quad \forall s \neq t \in \mathbb{N}. \tag{3.4}$$

Since $\rho \notin \mathbb{O}$ and $k_s \neq k_t$ when $s \neq t$, it is easy to see that $r_{s,t} \neq k_s, k_t$ when $s \neq t$.

Claim. *$r_{s,j} \to +\infty$ as $j \to +\infty$ for all $s \in \mathbb{N}$.*

Proof of the Claim. For any given $s, a \in \mathbb{N}$, if there are infinitely many j such that $r_{s,j} \le a$, then there are infinitely many identical $r_{s,j}$'s. Hence there exist $j_2 > j_1 > s$ so that $m|(v_{s,j_1} - v_{s,j_2})$ and $r_{s,j_1} = r_{s,j_2}$. This means that $\frac{\ell_{j_1}}{m\rho^{k_{j_1}}} - \frac{\ell_{j_2}}{m\rho^{k_{j_2}}} = \frac{v_{s,j_2} - v_{s,j_1}}{m\rho^{r_{s,j_1}}}$ and $\frac{v_{s,j_2} - v_{s,j_1}}{m\rho^{r_{s,j_1}}}$ is not a zero of $\hat{\mu}_\rho(t)$ by $\rho \notin \mathbb{O}$, a contradiction. Therefore, there are only finite j such that $r_{s,j} \le a$, the claim follows.

For any $s \ge 1$, the claim implies that, we can choose a sufficiently large t so that $k_t, r_{s,t} > k_s$. (3.4) implies that ρ^{-1} is a zero of the integral polynomial

$$\ell_s - \ell_t x^{k_t - k_s} - v_{s,t} x^{r_{s,t} - k_s}.$$

Hence Lemma 3.2 implies

$$a_0 | \ell_s, \quad s = 1, 2, \ldots. \tag{3.5}$$

For any given $s \neq t \in \mathbb{N}$, since $\lambda_s - \lambda_t = (\lambda_s - \lambda_u) - (\lambda_t - \lambda_u)$, so (3.4) implies

$$\frac{v_{s,t}}{m\rho^{r_{s,t}}} = \frac{v_{s,u}}{m\rho^{r_{s,u}}} - \frac{v_{t,u}}{m\rho^{r_{t,u}}}$$

for all $u \in \mathbb{N}$. By the above claim, we can choose a sufficiently large u so that $r_{s,u}, r_{t,u} > r_{s,t}$. Hence ρ^{-1} is a zero of the integral polynomial

$$v_{s,t} - v_{s,u} x^{r_{s,u} - r_{s,t}} + v_{t,u} x^{r_{t,u} - r_{s,t}}.$$

Hence Lemma 3.2 implies

$$a_0 | v_{s,t}, \quad s \neq t = 1, 2, \ldots. \tag{3.6}$$

Therefore, (3.4)–(3.6) imply that $E_{a_0^{-1}\Lambda}$ is also an infinite orthonormal set in $L^2(\mu_\rho)$. Replace Λ by $a_0^{-1}\Lambda$, the above proof implies that $E_{a_0^{-2}\Lambda}$ is an infinite orthonormal set in $L^2(\mu_\rho)$. Continuing in this way, it can be proven that $E_{a_0^{-n}\Lambda}$ is an infinite orthonormal set in $L^2(\mu_\rho)$ for all $n > 0$. It must implies that $a_0 = 1$ necessary by using $a_0 \in \mathbb{N}$. The lemma is proven. □

For the minimal polynomial of ρ^{-1} defined in (3.3) with $a_0 = 1$, we define an $(n+1) \times (n+1)$ invertible matrix

$$E = \begin{pmatrix} 1 & 0 \\ -\mathbf{a} & I_n \end{pmatrix}, \tag{3.7}$$

where I_n is the $n \times n$ identical matrix, $\mathbf{a} = (a_1, \dots, a_n)^t$. For this matrix E, we define a sequence of $(n+1) \times 1$ vectors as following:

$$\alpha_0 \doteq (0, 1, 0, \dots, 0)^t$$

and

$$\alpha_{k+1} \doteq (\alpha_{k+1,0}, \alpha_{k+1,1}, \dots, \alpha_{k+1,n})^t = E(\alpha_{k,1}, \alpha_{k,2}, \dots, \alpha_{k,n}, 0)^t \tag{3.8}$$

for $k \ge 0$.

Lemma 3.5. *Assume m is a prime, then*

(i) $\alpha_{s,n} = -a_n\alpha_{s,0}$ *for any integer* $s \ge 0$*;*

(ii) *There is an integer* $\ell > 0$ *so that* $\alpha_{i+\ell} \equiv \alpha_i \pmod m$ *for all* $i \ge 0$*;*

(iii) $\alpha_{k-n} \equiv (\alpha_{k-n,0}, 0, \dots, 0, 1)^t \pmod m$ *when* $k = \ell j + 1 > n$ *for some integer* j*, where* $\alpha_{k-n,0}$ *satisfies* $a_n\alpha_{k-n,0} \equiv 1 \pmod m$.

Proof. (i) It follows directly from the definition.

(ii) Consider the residue class of $\mathbb{Z}^{n+1}$ modulo m, we see that there exist integers i and $\ell > 0$ so that $\alpha_{i+\ell} \equiv \alpha_i \pmod m$. If $i > 0$, then

$$E(\alpha_{i+\ell-1,1}, \alpha_{i+\ell-1,2}, \dots, \alpha_{i+\ell-1,n}, 0)^t \equiv E(\alpha_{i-1,1}, \alpha_{i-1,2}, \dots, \alpha_{i-1,n}, 0)^t \pmod m.$$

It is easy to see that $(\alpha_{i+\ell-1,1}, \alpha_{i+\ell-1,2}, \dots, \alpha_{i+\ell-1,n}, 0)^t \equiv (\alpha_{i-1,1}, \alpha_{i-1,2}, \dots, \alpha_{i-1,n}, 0)^t \pmod m$. By the conclusion (i) and the assumption that m is a prime, we see that $\alpha_{i+\ell-1,n} \equiv \alpha_{i-1,n} \pmod m$ implies $\alpha_{i+\ell-1,0} \equiv \alpha_{i-1,0} \pmod m$. Therefore, $\alpha_{i+\ell-1} \equiv \alpha_{i-1} \pmod m$. Continuing in this way implies the conclusion (ii).

(iii) The conclusion (ii) implies $\alpha_{k-1} \equiv \alpha_0 = (0, 1, 0, \dots, 0)^t \pmod m$. Note

$$E^{-1} = \begin{pmatrix} 1 & 0 \\ \mathbf{a} & I_n \end{pmatrix},$$

(3.8) implies $(\alpha_{k-2,1}, \alpha_{k-2,2}, \dots, \alpha_{k-2,n}, 0)^t \equiv E^{-1}(0, 1, 0, \dots, 0)^t \equiv (0, 1, 0, \dots, 0)^t \pmod m$. Since m, a_n are co-prime, $\alpha_{k-2,n} = a_n\alpha_{k-2,0}$, so $\alpha_{k-2} \equiv (0, 0, 1, 0, \dots, 0)^t \pmod m$ if $n > 2$. Continuing in this way proves $(\alpha_{k-n,1}, \alpha_{k-n,2}, \dots, \alpha_{k-n,n}, 0)^t \equiv (0, 0, \dots, 0, 1)^t \pmod m$. Note that m, a_n are co-prime, we see that the conclusion $a_n\alpha_{k-n,0} \equiv 1 \pmod m$ follows and so (iii) is proven. □

Proposition 3.6. *If $\rho \notin \mathbb{O}$, m is a prime and E_Λ is an orthonormal set in $L^2(\mu_\rho)$ with $0 \in \Lambda$, then Λ is a finite set.*

Proof. Assume on the contrary that Λ is an infinite set. Lemma 3.4 implies that ρ^{-1} has a minimal polynomial $g(x)$ as (3.3) with $a_0 = 1$.

Since m is a prime, the assumption of Lemma 3.5 holds, m, c_j are co-prime.

Claim. *There are infinitely many families of integers $\{c_1, c_2, c_3, s, k\}$ such that: ρ^{-1} is a zero of the integral polynomial $f(x) = c_1 - c_2x^s - c_3x^{k+n}$, $k + n > s > 0$, $c_1, c_2, c_3 \in \mathbb{Z} \setminus m\mathbb{Z}$ and, either $c_1 \equiv c_2 \pmod m$ and $\ell | s$ or $c_1 \equiv c_3 \pmod m$ and $\ell | (k+n)$, where ℓ is defined in Lemma 3.5.*

Proof of the Claim. By Lemma 3.3, there exist infinitely many $\frac{\ell_s}{m\rho^{k_s}} \in \Lambda$ such that $\ell_s \in \mathbb{Z} \setminus m\mathbb{Z}$ and $0 < k_j < k_{j+1}$. By (2.2) in Lemma 2.3(ii), there exist $v_{s,t} \in \mathbb{Z} \setminus m\mathbb{Z}$ and $r_{s,t} \in \mathbb{N}$ such that

$$\frac{\ell_s}{m\rho^{k_s}} - \frac{\ell_t}{m\rho^{k_t}} = \frac{v_{s,t}}{m\rho^{r_{s,t}}}, \quad s \ne t \ge 1. \tag{3.9}$$

(a) It is easy to see that there is an i such that $\ell_i \equiv \ell_s \pmod m$ for infinitely many s. By choosing a subclass of Λ, without loss of generality, we can assume that $\ell_1 \equiv \ell_s \pmod m$ for all $s > 1$.

(b) For the ℓ defined in Lemma 3.5, there is an i such that $\ell | (k_t - k_i)$ for infinitely many t. By choosing a subclass of Λ, without loss of generality, we can assume that $\ell | (k_t - k_1)$ for all $t > 1$.

(c) By the claim in the proof of Lemma 3.4, we can choose a t_0 such that $r_{1,t} > k_1$ for all $t \ge t_0$. Hence (3.9) implies that ρ^{-1} is a zero of the integral polynomial $\ell_1 - \ell_t x^{k_t-k_1} - v_{s,t}x^{r_{1,t}-k_1}$ with $\ell|(k_t - k_1)$.

Since $\rho \notin \mathbb{O}$ implies that either $r_{1,t} - k_1 > k_t - k_1 > 0$ or $0 < r_{1,t} - k_1 < k_t - k_1$, the claim follows.

By Lemma 3.4, let $g(x) = 1 + a_1x + \cdots + a_nx^n$ be the minimal polynomial of ρ^{-1}, then $f(x) = g(x)h(x)$ for an integral polynomial $h(x)$ with order k.

For the above $g(x), f(x)$ and $h(x)$, we use the following notations. Let $h(x) = b_0 + b_1x + \cdots + b_kx^k$, $\mathbf{a} = (1, a_1, \ldots, a_n)^t$, $\mathbf{b} = (b_0, b_1, \ldots, b_k)^t$. Let $\mathbf{e}_j$ be the j-th column of the $(k+n+1) \times (k+n+1)$ identical matrix I_{k+n+1}.

Define a $(k+n+1) \times (k+1)$ matrix $A = (u_{t,j})_{0\le t\le k+n, 0\le j\le k}$ by $u_{j,j} = 1$ for $0 \le j \le k$, $u_{j+i,j} = a_i$ for $0 \le j \le k$ and $1 \le i \le n$, and $u_{j,t} = 0$ for other (j, t).

Then $f(x) = g(x)h(x)$ if and only if $A\mathbf{b} = c_1\mathbf{e}_1 - c_2\mathbf{e}_{s+1} - c_3\mathbf{e}_{k+n+1}$. Define $(k+n+1) \times (k+n+1)$ matrices:

$$E_1 = \begin{pmatrix} E & 0 \\ 0 & I_k \end{pmatrix}, \ldots, E_j = \begin{pmatrix} I_{j-1} & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I_{k-j+1} \end{pmatrix}, \ldots, E_{k+1} = \begin{pmatrix} I_k & 0 \\ 0 & E \end{pmatrix},$$

where I_j is the $j \times j$ identical matrix, E is defined in (3.7).

It is clear that $E(1, a_1, \ldots, a_n)^t = (1, 0, \ldots, 0)^t$. It follows that E_1A reduces the first column of A to $(1, 0, \ldots, 0)^t$ and the other columns remain unchanged; E_2E_1A reduces the second column of E_1A to $(0, 1, 0, \ldots, 0)^t$ and the other columns remain unchanged. Finally $E_{k+1}\cdots E_1A$ is the matrix with 1 on the diagonal and 0 elsewhere.

Furthermore, the last $n+1$ entries of $E_{k+1}\cdots E_1\mathbf{e}_1$ generate the vector α_{k+1} defined in Lemma 3.5. Hence $E_{k+1}E_k\cdots E_1\mathbf{e}_1 \equiv (v_0, v_1, \ldots, v_{k-1}, \alpha_{k+1,0}, \alpha_{k+1,1}, \ldots, \alpha_{k+1,n})^t \pmod m$ for some $v_0, v_1, \ldots, v_{k-1}$.

For $\mathbf{e}_{s+1}$, we have

$$E_{k+1}\cdots E_1\mathbf{e}_{s+1} = \begin{cases} E_{k+1}\cdots E_{s+1}\mathbf{e}_{s+1}, & s \le k, \\ \mathbf{e}_{s+1}, & s > k. \end{cases} \tag{3.10}$$

Hence the last $n+1$ entries of $E_{k+1}\cdots E_1\mathbf{e}_{s+1}$ generate the vector α_{k+1-s} when $s \le k$ and the vector $(0, \ldots, 0, 1, 0, \ldots, 0)^t$ with the 1 at the $s-k+1$-th entry when $s > k$.

Since $f(x) = g(x)h(x)$, so the last n entries of $E_{k+1}\cdots E_1(c_1\mathbf{e}_1 - c_2\mathbf{e}_{s+1} - c_3\mathbf{e}_{k+n+1})$ are all zero. In the following, we will consider the $(n+1) \times 1$ vector generated by the last $n+1$ entries of $E_{k+1}\cdots E_1(c_1\mathbf{e}_1 - c_2\mathbf{e}_{s+1} - c_3\mathbf{e}_{k+n+1})$ by using the above claim. Note that the last $n+1$ entries of $E_{k+1}\cdots E_1A\mathbf{b}$ generates the vector$(b_k, 0, \ldots, 0)^t$. Hence

$$\begin{cases} c_1\alpha_{k+1} - c_2\alpha_{k+1-s} - c_3(0, \ldots, 0, 1) = (b_k, 0, \ldots, 0)^t, & \text{if } s \le k, \\ c_1\alpha_{k+1} - c_2(0, \ldots, 0, 1, 0, \ldots, 0)^t - c_3(0, \ldots, 0, 1) = (b_k, 0, \ldots, 0)^t, & \text{if } s > k. \end{cases} \tag{3.11}$$

Case 1. If $\ell|s$ and $k+n > s > k$. Then, use (3.10), the vector generated by the last $n+1$ entries of $E_{k+1}\cdots E_1(c_1\mathbf{e}_1 - c_2\mathbf{e}_{s+1} - c_3\mathbf{e}_{k+n+1})$ is $c_1\alpha_{k+1} - (0, \ldots, 0, c_2, 0, \ldots, 0, c_3)^t$ with the c_2 at the $s-k+1$-th entry. Hence (3.11) implies $c_1\alpha_{k+1} = (b_k, 0, \ldots, 0, c_2, 0, \ldots, 0, c_3)^t$, so c_1 is a common divisor of b_k, c_2 and c_3, so $\alpha_{k+1} = (b_k/c_1, 0, \ldots, 0, c_2/c_1, 0, \ldots, 0, c_3/c_1)^t$. By the definition of α_j, we have $\alpha_{s+1} = E(c_2/c_1, 0, \ldots, 0, c_3/c_1, \ldots, 0,)^t$. This contradicts $\alpha_{s+1} \equiv E(1, 0, \ldots, 0)^t \pmod m$, since $\ell|s$ and $c_2, c_3 \not\equiv 0 \pmod m$.

Case 2. If $\ell|s$, $s \le k$ and $c_1 \equiv c_2 \pmod m$. Then the last $n+1$ entries of $E_{k+1}\cdots E_1\mathbf{e}_{s+1}$ generate the vector α_{k+1-s} by using the definition of E_j and (3.10). Hence the last $n+1$ entries of $E_{k+1}\cdots E_1(c_1\mathbf{e}_1 - c_2\mathbf{e}_{s+1})$ generate the vector $c_1\alpha_{k+1} - c_2\alpha_{k+1-s} \equiv 0 \pmod m$ by using Lemma 3.5(ii) and $c_1 \equiv c_2 \pmod m$.

Therefore, we have

$$E_{k+1}E_k\cdots E_1(c_1\mathbf{e}_1 - c_2\mathbf{e}_{s+1} - c_3\mathbf{e}_{k+n+1}) \equiv (c_1u_0, \ldots, c_1u_{s-1}, 0, \ldots, 0, -c_3)^t \pmod m$$

for some $u_0, u_1, \ldots, u_{s-1}$. Since $c_3 \not\equiv 0 \pmod m$, it contradicts to the conclusion that the last n entries of $E_{k+1}\cdots E_1(c_1\mathbf{e}_1 - c_2\mathbf{e}_{s+1} - c_3\mathbf{e}_{k+n+1})$ are all zero.

Case 3. If $\ell|(k+n)$ and $c_1 \equiv c_3 \pmod m$. Then $E_{k+1}E_k\cdots E_1(c_1\mathbf{e}_1) \equiv (v_0, v_1, \ldots, v_{k-1}c_1\alpha_{k+1,0}, 0, \ldots, 0, c_1)^t \pmod m$ for some $v_0, v_1, \ldots, v_{k-1}$ by using Lemma 3.5(ii). Note $E_{k+1}E_k\cdots E_1\mathbf{e}_{k+n+1} = \mathbf{e}_{k+n+1}$, we see

$$E_{k+1}E_k\cdots E_1(c_1\mathbf{e}_1 - c_3\mathbf{e}_{k+n+1}) \equiv (v_0, v_1, \ldots, v_{k-1}, c_1\alpha_{k+1,0}, 0, \ldots, 0)^t \pmod m.$$

Since the last n entries of $E_{k+1}\cdots E_1(c_1\mathbf{e}_1 - c_2\mathbf{e}_{s+1} - c_3\mathbf{e}_{k+n+1})$ are all zero, the above equality implies

$$E_{k+1}E_k\cdots E_1\mathbf{e}_{s+1} \equiv (w_0, w_1, \ldots, w_k, 0, 0, \ldots, 0)^t \pmod m$$

for some $w_0, w_1, \ldots, w_k$. If $s \le k$, (3.10) implies $\alpha_{k-s+1} \equiv (w_k, 0, 0, \ldots, 0)^t \pmod m$, so Lemma 3.5(i) implies $a_nw_k = a_n\alpha_{k-s+1,0} = \alpha_{k-s+1,n} \equiv 0 \pmod m$. Since $f(x) = g(x)h(x)$, so $a_n|c_3$. Since $c_3 \in \mathbb{Z} \setminus m\mathbb{Z}$ and m is a prime, so $\alpha_{k-s+1} \equiv 0 \pmod m$. Hence the definition of α_j implies $\alpha_j \equiv 0 \pmod m$ for all sufficiently large j, a contradiction to Lemma 3.5(ii). If $s > k$, (3.10) implies $E_{k+1}E_k\cdots E_1\mathbf{e}_{s+1} = \mathbf{e}_{s+1}$, so the above equality implies that $(w_k, 0, 0, \ldots, 0)^t \equiv (0, \ldots, 0, 1, 0, \ldots, 0)^t \pmod m$ with the 1 at the $s-k+1$-th entry, This is obviously impossible.

Therefore, Λ is finite. $\square$

Proof of Theorem 1.2. The sufficiency follows from Proposition 3.1. For the necessity, Proposition 3.6 implies $\rho \in \mathbb{O}$, then the necessity follows from Proposition 3.1. $\square$

4. Proof of Theorem 1.1

The idea of the following theorem comes from [1].

Theorem 4.1. *If $|\rho| = q/p$ with $1 < q < p$ being co-prime, then there exist constants $a > 0$, $C_0 > 0$ so that*

$$\sup_{t\in\mathbb{R}}\{|\hat{\mu}_\rho(t)|(\ln(2+|t|))^a\} \le C_0 < +\infty.$$

Proof. Claim. *There is a positive constant $b < 1$ such that: assume $|\xi| > 1$, then we can find an η such that $|\rho\xi| \ge |\eta| \ge \rho^2|\xi|^{\ln q/\ln p}$ and*

$$|\hat{\mu}_\rho(\xi)| \le b|\hat{\mu}_\rho(\eta)|.$$

Proof of the Claim. For any real number x, let $\lceil x \rceil$ be the number in $(-\frac{1}{2}, \frac{1}{2}]$ such that $x - \lceil x \rceil \in \mathbb{Z}$.

(a) If $\lceil \rho\xi \rceil \notin (-\frac{1}{2p}, \frac{1}{2p})$, then $|1 + \exp\{2\pi\lceil\rho\xi\rceil i\}| \le |1 + \exp\{\frac{\pi}{p} i\}| < 2$. Use (2.1) and $|\exp\{2\pi j\lceil\rho\xi\rceil i\}| \le 1$, we have

$$\begin{aligned}
|\hat{\mu}_\rho(\xi)| &= |\hat{\mu}_\rho(\rho\xi)| \cdot \left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho\xi i\})^j\right| = |\hat{\mu}_\rho(\rho\xi)| \cdot \left|\frac{1}{m}\sum_{j=0}^{m-1}\exp\{2\pi j\lceil\rho\xi\rceil i\}\right| \\
&\le |\hat{\mu}_\rho(\rho\xi)| \cdot \frac{m-2+|1+\exp\{2\pi\lceil\rho\xi\rceil i\}|}{m} \le |\hat{\mu}_\rho(\rho\xi)| \cdot \frac{m-2+\left|1+\exp\left\{\frac{\pi}{p}i\right\}\right|}{m}.
\end{aligned}$$

Let $b = \frac{m-2+|1-\exp\{\frac{\pi}{p}i\}|}{m}$, $\eta = \rho\xi$, the claim holds.

(b) If $\lceil\rho\xi\rceil \in (-\frac{1}{2p}, \frac{1}{2p})$. Since $|\xi| > 1$, so $|\rho\xi| > |\rho| \ge \frac{1}{p}$ by $|\rho| = q/p$. Hence, we can write $|\rho\xi| = \lceil\rho\xi\rceil + \sum_{k=s}^{t} e_k p^k$ with $e_j \in \{0, 1, \dots, p-1\}$, $e_s > 0$, $e_t > 0$ and $s \ge 0$. Hence

$$\lceil\rho^{s+2}\xi\rceil = \left\lceil (q/p)^{s+1}\lceil\rho\xi\rceil + q^{s+1}/p\sum_{k=s}^{t} e_k p^{k-s}\right\rceil = \left\lceil (q/p)^{s+1}\lceil\rho\xi\rceil + \frac{e_s q^{s+1}}{p}\right\rceil.$$

Since $\lceil\rho\xi\rceil \in (-\frac{1}{2p}, \frac{1}{2p})$ and $0 < (q/p)^{s+1} < 1$, so $(q/p)^{s+1}\lceil\rho\xi\rceil \in (-\frac{1}{2p}, \frac{1}{2p})$. Hence $\lceil\rho^{s+2}\xi\rceil \notin (-\frac{1}{2p}, \frac{1}{2p})$ by using the above equality. Therefore, similar to the above conclusion (a), we have

$$\begin{aligned}
|\hat{\mu}_\rho(\xi)| &= \left|\prod_{k=1}^{s+2}\left[\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^k\xi i\})^j\right]\hat{\mu}_\rho(\rho^{s+2}\xi)\right| \\
&\le \left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^{s+2}\xi i\})^j| \cdot |\hat{\mu}_\rho(\rho^{s+2}\xi)\right| \\
&\le \frac{m-2+\left|1-\exp\left\{\frac{\pi}{p}i\right\}\right|}{m}|\hat{\mu}_\rho(\rho^{s+2}\xi)|.
\end{aligned}$$

Since $|\rho\xi| = \lceil|\rho\xi|\rceil + \sum_{k=s}^{t} e_k p^k \ge e_s p^s - \frac{1}{p} \ge |\rho|p^s$, so $s \le \log_p|\xi|$. Hence $|\rho^{s+2}\xi| \ge |\xi||\rho^2||\rho|^{\log_p|\xi|} = |\xi||\rho^2||\xi|^{\log_p|\rho|} = \rho^2|\xi|^{\ln q/\ln p}$. Let $\eta = \rho^{s+2}\xi$ and $b = \frac{m-2+|1-\exp\{\frac{\pi}{p}i\}|}{m}$, then we have

$$|\hat{\mu}_\rho(\xi)| \le b|\hat{\mu}_\rho(\eta)|, \qquad |\rho\xi| \ge |\eta| \ge \rho^2|\xi|^{\ln q/\ln p}$$

the claim also holds. The proof of the claim is completed.

For any real number t with $|t| > 1$, the claim shows that, there exist an integer n and real numbers ξ_j with $\xi_0 = t$, $|\xi_{n-1}| > 1 \ge |\xi_n|$ such that, $|\rho\xi_j| \ge |\xi_{j+1}| \ge \rho^2|\xi_j|^{\ln q/\ln p}$ and $|\hat{\mu}_\rho(\xi_j)| \le b|\hat{\mu}_\rho(\xi_{j+1})|$.

The second inequality implies $|\hat{\mu}_\rho(t)| \le b^n|\hat{\mu}_\rho(\xi_n)|$. While, the first inequality implies

$$\begin{aligned}
1 &\ge |\xi_n| \ge \rho^2|\xi_{n-1}|^{\ln q/\ln p} \ge \rho^{2+2(\ln q/\ln p)}|\xi_{n-2}|^{(\ln q/\ln p)^2} \\
&\ge \cdots \ge \rho^{2+2(\ln q/\ln p)+\cdots+2(\ln q/\ln p)^{n-1}}|\xi_0|^{(\ln q/\ln p)^n} \\
&\ge \rho^{2\ln p/(\ln p-\ln q)}|t|^{(\ln q/\ln p)^n} \\
&= p^{-2}|t|^{(\ln q/\ln p)^n}
\end{aligned}$$

by using the equality $x^{\frac{1}{-\ln x}} = e^{-1}$ for $0 < x < 1$. Hence

$$\ln p^2 \geq \ln|t| \cdot (\ln q/\ln p)^n = \ln|t| \cdot b^{n\ln(\ln q/\ln p)/\ln b},$$

so

$$(\ln p^2)^{\ln b/\ln(\ln q/\ln p)} \geq (\ln|t|)^{\ln b/\ln(\ln q/\ln p)} \cdot b^n, \quad \forall |t| > 1.$$

Let $a = \ln b/\ln(\ln q/\ln p)$, then $a > 0$ and

$$\begin{aligned}|\hat{\mu}_\rho(t)|(\ln(2+|t|))^a &\leq b^n|\hat{\mu}_\rho(\xi_n)|(\ln(2+|t|))^a = (\ln|t|)^{\ln b/\ln(\ln q/\ln p)} \cdot b^n|\hat{\mu}_\rho(\xi_n)|\left(\frac{\ln(2+|t|)}{\ln|t|}\right)^a \\ &\leq (\ln p^2)^{\ln b/\ln(\ln q/\ln p)}|\hat{\mu}_\rho(\xi_n)|\left(\frac{\ln(2+|t|)}{\ln|t|}\right)^a, \quad \forall |t| > 1.\end{aligned}$$

Since

$$\sup_{|t|\geq 2}\left\{\left(\frac{\ln(2+|t|)}{\ln|t|}\right)^a\right\} < +\infty, \qquad \sup_{|t|\leq 2}\{|\hat{\mu}_\rho(t)|(\ln(2+|t|))^a\} < +\infty,$$

so

$$\sup_{t\in\mathbb{R}}\{|\hat{\mu}_\rho(t)|(\ln(2+|t|))^a\} < +\infty$$

by using the fact $|\hat{\mu}_\rho(\xi)| \leq 1$. The theorem is proven. $\square$

Lemma 4.2. *Assume $|\rho|$ admits the minimal polynomial $px^r - q$, where p, q are co-prime and $r \in \mathbb{N}$. If E_Λ is an orthonormal basis of $L^2(\mu_\rho)$, then*

$$\Lambda - \Lambda \subseteq \{0\} \cup \left[\bigcup_{k\geq 0} \frac{\mathbb{Z}\setminus m\mathbb{Z}}{m|\rho|^{1+kr}}\right],$$

$$(\Lambda - \Lambda) \cap \left[\frac{\mathbb{Z}\setminus m\mathbb{Z}}{m|\rho|}\right] \neq \emptyset.$$

Proof. Since E_Λ is an orthonormal basis of $L^2(\mu_\rho)$, so $\Lambda - \Lambda \subseteq \mathcal{Z}_\rho$. By Lemmas 2.2 and 2.3, we can choose the smallest positive integer k_0 so that $(\Lambda - \Lambda) \cap \frac{\mathbb{Z}\setminus m\mathbb{Z}}{m|\rho|^{k_0}} \neq \emptyset$. Hence there exist $\lambda_1, \lambda_0 \in \Lambda$ such that

$$\lambda_1 - \lambda_0 = \frac{mz+s}{m|\rho|^{k_0}}, \quad 1 \leq s < m. \tag{4.1}$$

For any $\lambda \in \Lambda \setminus \{\lambda_0, \lambda_1\}$, Lemma 2.3 shows that $\lambda - \lambda_j = \frac{mz_j+s_j}{m|\rho|^{n_j}}$ with $1 \leq s_j < m$. The definition of k_0 shows that $n_j \geq k_0$. Hence

$$\frac{mz_0+s_0}{m|\rho|^{n_0}} - \frac{mz_1+s_1}{m|\rho|^{n_1}} = \lambda_1 - \lambda_0 = \frac{mz+s}{m|\rho|^{k_0}}$$

for some integers $z_0, z_1, z, s_0, s_1, s, n_0, n_1$ with $n_j \geq k_0$ and $1 \leq s_j < m$. Hence Lemma 2.4 shows $r|(n_j - k_0)$, $j = 0, 1$. Therefore, $\lambda = \lambda_0 + \frac{mz_0+s_0}{m|\rho|^{k_0+rk}}$ for some $k \geq 0$.

Now, for any $\gamma \in \Lambda$ different from λ, the above results show that γ has the form: $\gamma = \lambda_0 + \frac{mz+s}{m|\rho|^{k_0+rn}}$ and satisfies

$$\gamma - \lambda = \frac{mz+s}{m|\rho|^{k_0+rn}} - \frac{mz_0+s_0}{m|\rho|^{k_0+rk}} = \frac{mz'+s'}{m|\rho|^N}$$

for some integers z', s' and N with $1 \leq s' < m$ (since $\gamma - \lambda \in \mathcal{Z}_\rho$). Hence Lemma 2.4 shows $r|(N - k_0)$. Therefore, $\lambda - \gamma \in [\bigcup_{k\geq 0} \frac{\mathbb{Z}\setminus m\mathbb{Z}}{m|\rho|^{k_0+kr}}]$. Hence

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \bigcup_{k\geq 0} \frac{\mathbb{Z}\setminus m\mathbb{Z}}{m|\rho|^{k_0+kr}}.$$

Noting (4.1), we see that, in order to prove the lemma, we need only prove $k_0 = 1$. Assume $k_0 > 1$ on the contrary.

Let $\tilde{\Lambda} = \rho^{k_0-1}(\Lambda - \lambda_0)$. The definition of k_0 and Λ shows that $E_{\tilde{\Lambda}}$ is also an orthonormal set of $L^2(\mu_\rho)$. For any $\lambda = \lambda_0 + \frac{mz+s}{m|\rho|^{k_0+rk}} \in \Lambda$, let $\tilde{\lambda} = \rho^{k_0-1}(\lambda - \lambda_0)$ and $\tilde{t} = (t - \lambda_0)\rho^{k_0-1}$, then (2.1) implies

$$\hat{\mu}_\rho(t-\lambda) = \prod_{k=1}^{k_0-1}\left[\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^k(t-\lambda)i\})^j\right]\hat{\mu}_\rho(\tilde{t}-\tilde{\lambda}).$$

Hence

$$\sum_{\lambda\in\Lambda}|\hat{\mu}_\rho(t-\lambda)|^2 = \sum_{\tilde{\lambda}\in\tilde{\Lambda}}|\hat{\mu}_\rho(\tilde{t}-\tilde{\lambda})|^2\cdot\left|\prod_{k=1}^{k_0-1}\left[\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^k(t-\lambda)i\})^j\right]\right|^2.$$

It is easy to see that there exist a $t \in \mathbb{R}$ and a $\lambda \in \Lambda$ so that $|\prod_{k=1}^{k_0-1}[\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^k(t-\lambda)i\})^j]| < 1$ and $\hat{\mu}_\rho(\tilde{t}-\tilde{\lambda}) \neq 0$. Hence

$$\sum_{\lambda\in\Lambda}|\hat{\mu}_\rho(t-\lambda)|^2 < \sum_{\tilde{\lambda}\in\tilde{\Lambda}}|\hat{\mu}_\rho(\tilde{t}-\tilde{\lambda})|^2 \le 1,$$

where the last inequality follows from Parseval's identity and the fact that $E_{\tilde{\Lambda}}$ is also an orthonormal set of $L^2(\mu_\rho)$.

Therefore, E_Λ is not an orthonormal basis of $L^2(\mu_\rho)$ by using Parseval's identity. A contradiction, hence $k_0 = 1$. The proof completes. □

Now, we prove Theorem 1.1 by using the conclusions proven in the above part. Assume Λ is a spectrum of μ_ρ. It is easy to see that Λ is infinite, so Theorem 1.2 shows that $|\rho| = (\frac{q}{p})^{\frac{1}{r}}$, admit a minimal polynomial $px^r - q$ such that $p > q > 0$ are co-prime and $m|p$. We first prove $r = 1$, then prove $q = 1$.

As an extension of [3, Theorem 4.4], we have the following.

Proposition 4.3. *If μ_ρ is a spectral measure, then ρ is a rational.*

Proof. By Theorem 1.2, $|\rho|$ has a minimal polynomial as $px^r - q$. Assume $r > 1$, on the contrary. Define μ_{ρ^k} with $k \in \mathbb{N}$ to be the self-similar measure satisfying

$$\mu_{\rho^k}(A) = \frac{1}{m}\sum_{j=0}^{m-1}\mu_{\rho^k}(\rho^{-1}(A)-j).$$

Let Λ be a spectrum of μ_ρ, use Lemma 4.2, we see that

$$(\Lambda-\Lambda)\setminus\{0\} \subseteq \bigcup_{k\ge 0}\frac{\mathbb{Z}\setminus m\mathbb{Z}}{m|\rho|^{1+kr}}.$$

Hence

$$\rho^{-r+1}(\Lambda-\Lambda)\setminus\{0\} \subseteq \bigcup_{k\ge 1}\frac{\mathbb{Z}\setminus m\mathbb{Z}}{m|\rho|^{kr}} = \mathcal{Z}_{\rho^r}. \tag{4.2}$$

Since $|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^k ti\})^j| \le 1$, $\mathbb{N}+r-1 \supset \{r+1\}\cup(r\mathbb{N})$ by using $r > 1$, so

$$\begin{aligned}
|\hat{\mu}_\rho(t)| &= \prod_{k=1}^{+\infty}\left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^k ti\})^j\right| \\
&= \prod_{k=1}^{+\infty}\left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^{k+r-1}\rho^{1-r}ti\})^j\right| \\
&\le \left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^2 ti\})^j\right|\prod_{n=1}^{+\infty}\left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^{nr}\rho^{1-r}ti\})^j\right| \\
&= \left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^2 ti\})^j\right|\cdot|\hat{\mu}_{\rho^r}(\rho^{1-r}t)|.
\end{aligned}$$

Choose a $\lambda_0 \in \Lambda$ and a $t_0 \in \mathbb{R}$ so that $|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^2(t_0-\lambda_0)i\})^j| < 1$ and $|\hat{\mu}_{\rho^r}(\rho^{1-r}(t_0-\lambda_0))| > 0$. Since E_Λ is an orthonormal basis of $L^2(\mu_\rho)$, so the above inequality and Parseval's identity imply

$$1 = \sum_{\lambda\in\Lambda}|\hat{\mu}_\rho(t_0-\lambda)|^2 < \sum_{\lambda\in\Lambda}|\hat{\mu}_{\rho^r}(\rho^{1-r}t_0-\rho^{1-r}\lambda)|^2 = \sum_{\gamma\in\rho^{1-r}\Lambda}|\hat{\mu}_{\rho^r}(\rho^{1-r}t_0-\gamma)|^2 \le 1,$$

a contradiction, where the last inequality "$\le$" follows from (4.2) which implies that $E_{\rho^{1-r}\Lambda}$ is an orthonormal set of $L^2(\mu_{\rho^r})$.

Therefore, $r = 1$, the proposition follows. $\square$

Lemma 4.4. *If Λ is a spectrum of μ_ρ and $|\rho| = q/p$ with* $\gcd(q, m) = 1$. *Then*

$$(\Lambda - \Lambda) \setminus \{0\} \subset \bigcup_{k\ge 0} \frac{p^k(\mathbb{Z} \setminus m\mathbb{Z})}{m|\rho|}.$$

Proof. We first prove a weaker conclusion:

$$(\Lambda - \lambda_0) \setminus \{0\} \subset \bigcup_{k\ge 0} \frac{p^k(\mathbb{Z} \setminus m\mathbb{Z})}{m|\rho|^\imath} \tag{4.3}$$

for some positive integer $\imath$ and $\lambda_0 \in \Lambda$.

For any $a \in \mathcal{Z}_\rho$, it can be written as $a = \frac{\ell}{m|\rho|^k}$ for some $k \in \mathbb{N}$, $\ell \in \mathbb{Z} \setminus m\mathbb{Z}$. We can find a nonnegative integer $n \ge 0$ so that $\ell = p^n \ell_1$ with $\ell_1 \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z}$, then $a = \frac{q^n \ell_1}{m|\rho|^{k+n}}$ with $q^n \ell_1 \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z}$ by using the assumption $\gcd(p, q) = 1$ and $\gcd(q, m) = 1$. For any $\lambda \in \Lambda \setminus \{\lambda_0\}$, we have $\lambda - \lambda_0 \in \mathcal{Z}_\rho$. Hence

$$\Lambda \setminus \{\lambda_0\} \subset \lambda_0 + \mathcal{Z}_\rho \subset \lambda_0 + \left\{ \frac{\ell}{m|\rho|^k} : k \in \mathbb{N}, \ell \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z} \right\}. \tag{4.4}$$

Since $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}_\rho$, let $\imath$ be the smallest positive integer such that there is a $\lambda_1 \in \Lambda$ so that $\lambda_1 - \lambda_0 = \frac{\ell_1}{m|\rho|^\imath} \in \mathcal{Z}_\rho$ for some $\ell_1 \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z}$.

For any $\lambda \in \Lambda \setminus \{\lambda_0, \lambda_1\}$, by (4.4), we have $\lambda - \lambda_0 = \frac{\ell_2}{m|\rho|^n} \in \Lambda$ with $n \in \mathbb{N}$, $\ell_2 \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z}$. Furthermore, Lemma 2.3 and the above (4.4) imply that there exists $\frac{v}{m|\rho|^u} \in \mathcal{Z}_\rho$ with $v \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z}$ such that

$$\frac{\ell_1}{m|\rho|^\imath} - \frac{\ell_2}{m|\rho|^n} = \lambda_1 - \lambda = \frac{v}{m|\rho|^u}. \tag{4.5}$$

Let $n = t + \imath$, $u = s + \imath$. The definition of $\imath$ implies that either $t = 0$ or $t > 0$. If $t = 0$, i.e. $n = \imath$, then $\lambda - \lambda_0$ belongs to the right hand side of (4.3). Assume $t > 0$, i.e. $n > \imath$. (4.5) implies

$$\ell_1 - \frac{\ell_2 p^t}{q^t} = \frac{v p^s}{q^s}.$$

If $s > 0$, since p, q are co-prime, so $p|\ell_1$, a contradiction, hence $s \le 0$. If $s < 0$, then the above equality shows $p^{|s|}|q^{|s|+t}v$, also a contradiction. Hence $s = 0$, the above equality implies that $q^t|\ell_2$, so $\lambda - \lambda_0 = \frac{p^t \ell_3}{m|\rho|^\imath}$ for some $\ell_3 \in (\mathbb{Z} \setminus p\mathbb{Z}) \setminus m\mathbb{Z}$. Therefore, (4.3) holds for some $\imath \in \mathbb{N}$.

Assume $\imath > 1$. Note that we have assumed $\gcd(q, m) = 1$. For any distinct $\lambda, \lambda' \in \Lambda$, the above equality shows $\lambda - \lambda' = \frac{p^s x - p^t y}{m|\rho|^\imath}$ for some integers $x, y \in \mathbb{Z} \setminus p\mathbb{Z}$ and $t \ge s \ge 0$. Since $\lambda - \lambda' \in \mathcal{Z}_\rho$, so

$$\lambda - \lambda' = \frac{p^s(x - p^{t-s}y)}{m|\rho|^\imath} = \frac{v}{m|\rho|^{\imath+n}} \tag{4.6}$$

for some $v \in \mathbb{Z} \setminus m\mathbb{Z}$ and $n > -\imath$. We can find $k \ge 0$ so that $v = p^k(mu + w')$ with $p \nmid (mu + w')$ and $1 \le w' < m$. Since $|\rho| = q/p$ and p, q are co-prime, the second equality of (4.6) shows $k + n \ge s \ge 0$. Noting $\gcd(q, m) = 1$, we see that $q^k(mu + w')$ belongs to $\mathbb{Z} \setminus m\mathbb{Z}$. Hence

$$|\rho|(\lambda - \lambda') = \frac{p^k(mu + w')}{m|\rho|^{\imath+n-1}} = \frac{q^k(mu + w')}{m|\rho|^{\imath+n+k-1}} \in \mathcal{Z}_\rho$$

by using $\gcd(q, m) = 1$, $k + n \ge s \ge 0$ and $\imath > 1$.

Therefore, $\rho(\Lambda - \Lambda) \subseteq \mathcal{Z}_\rho \cup \{0\}$, i.e. $E_{\rho\Lambda}$ is an orthonormal set of $L^2(\mu_\rho)$ when $\imath > 1$. Since Λ is a spectrum, Parseval's identity implies

$$1 = \sum_{\lambda\in\Lambda} |\hat{\mu}_\rho(t - \lambda)|^2 = \sum_{\gamma\in\rho\Lambda} \left| \frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi i(\rho t - \gamma)\}) \right| \cdot |\hat{\mu}_\rho(\rho t - \gamma)|^2.$$

It is easy to see that there exist $t \in \mathbb{R}$ and $\gamma \in \rho\Lambda$ so that $|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi i(\rho t-\gamma)\})| \cdot |\hat{\mu}_\rho(\rho t-\gamma)|^2 < |\hat{\mu}_\rho(\rho t-\gamma)|^2$. Hence, by $|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi i(\rho t-\gamma)\})| \le 1$, we have

$$1 = \sum_{\lambda\in\Lambda} |\hat{\mu}_\rho(t-\lambda)|^2 < \sum_{\gamma\in\rho\Lambda} |\hat{\mu}_\rho(\rho t-\gamma)|^2 \le 1,$$

a contradiction, where the last inequality follows from the proven fact that $E_{\rho\Lambda}$ is an orthonormal set of $L^2(\mu_\rho)$. Therefore, $\imath = 1$.

By (4.3) and (4.6) and $\imath = 1$, we see that, for any two different $\lambda, \lambda' \in \Lambda$, the v and n in (4.6) satisfy $n \ge 0$ and $q^n|v$. Hence $\lambda - \lambda'$ belongs to the set $\frac{p^n(\mathbb{Z}\setminus m\mathbb{Z})}{m|\rho|}$ with $n \ge 0$. The lemma follows. □

The following theorem describes the structure of a spectrum.

Theorem 4.5. *If Λ is a spectrum of μ_ρ and $|\rho| = q/p$ with $m|p$ and $\gcd(p,q) = 1$, then there exist spectrums $\Gamma_k \subset \mathbb{Z}$ and integers z_k such that $\Lambda = \bigcup_{k=0}^{m-1}(\lambda_0 + \frac{k+mz_k}{m|\rho|} + |\rho|^{-1}\Gamma_k)$ with $\lambda_0 \in \Lambda$ and $0 \in \Gamma_k$. Furthermore, the union is a disjoint union.*

Proof. Use the notations in Lemma 4.4. Let

$$B = \left\{k \in \{0,1,\ldots,m-1\} : \left(\lambda_0 + \frac{k+m\mathbb{Z}}{m|\rho|}\right) \cap \Lambda \ne \emptyset\right\}.$$

Using Lemmas 4.2 and 4.4, there are integers z_k, for all $k \in B$, such that

$$\left|\frac{mz_k+k}{m|\rho|}\right| = \min\left\{\left|\frac{mz+k}{m|\rho|}\right| : z \in \mathbb{Z}, \lambda_0 + \frac{k}{m|\rho|} + \frac{mz}{m|\rho|} \in \Lambda\right\} \tag{4.7}$$

and

$$\Lambda = \bigcup_{k\in B}\left[\left(\lambda_0 + \frac{mz_k+k}{m|\rho|} + \frac{m\mathbb{Z}}{m|\rho|}\right) \cap \Lambda\right]$$

by noting that Lemma 4.4 implies $\Lambda - \lambda_0 \subset \frac{\mathbb{Z}}{m|\rho|}$. Furthermore, the above union is disjoint.

Let

$$\Gamma_k = \mathbb{Z} \cap \left(|\rho|\Lambda - |\rho|\lambda_0 - \frac{k+mz_k}{m}\right), \quad k \in B,$$

then

$$\Lambda = \bigcup_{k\in B}\left(\lambda_0 + \frac{k+mz_k}{m|\rho|} + |\rho|^{-1}\Gamma_k\right), \quad \Gamma_k \subset \mathbb{Z}.$$

We will prove that E_{Γ_k} is an orthonormal basis of $L^2(\mu_\rho)$ for each $k \in B$ and $B = \{0,1,\ldots,m-1\}$.

For any $k \in B$, let $z, \ell \in \Gamma_k$ be distinct. Let

$$a = \lambda_0 + \frac{mz_k+k}{m|\rho|} + \frac{mz}{m|\rho|}, \qquad b = \lambda_0 + \frac{mz_k+k}{m|\rho|} + \frac{m\ell}{m|\rho|}.$$

Then a, b belong to Λ, so $a - b = \frac{mt+s}{m|\rho|^n}$ for some integers t, s, n such that $0 < s < m$ and $n > 0$, hence $z - \ell = |\rho|(a-b) = \frac{mt+s}{m|\rho|^{n-1}}$. Noting $|\rho| = q/p$, we see that $m|p^{n-1}(mt+s)$, so $n > 1$ by using $0 < s < m$. Hence $z - \ell \in \mathcal{Z}_\rho$ by Lemma 2.2. This means that E_{Γ_k} is an orthonormal set of $L^2(\mu_\rho)$ for all $k \in B$ by Lemma 2.3.

Since

$$\sum_{k=0}^{m-1}\exp\left\{2\pi\frac{(j_1-j_2)k}{m}i\right\} = \begin{cases} 0, & j_1 \ne j_2 \\ m, & j_1 = j_2 \end{cases} \tag{4.8}$$

for any $j_1, j_2 \in \{0,1,\ldots,m-1\}$, so

$$\begin{aligned}\sum_{k=0}^{m-1}\left|\frac{1}{m}\sum_{j=0}^{m-1}\left(\exp\left\{2\pi\left(\rho t - \rho\lambda_0 - \frac{k}{m}\right)i\right\}\right)^j\right|^2 &= \frac{1}{m^2}\sum_{k=0}^{m-1}\sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1}\exp\left\{2\pi(j_1-j_2)\left(\rho t-\rho\lambda_0-\frac{k}{m}\right)i\right\}\\ &= \frac{1}{m^2}\sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1}\exp\{2\pi(j_1-j_2)(\rho t-\rho\lambda_0)i\}\sum_{k=0}^{m-1}\exp\left\{2\pi\frac{(j_2-j_1)k}{m}i\right\}\\ &= \frac{1}{m^2}\sum_{j_1=0}^{m-1} m = 1\end{aligned}$$

for all t. Hence, by $\Gamma_k \subseteq \mathbb{Z}$, we have

$$\begin{aligned}
1 &= \sum_{\lambda\in\Lambda} |\hat{\mu}(t-\lambda)||^2 = \sum_{\lambda\in\Lambda} |\hat{\mu}_\rho(\rho t-\rho\lambda)||^2 \left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi(\rho t-\rho\lambda)i\})^j\right|^2 \\
&= \sum_{k\in B}\sum_{\gamma\in\Gamma_k}\left|\hat{\mu}_\rho\left(\rho t-\rho\lambda_0-\frac{k+mz_k}{m}-\gamma\right)\right|\Bigg|^2 \left|\frac{1}{m}\sum_{j=0}^{m-1}\left(\exp\left\{2\pi\left(\rho t-\rho\lambda_0-\frac{k+mz_k}{m}-\gamma\right)i\right\}\right)^j\right|^2 \\
&= \sum_{k\in B}\sum_{\gamma\in\Gamma_k}\left|\hat{\mu}_\rho\left(\rho t-\rho\lambda_0-\frac{k+mz_k}{m}-\gamma\right)\right|\Bigg|^2 \left|\frac{1}{m}\sum_{j=0}^{m-1}\left(\exp\left\{2\pi\left(\rho t-\rho\lambda_0-\frac{k}{m}\right)i\right\}\right)^j\right|^2 \\
&\le \sum_{k\in B}\left|\frac{1}{m}\sum_{j=0}^{m-1}\left(\exp\left\{2\pi\left(\rho t-\rho\lambda_0-\frac{k}{m}\right)i\right\}\right)^j\right|^2 \\
&\le \sum_{k=0}^{m-1}\left|\frac{1}{m}\sum_{j=0}^{m-1}\left(\exp\left\{2\pi\left(\rho t-\rho\lambda_0-\frac{k}{m}\right)i\right\}\right)^j\right|^2 = 1
\end{aligned}$$

for all t. Therefore, the above two inequalities are equalities. The first one means that Γ_j is a spectrum of $L^2(\mu_\rho)$ for each $j \in B$. The second one means that $B = \{0, 1, \dots, m-1\}$.

While the conclusion $0 \in \Gamma_k$ follows from the assumption $\lambda_0 + \frac{mz_k+k}{m|\rho|} \in \Lambda$. The theorem is proven. □

Repeated application of Theorem 4.4 shows, there exist spectrums $\Gamma_{k_1,\dots,k_n}$ containing zero such that

$$\Lambda = \bigcup_{k_1=0}^{m-1}\bigcup_{k_2=0}^{m-1}\cdots\bigcup_{k_n=0}^{m-1}\left(\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} + |\rho|^{-n}\Gamma_{k_1,\dots,k_n}\right) \tag{4.9}$$

and the union is a disjoint union, where integers $z_{k_1,\dots,k_j}$ $(j = 1, 2, \dots, n)$ are chosen to satisfy $\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_j+mz_{k_1,\dots,k_j}}{m|\rho|^j} \in \Lambda$ for $j = 1, 2, \dots, n$.

First, by (4.9) and Lemma 4.4, we have $|\rho|^{-n}\Gamma_{k_1,\dots,k_n} \subseteq \Lambda - \Lambda \subseteq \frac{\mathbb{Z}}{m|\rho|}$. Since $|\rho| = q/p$ and p, q are co-prime, we see $\Gamma_{k_1,\dots,k_n} \subseteq q^n\frac{\mathbb{Z}}{m|\rho|}$ and

$$\frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} \in |\rho|^{1-n}\Gamma_{k_1,\dots,k_{n-1}} \subseteq \frac{p^{n-1}\mathbb{Z}}{m|\rho|}. \tag{4.10}$$

Second, as we have assumed (4.7) for the case $n = 1$, repeated application of Theorem 4.4 shows that we can choose $z_{k_1,\dots,k_j}$ so that $\frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}$ has the minimal absolute value in the set $\frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} + |\rho|^{-n}\Gamma_{k_1,\dots,k_n}$. This implies that $z_{k_1,\dots,k_j} = 0$ whenever $k_j = 0$.

For any given sequence $\{k_j\}_j$ of $\{0, 1, \dots, m-1\}$, denote $a_n = \lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}$, then $|a_n| \ge |a_{n-1}|$.

Claim. *There exists a constant $c \in \mathbb{N}$ so that $|a_n| \ge p^{n-c}$ whenever $k_n \ne 0$.*

Proof of the Claim. If a_n, a_{n-1} have the same sign, then $|a_n| = |a_{n-1}| + |\frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}|$ by using $|a_n| \ge |a_{n-1}|$. Using (4.10) implies $a_n - a_{n-1} = \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} \in \frac{p^{n-1}\mathbb{Z}}{m|\rho|}$ is non-zero, so $|a_n| \ge \frac{p^{n-1}}{m|\rho|}$. Otherwise, a_n, a_{n-1} have different signs. Since $|a_n| \ge |a_{n-1}|$, so $|a_n| \ge \frac{1}{2}|\frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}|$. Using (4.10) implies $\frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} \in \frac{p^{n-1}\mathbb{Z}}{m|\rho|}$ is non-zero, so $|a_n| \ge \frac{p^{n-1}}{2m|\rho|}$. The claim is proven.

For any $\lambda \in \Lambda\setminus\{\lambda_0\}$, we can find an integer n_0 so that $|\lambda| < p^{n_0-c}$, hence the above claim shows that λ does not belong to the set $\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} + |\rho|^{-n}\Gamma_{k_1,\dots,k_n}$ for any $n \ge n_0$ with $k_n \ne 0$. On the other hand, however, (4.9) shows that $\lambda \in \lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n} + |\rho|^{-n}\Gamma_{k_1,\dots,k_n}$ for some n. Hence λ can be written as the form $\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\dots,k_n}}{m|\rho|^n}$ for some n.

Therefore, we have proven the following theorem.

Theorem 4.6. *If μ_ρ is a spectral measure with spectrum Λ and $|\rho| = q/p$ with $\gcd(q, m) = 1$, then $m|p$ and Λ has the form*

$$\Lambda = \left\{\lambda_0 + \frac{k_1 + mz_{k_1}}{m|\rho|} + \frac{k_2 + mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n + mz_{k_1,\ldots,k_n}}{m|\rho|^n} : k_j = 0, 1, \ldots, m-1; n > 0\right\} \tag{4.11}$$

such that $|\lambda_0 + \frac{k_1+mz_{k_1}}{m|\rho|} + \frac{k_2+mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n+mz_{k_1,\ldots,k_n}}{m|\rho|^n}| \geq p^{n-c}$ if $k_n \neq 0$ and this sequence of absolute values increases as n increases for any given sequence $\{k_j\}_j$ of $\{0, 1, \ldots, m-1\}$.

Proof of Theorem 1.1. Assume μ_ρ is a spectral measure with spectrum Λ. Since m is a prime, Theorem 1.2 implies that $|\rho| = (q/p)^{\frac{1}{r}}$ with $\gcd(p, q) = 1$ and $m|p$. Proposition 4.3 shows that $r = 1$. $m|p$ and $\gcd(p, q) = 1$ imply $\gcd(m, q) = 1$. Hence Theorems 4.5 and 4.6 applicable. Hence Λ has the form as (4.11).

Therefore, we need only prove $q = 1$. Assume, on the contrary, that $q > 1$.
Use Theorem 4.6, we define

$$\Lambda_n = \left\{\lambda_0 + \frac{k_1 + mz_{k_1}}{m|\rho|} + \frac{k_2 + mz_{k_1,k_2}}{m|\rho|^2} + \cdots + \frac{k_n + mz_{k_1,\ldots,k_n}}{m|\rho|^n} : k_j = 0, 1, \ldots, m-1\right\} \tag{4.12}$$

for $n = 1, 2, \ldots$. Then Theorem 4.6 implies

$$\lambda \in \Lambda \setminus \Lambda_n \Longrightarrow |\lambda| \geq p^{n+1-c}. \tag{4.13}$$

Using Theorems 4.5 and 4.6, we see that $\frac{k_n+mz_{k_1,\ldots,k_n}}{m|\rho|^n} \in |\rho|^{-1}\mathbb{Z}$ for $n > 1$. Hence $q^{n-1}|(k_n + mz_{k_1,\ldots,k_n})$. Since we have proven $|\rho| = q/p$ and $m|p$, so

$$\rho^\ell \frac{k_n + mz_{k_1,\ldots,k_n}}{m|\rho|^n} \in \mathbb{Z}, \quad \forall 1 \leq \ell < n. \tag{4.14}$$

Using (4.8), we have

$$\begin{aligned}
\sum_{k_n=0}^{m-1}\left|\left[\frac{1}{m}\sum_{j=0}^{m-1}\left(\exp\left\{2\pi j\left(t - \frac{k_n}{m}\right)i\right\}\right)\right]\right|^2 &= \frac{1}{m^2}\sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1}\sum_{k_n=0}^{m-1}\exp\left\{2\pi(j_1 - j_2)\left(t - \frac{k_n}{m}\right)i\right\} \\
&= \frac{1}{m^2}\sum_{j_1=0}^{m-1}\sum_{j_2=0}^{m-1}\exp\{2\pi(j_1 - j_2)ti\}\sum_{k_n=0}^{m-1}\exp\left\{2\pi\frac{(j_2 - j_1)k_n}{m}i\right\} \\
&= \frac{1}{m^2}\sum_{j_1=0}^{m-1} m = 1
\end{aligned}$$

for all t. Hence, by (2.1), Theorem 4.6 and (4.14), we have

$$\begin{aligned}
&\sum_{\lambda\in\Lambda_n}\prod_{\ell=1}^{n}\left|\left[\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^\ell j(t-\lambda)i\})\right]\right|^2 \\
&= \sum_{\gamma\in\Lambda_{n-1}}\sum_{k_n=0}^{m-1}\prod_{\ell=1}^{n}\left|\frac{1}{m}\sum_{j=0}^{m-1}\exp\left\{2\pi\rho^\ell j\left(t - \gamma - \frac{k_n + mz_{k_1,\ldots,k_n}}{m|\rho|^n}\right)i\right\}\right|^2 \\
&= \sum_{\gamma\in\Lambda_{n-1}}\sum_{k_n=0}^{m-1}\prod_{\ell=1}^{n-1}\left|\frac{1}{m}\sum_{j=0}^{m-1}\exp\{2\pi\rho^\ell j(t-\gamma)\}\right|^2 \cdot \left|\frac{1}{m}\sum_{j=0}^{m-1}\exp\left\{2\pi j\left(\rho^n(t-\gamma) - \frac{k_n + mz_{k_1,\ldots,k_n}}{m}\right)i\right\}\right|^2 \\
&= \sum_{\gamma\in\Lambda_{n-1}}\prod_{\ell=1}^{n-1}\left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^\ell j(t-\gamma)i\})\right|^2 \cdot \left|\frac{1}{m}\sum_{j=0}^{m-1}\exp\left\{2\pi j\left(\rho^n(t-\gamma) - \frac{k_n}{m}\right)i\right\}\right|^2 \\
&= \sum_{\gamma\in\Lambda_{n-1}}\prod_{\ell=1}^{n-1}\left|\frac{1}{m}\sum_{j=0}^{m-1}(\exp\{2\pi\rho^\ell j(t-\gamma)i\})\right|^2
\end{aligned}$$

for all t and $n > 1$.

Repeatedly using the above equality gives

$$\sum_{\lambda\in\Lambda_n}\prod_{\ell=1}^{n}\left|\frac{1}{m}\sum_{j=0}^{m-1}\exp\{2\pi\rho^\ell j(t-\lambda)i\}\right|^2 = \sum_{k_1=0}^{m-1}\left|\frac{1}{m}\sum_{j=0}^{m-1}\exp\left\{2\pi\rho j\left(t - \lambda_0 - \frac{k_1 + mz_{k_1}}{m|\rho|}\right)i\right\}\right|^2 = 1. \tag{4.15}$$

Let N be an integer with $N > a^{-1}$, where a is defined in Theorem 4.1. Let

$$Q_\ell(t) = \sum_{\lambda \in \Lambda_{\ell^N}} |\hat{\mu}_\rho(t-\lambda)|^2, \quad \ell \in \mathbb{N}.$$

Then

$$\begin{aligned}
Q_{\ell+1}(t) - Q_\ell(t) &= \sum_{\lambda \in \Lambda_{(\ell+1)^N} \setminus \Lambda_{\ell^N}} |\hat{\mu}_\rho(t-\lambda)|^2 \\
&= \sum_{\lambda \in \Lambda_{(\ell+1)^N} \setminus \Lambda_{\ell^N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi \rho^s j(t-\lambda) i\}) \right] \right|^2 \cdot |\hat{\mu}_\rho(\rho^{(\ell+1)^N}(t-\lambda))|^2 \\
&\le C_0^2 \sum_{\lambda \in \Lambda_{(\ell+1)^N} \setminus \Lambda_{\ell^N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi \rho^s j(t-\lambda) i\}) \right] \right|^2 \cdot (\ln(2 + |\rho^{(\ell+1)^N}(t-\lambda)|))^{-2a} \\
&\le C_0^2 \sum_{\lambda \in \Lambda_{(\ell+1)^N} \setminus \Lambda_{\ell^N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi \rho^s j(t-\lambda) i\}) \right] \right|^2 \cdot (\ln(2 + |\rho^{(\ell+1)^N} p^{\ell^N - c - 1}|))^{-2a} \\
&= C_0^2 \left[1 - \sum_{\lambda \in \Lambda_{\ell^N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi \rho^s j(t-\lambda) i\}) \right] \right|^2 \right] (\ln(2 + |\rho^{(\ell+1)^N} p^{\ell^N - c - 1}|))^{-2a}
\end{aligned}$$

for any $t \in (-p^{-c-1}, p^{-c-1})$, where the first equality follows from (2.1), the first inequality follows from Theorem 4.1, the second inequality follows from (4.13), the last equality follows from (4.15).

By (2.1), we have

$$\begin{aligned}
Q_\ell(t) &= \sum_{\lambda \in \Lambda_{\ell^N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi \rho^s j(t-\lambda) i\}) \right] \right|^2 \cdot |\hat{\mu}_\rho(\rho^{(\ell+1)^N}(t-\lambda))|^2 \\
&\le \sum_{\lambda \in \Lambda_{\ell^N}} \prod_{s=1}^{(\ell+1)^N} \left| \left[\frac{1}{m} \sum_{j=0}^{m-1} (\exp\{2\pi \rho^s j(t-\lambda) i\}) \right] \right|^2 .
\end{aligned}$$

Therefore,

$$1 - Q_{\ell+1}(t) \ge [1 - Q_\ell(t)][1 - C_0^2 (\ln(2 + |\rho^{(\ell+1)^N} p^{\ell^N - c - 1}|))^{-2a}],$$

and so

$$1 - Q_{\ell+1}(t) \ge [1 - Q_n(t)] \prod_{k=n}^{\ell} [1 - C_0^2 (\ln(2 + |\rho^{(k+1)^N} p^{k^N - c - 1}|))^{-2a}], \quad \forall \ell > n. \tag{4.16}$$

The assumption $N > \frac{1}{a}$ shows that $\sum_{k=1}^{\infty} k^{-2Na} < +\infty$. Hence $\prod_{k=1}^{\ell} [1 - k^{-2Na}]$ converges to a positive number. Since

$$\lim_{\ell \to +\infty} \frac{(\ln(2 + |\rho^{(\ell+1)^N} p^{\ell^N - c - 1}|))^{-2a}}{\ell^{-2Na}} = \lim_{\ell \to +\infty} \left(\frac{(\ell+1)^N \ln|\rho| + \ell^N \ln p}{\ell^N} \right)^{-2a} = (\ln q)^{-2a} > 0.$$

Hence, we can find $\ell_0 > 0$ so that

$$\prod_{k=\ell_0}^{+\infty} [1 - C_0^2 (\ln(2 + |\rho^{(k+1)^N} p^{k^N - c - 1}|))^{-2a}] = a_0 \in (0, 1).$$

Therefore, by noting that Λ is a spectrum of μ_ρ, (4.16) shows

$$0 = 1 - \sum_{\lambda \in \Lambda} |\hat{\mu}_\rho(t-\lambda)|^2 = \lim_{\ell \to +\infty} [1 - Q_{\ell+1}(t)] \ge a_0 [1 - Q_{\ell_0}(t)] \ge 0$$

for any $t \in (-p^{-c-1}, p^{-c-1})$. Hence $Q_{\ell_0}(t) = 1$ for all $t \in (-p^{-c-1}, p^{-c-1})$. Since $Q_{\ell_0}(t)$ can be extended to an analytic function on the complex plane, so $Q_{\ell_0}(t) = 1$ for all $t \in \mathbb{R}$. Hence $\Lambda_{\ell_0^N}$ is a spectrum. It is obviously impossible, as $\Lambda_{\ell_0^N}$ is a finite set.

Therefore, $q = 1$, so Theorem 1.1 is proven.

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