J. Math. Anal. Appl. 409 (2014) 1159–1163

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

A double Mertens type evaluation

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ARTICLE INFO

Article history:
Received 14 December 2012
Available online 31 July 2013
Submitted by B.C. Berndt

Keywords:
Arithmetic functions
Asymptotic results
Mertens type evaluations

ABSTRACT

We prove the formula

$$\sum_{pq\le x}\frac{1}{pq}=(\ln(\ln x)+B)^2-\ln^2 2+2\int_{0+0}^{\frac{1}{2}}\frac{\ln(1-x)}{x}dx+O\left(\frac{\ln(\ln x)}{\ln x}\right),$$

where B is the Mertens constant.

1. Introduction and notation

The starting point for this paper was the following problem from the book of G. Tenenbaum, (see [9, Problem 12, p. 22]): Evaluate $\sum_{pq\le x}\frac{1}{pq}$, where p and q denote the primes. The evaluation of this sum was stated in the Abstract and will be proved in Theorem 1. As far as we know, in the literature there exist two less precise formulas:

$$\sum_{pq\le x}\frac{1}{pq}=(\log\log x)^2+2B\log\log x+O(1)$$

and

$$\sum_{pq\le x}\frac{1}{pq}\backsim(\log\log x)^2,$$

which appear in [7, p. 23; solution on pp. 60–62] and [6, p. 315].

Let us fix some notations and notions. By e we denote the Euler number and $\ln x=\log_e x$, $x>0$. Let $a\in\mathbb{R}\cup\{-\infty\}$, $g:(a,\infty)\to\mathbb{R}$ be such that there exists $b\ge a$ with $g(x)\ne 0$ for $x\in(b,\infty)$. If $f:(a,\infty)\to\mathbb{R}$ is a function, we write $f(x)\sim g(x)$ if and only if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$.

Throughout this paper, we use the notation $\sum_{n\le x}$ to mean $\sum_{n\le x;n\in\mathbb{N}}$, the notation $\sum_{p\le x}$ to mean $\sum_{p\le x;p\text{ prime}}$ and the notation $\sum_{pq\le x}$ to mean $\sum_{pq\le x;p,q\text{ prime}}$. We recall the following (see [1,3,4,9] and the more recent [5]).

Mertens' second Theorem. There exists a constant B, called the Mertens constant, such that

$$\sum_{p\le x}\frac{1}{p}=\ln(\ln x)+B+O\left(\frac{1}{\ln x}\right).$$

All notations and notions used and not defined in this paper are standard, (see [1–4,9]).

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http://dx.doi.org/10.1016/j.jmaa.2013.07.058

2. The results

We recall, (see [9, pp. 14–15])
Mertens' first Theorem. We have

$$-1-\ln 4<\sum_{p\le x}\frac{\ln p}{p}-\ln x<\ln 4 \quad \text{for } x\ge 1.$$

The following result is a precision of a well-known result, (see [8, Lemma 7, p. 491] or [6, p. 281, Exercise 4]).

Proposition 1. *Let $a, b\in\mathbb{R}$, $a<b$ be such that*

$$a\le \sum_{p\le x}\frac{\ln p}{p}-\ln x\le b \quad \textit{for } x\ge 1.$$

Then:

(i) *There exists $V_1:[1,\infty)\to\mathbb{R}$ such that*

$$\sum_{p\le\sqrt{x}}\frac{\ln p}{p}=\frac{\ln x}{2}+V_1(x) \quad \textit{for } x\ge 1$$

and

$$a\le V_1(x)\le b \quad \textit{for } x\ge 1.$$

(ii) *For $k\ge 2$, there exists $T_k:[1,\infty)\to\mathbb{R}$ such that*

$$\sum_{p\le x}\frac{(\ln p)^k}{p}=\frac{1}{k}(\ln x)^k+T_k(x) \quad \textit{for } x\ge 1$$

and

$$|T_k(x)|\le (b-a)(\ln x)^{k-1} \quad \textit{for } x\ge 1.$$

In particular, for $k\ge 2$ there exists $V_k:[1,\infty)\to\mathbb{R}$ such that

$$\sum_{p\le\sqrt{x}}\frac{(\ln p)^k}{p}=\frac{1}{k\cdot 2^k}(\ln x)^k+V_k(x) \quad \textit{for } x\ge 1$$

and

$$|V_k(x)|\le (b-a)\frac{1}{2^{k-1}}(\ln x)^{k-1} \quad \textit{for } x\ge 1.$$

Proof. (i) Let $R:[1,\infty)\to\mathbb{R}$ be defined by $R(x)=\sum_{p\le x}\frac{\ln p}{p}-\ln x$ and note that, by hypothesis, $\sum_{p\le x}\frac{\ln p}{p}=\ln x+R(x)$ and $a\le R(x)\le b$ for $x\ge 1$. Replacing x with $\sqrt{x}$ we get (i) with $V_1(x)=R\left(\sqrt{x}\right)$.

(ii) Define $G:(0,\infty)\to[0,\infty)$ by $G(x)=\sum_{p\le x}\frac{\ln p}{p}$. Let $x\ge 1$. From the Abel summation formula see [1, Theorem 4.2] or [3, Theorem A], we have

$$\begin{aligned}\sum_{p\le x}\frac{(\ln p)^k}{p}&=\sum_{p\le x}\frac{\ln p}{p}\cdot(\ln p)^{k-1}\\&=(\ln x)^{k-1}G(x)-(k-1)\int_1^x\frac{(\ln t)^{k-2}}{t}G(t)dt.\end{aligned}$$

With the same notation as in (i), we have

$$\begin{aligned}\sum_{p\le x}\frac{(\ln p)^k}{p}&=(\ln x)^k+(\ln x)^{k-1}R(x)-(k-1)\int_1^x\frac{(\ln t)^{k-2}}{t}(\ln t+R(t))\,dt\\&=\frac{1}{k}(\ln x)^k+T_k(x),\end{aligned}$$

where $T_k(x)=(\ln x)^{k-1}R(x)-(k-1)\int_1^x\frac{(\ln t)^{k-2}}{t}R(t)dt$.

From $a\le R(t)\le b$ for $t\ge 1$ we deduce

$$\frac{a(\ln t)^{k-2}}{t}\le\frac{(\ln t)^{k-2}}{t}R(t)\le\frac{b(\ln t)^{k-2}}{t} \quad \text{for } t\ge 1$$

and by integration

$$a\,(\ln x)^{k-1} \le (k-1)\int_1^x \frac{(\ln t)^{k-2}}{t}R(t)dt \le b\,(\ln x)^{k-1} \quad \text{for } x \ge 1.$$

We obtain $(a-b)\,(\ln x)^{k-1} \le T_k(x) \le (b-a)\,(\ln x)^{k-1}$ for $x \ge 1$.

For the second part in (ii) we replace x with $\sqrt{x}$ and take $V_k(x) = T_k\left(\sqrt{x}\right)$. □

Proposition 2. *Let $a, b \in \mathbb{R}$, $a < b$ be such that*

$$a \le \sum_{p \le x} \frac{\ln p}{p} - \ln x \le b \quad \textit{for } x \ge 1.$$

Then there exists $h : (1, \infty) \to \mathbb{R}$ such that

$$\sum_{p \le \sqrt{x}} \frac{1}{p} \ln\left(1 - \frac{\ln p}{\ln x}\right) = \int_{0+0}^{\frac{1}{2}} \frac{\ln(1-x)}{x}dx + h(x) \quad \textit{for } x > 1$$

and

$$\frac{2\,(b-a)\ln 2 + a}{\ln x} \le h(x) \le \frac{2\,(b-a)\ln 2 + b}{\ln x} \quad \textit{for } x > 1.$$

Proof. We use the well-known formula: $\ln(1-u) = -\sum_{k=1}^{\infty} \frac{u^k}{k}$ for $-1 < u < 1$. Let $x > 1$. From the equality $\ln\left(1 - \frac{\ln p}{\ln x}\right) = -\sum_{k=1}^{\infty} \frac{(\ln p)^k}{k(\ln x)^k}$ for $p \le \sqrt{x}$ we get

$$\sum_{p \le \sqrt{x}} \frac{1}{p} \ln\left(1 - \frac{\ln p}{\ln x}\right) = -\sum_{p \le \sqrt{x}} \frac{1}{p} \sum_{k=1}^{\infty} \frac{(\ln p)^k}{k\,(\ln x)^k} = -\sum_{k=1}^{\infty} \frac{1}{k} \sum_{p \le \sqrt{x}} \frac{(\ln p)^k}{p\,(\ln x)^k}.$$

By Proposition 1 we deduce

$$\begin{aligned}\sum_{p \le \sqrt{x}} \frac{1}{p} \ln\left(1 - \frac{\ln p}{\ln x}\right) &= -\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{k \cdot 2^k} + \frac{1}{(\ln x)^k}V_k(x)\right) \\ &= -\sum_{k=1}^{\infty} \frac{1}{k^2 \cdot 2^k} + h(x)\end{aligned}$$

where $h(x) = -\sum_{k=1}^{\infty} \frac{1}{k(\ln x)^k}V_k(x)$. By Proposition 1, we have

$$\begin{aligned}\left|\sum_{k=2}^{\infty} \frac{1}{k\,(\ln x)^k}V_k(x)\right| &\le \sum_{k=2}^{\infty} \frac{1}{k\,(\ln x)^k}\,|V_k(x)| \\ &\le \frac{b-a}{\ln x}\sum_{k=2}^{\infty} \frac{1}{k \cdot 2^{k-1}} \\ &= \frac{2\,(b-a)}{\ln x}\left(\ln 2 - \frac{1}{2}\right)\end{aligned}$$

and thus $\left|h(x) + \frac{V_1(x)}{\ln x}\right| \le \frac{2(b-a)}{\ln x}\left(\ln 2 - \frac{1}{2}\right)$. We obtain

$$-\frac{V_1(x)}{\ln x} - \frac{2\,(b-a)}{\ln x}\left(\ln 2 - \frac{1}{2}\right) \le h(x) \le -\frac{V_1\,(x)}{\ln x} + \frac{2\,(b-a)}{\ln x}\left(\ln 2 - \frac{1}{2}\right)$$

or

$$-\frac{b}{\ln x} - \frac{2\,(b-a)}{\ln x}\left(\ln 2 - \frac{1}{2}\right) \le h(x) \le -\frac{a}{\ln x} + \frac{2\,(b-a)}{\ln x}\left(\ln 2 - \frac{1}{2}\right)$$

which after some simple calculations gives the second evaluation from the statement.

Let us define $A : \left[0, \frac{1}{2}\right] \to \mathbb{R}$ by $A(u) = \begin{cases} \frac{\ln(1-u)}{u}, & u \ne 0 \\ -1, & u = 0 \end{cases}$ and note the equality $A(u) = -\sum_{k=1}^{\infty} \frac{u^{k-1}}{k}$ on $\left[0, \frac{1}{2}\right]$. Since the series is uniformly convergent on $\left[0, \frac{1}{2}\right]$, by integration we get the equality $\int_{0+0}^{\frac{1}{2}} \frac{\ln(1-x)}{x}dx = -\sum_{k=1}^{\infty} \frac{1}{k^2 \cdot 2^k}$. □

From Mertens' first theorem and Proposition 2 we deduce the following result which will play a key role in the proof of Theorem 1.

Corollary 1. *There exists* $h : (1, \infty) \to \mathbb{R}$ *such that*

$$\sum_{p \le \sqrt{x}} \frac{1}{p} \ln\left(1 - \frac{\ln p}{\ln x}\right) = \int_{0+0}^{\frac{1}{2}} \frac{\ln(1-x)}{x} dx + h(x) \quad \text{for } x > 1$$

and

$$\frac{2(2\ln 4 + 1)\ln 2 - 1 - \ln 4}{\ln x} \le h(x) \le \frac{2(2\ln 4 + 1)\ln 2 + \ln 4}{\ln x} \quad \text{for } x > 1.$$

We will need the following well-known result, which is the hyperbola method for the primes. We give only a sketch of the proof.

Proposition 3. *Let* $f, g : \mathbb{N} \to \mathbb{R}$ *be two functions and define* $F(x) = \sum_{p \le x} f(p)$ *and* $G(x) = \sum_{p \le x} g(p)$*. Then*

$$\sum_{pq \le x} f(p)g(q) = \sum_{p \le \sqrt{x}} f(p) G\left(\frac{x}{p}\right) + \sum_{p \le \sqrt{x}} g(p) F\left(\frac{x}{p}\right) - F\left(\sqrt{x}\right) G\left(\sqrt{x}\right).$$

Proof. Denote by $\mathbb{P}$ the set of all the primes and by $\chi_{\mathbb{P}} : \mathbb{N} \to \mathbb{R}$ the characteristic function of the set $\mathbb{P}$ i.e., $\chi_{\mathbb{P}}(n) = \begin{cases} 1 \text{ if } n \text{ is prime} \\ 0 \text{ if } n \text{ is not prime} \end{cases}$.

The statement follows from the well-known hyperbola method of Dirichlet applied to $f\chi_{\mathbb{P}}$ and $g\chi_{\mathbb{P}}$, (see [1,3,4,9]). □

Now, we are in the position to prove the main result of this paper.

Theorem 1. *We have*

$$\sum_{pq \le x} \frac{1}{pq} = (\ln(\ln x) + B)^2 - \ln^2 2 + 2\int_{0+0}^{\frac{1}{2}} \frac{\ln(1-x)}{x} dx + O\left(\frac{\ln(\ln x)}{\ln x}\right),$$

where B is the Mertens constant.

Proof. From Proposition 3 we have

$$\sum_{pq \le x} \frac{1}{pq} = 2\sum_{p \le \sqrt{x}} \frac{1}{p} F\left(\frac{x}{p}\right) - \left[F\left(\sqrt{x}\right)\right]^2$$

where $F(x) = \sum_{p \le x} \frac{1}{p}$. This equality appears in [7, p. 23; solution on pp. 59–60]. By Mertens' second theorem,

$$F(x) = \ln(\ln x) + B + C(x) \quad \text{and} \quad C(x) = O\left(\frac{1}{\ln x}\right).$$

Then

$$\begin{aligned} \left[F\left(\sqrt{x}\right)\right]^2 &= \left[\ln\left(\ln\sqrt{x}\right) + B + O\left(\frac{1}{\ln x}\right)\right]^2 \\ &= \left(\ln\left(\ln\sqrt{x}\right) + B\right)^2 + 2\left(\ln\left(\ln\sqrt{x}\right) + B\right) O\left(\frac{1}{\ln x}\right) + O\left(\frac{1}{\ln^2 x}\right) \\ &= \left(\ln\left(\ln\sqrt{x}\right) + B\right)^2 + O\left(\frac{\ln(\ln x)}{\ln x}\right). \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{p \le \sqrt{x}} \frac{1}{p} F\left(\frac{x}{p}\right) &= \sum_{p \le \sqrt{x}} \frac{1}{p}\left(\ln\left(\ln\frac{x}{p}\right) + B + C\left(\frac{x}{p}\right)\right) \\ &= \sum_{p \le \sqrt{x}} \frac{1}{p} \ln\left(\ln\frac{x}{p}\right) + BF\left(\sqrt{x}\right) + \sum_{p \le \sqrt{x}} \frac{1}{p} C\left(\frac{x}{p}\right). \end{aligned}$$

Since $\frac{1}{\ln x-\ln p} \leq \frac{2}{\ln x}$ for $p \leq \sqrt{x}$, again by Mertens' second theorem, we have

$$\left|\sum_{p\leq\sqrt{x}} \frac{1}{p} C\left(\frac{x}{p}\right)\right| \leq \sum_{p\leq\sqrt{x}} \frac{1}{p}\left|C\left(\frac{x}{p}\right)\right| \leq \sum_{p\leq\sqrt{x}} \frac{1}{p}\cdot\frac{1}{\ln\frac{x}{p}} = \sum_{p\leq\sqrt{x}} \frac{1}{p}\cdot\frac{1}{\ln x-\ln p}$$
$$\leq \frac{2}{\ln x}\sum_{p\leq\sqrt{x}}\frac{1}{p} = O\left(\frac{\ln(\ln x)}{\ln x}\right).$$

We obtain

$$\sum_{p\leq\sqrt{x}} \frac{1}{p} F\left(\frac{x}{p}\right) = \sum_{p\leq\sqrt{x}} \frac{1}{p}\ln\left(\ln\frac{x}{p}\right) + B\left(\ln\left(\ln\sqrt{x}\right) + B + O\left(\frac{1}{\ln x}\right)\right) + O\left(\frac{\ln(\ln x)}{\ln x}\right)$$
$$= \sum_{p\leq\sqrt{x}} \frac{1}{p}\ln\left(\ln\frac{x}{p}\right) + B\ln\left(\ln\sqrt{x}\right) + B^2 + O\left(\frac{\ln(\ln x)}{\ln x}\right).$$

Thus

$$\sum_{pq\leq x}\frac{1}{pq} = 2\sum_{p\leq\sqrt{x}} \frac{1}{p}\ln\left(\ln\frac{x}{p}\right) + 2B\ln\left(\ln\sqrt{x}\right) + 2B^2 - \left(\ln\left(\ln\sqrt{x}\right)+B\right)^2 + O\left(\frac{\ln(\ln x)}{\ln x}\right)$$
$$= 2\sum_{p\leq\sqrt{x}} \frac{1}{p}\ln\left(\ln\frac{x}{p}\right) - \left(\ln\left(\ln\sqrt{x}\right)\right)^2 + B^2 + O\left(\frac{\ln(\ln x)}{\ln x}\right).$$

Now, by Mertens' second theorem and Corollary 1, we have

$$\sum_{p\leq\sqrt{x}} \frac{1}{p}\ln\left(\ln\frac{x}{p}\right) = \sum_{p\leq\sqrt{x}} \frac{1}{p}\ln(\ln x-\ln p)$$
$$= \ln(\ln x)\sum_{p\leq\sqrt{x}}\frac{1}{p} + \sum_{p\leq\sqrt{x}}\frac{1}{p}\ln\left(1-\frac{\ln p}{\ln x}\right)$$
$$= [\ln(\ln x)]\left(\ln\left(\ln\sqrt{x}\right)+B+O\left(\frac{1}{\ln x}\right)\right) + \int_{0+0}^{\frac{1}{2}}\frac{\ln(1-x)}{x}dx + O\left(\frac{1}{\ln x}\right)$$
$$= [\ln(\ln x)]\left(\ln\left(\ln\sqrt{x}\right)+B\right) + \int_{0+0}^{\frac{1}{2}}\frac{\ln(1-x)}{x}dx + O\left(\frac{\ln(\ln x)}{\ln x}\right).$$

Then

$$\sum_{pq\leq x}\frac{1}{pq} = 2[\ln(\ln x)]\left(\ln\left(\ln\sqrt{x}\right)+B\right) + 2\int_{0+0}^{\frac{1}{2}}\frac{\ln(1-x)}{x}dx - \ln\left(\ln\sqrt{x}\right)^2 + B^2 + O\left(\frac{\ln(\ln x)}{\ln x}\right)$$
$$= (\ln(\ln x)+B)^2 - \ln^2 2 + 2\int_{0+0}^{\frac{1}{2}}\frac{\ln(1-x)}{x}dx + O\left(\frac{\ln(\ln x)}{\ln x}\right),$$

since

$$2\ln(\ln x)\left(\ln\left(\ln\sqrt{x}\right)\right) - \left(\ln\left(\ln\sqrt{x}\right)\right)^2 = 2\ln(\ln x)\left(\ln(\ln x)-\ln 2\right) - \left(\ln(\ln x)-\ln 2\right)^2$$
$$= [\ln(\ln x)]^2 - \ln^2 2. \quad \square$$

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