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Asymptotic inequalities for k-ranks and their cumulation functions

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ABSTRACT

Asymptotic formulas for the positive moments of rank and crank of partitions were obtained by K. Bringmann and K. Mahlburg recently. Motivated by their works, in this paper, we prove asymptotic formulas for the k-ranks and their cumulation functions. Asymptotic inequalities between these combinatorial objects are also discovered. In particular, we show that, for fixed integer l and sufficiently large N,

$$\mathcal{M}(l, N) \sim \mathcal{N}(l, N)$$

and

$$\mathcal{M}(l, N) < \mathcal{N}(l, N),$$

where $\mathcal{M}(l, N)$ (resp. $\mathcal{N}(l, N)$) denotes the number partitions of N with crank (resp. rank) l.

1. Introduction

The ranks and cranks of partitions provide combinatorial explanations for Ramanujan's famous congruences. Recently, A. O. L. Atkin and F. G. Garvan [4] studied the even moments of rank and crank which were modified and generalized by Andrews, Chan and Kim in [2]. Let j be a positive integer. Then the (modified) j-th moments of the rank and crank are defined by, respectively,

$$\overline{\mathcal{N}}_j(N) = \sum_{r=1}^{\infty} r^j \mathcal{N}(r, N),$$

$$\overline{\mathcal{M}}_j(N) = \sum_{r=1}^{\infty} r^j \mathcal{M}(r, N),$$

where $\mathcal{M}(r, N)$ (resp. $\mathcal{N}(r, N)$) denotes the number partitions of N with crank (resp. rank) r. In [18], Garvan proved that the even moments of crank were always larger than the ranks which were first conjectured also by Garvan [17]. Andrews, Chan and Kim [2] established these inequalities for all positive moments. K. Bringmann, K. Mahlburg and R. C. Rhoades [10] proved asymptotic formulas for the even moments and established these inequalities asymptotically. In [11], also by those authors, a strengthening asymptotic result for the even moments was obtained. Asymptotic results for the positive moments were first proved by Bringmann and Mahlburg in [9].

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On the other hand, in [16], Garvan studied a partition statistic called k-rank which generalized the rank and crank. For an integer l, he defined $\mathcal{N}_k(l,N)$ by

$$\sum_{N\geq 0}\mathcal{N}_k(l,N)q^N=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2+|l|n}(1-q^n), \tag{1.1}$$

where for $n\in\mathbb{N}_0\bigcup\{\infty\}$ we adopt the standard q-factorial notation $(a)_n=(a;q)_n:=\prod_{j=0}^{n-1}(1-aq^j)$. When $k=1$ (resp. $k=2$), this is the generating function for the crank (resp. rank). For $k\geq 3$, the interpretation of $\mathcal{N}_k(l,N)$ was also discussed in [16]. The even k-rank moments and the inequalities between them were also studied by A. Dixit and A. J. Yee in [13]. Moreover, with a similar method by Bringmann, Mahlburg and Rhoades in [11], M. Waldherr [21] obtained asymptotic formulas for the even k-rank moments. In this article, with a similar method used by Bringmann and Mahlburg in [9], we study the asymptotic properties of the k-ranks and their *cumulation functions* which will be defined later.

We begin with an interesting phenomenon discussed by K. Bringmann and K. Mahlburg in the concluding remarks of [8]. Following their notations, for a nonnegative integer l, we define the crank and rank *cumulation functions* by

$$\overline{\mathcal{M}}(l,N):=\sum_{r\leq -l}\mathcal{M}(r,N)=\sum_{r\geq l}\mathcal{M}(r,N)\quad\text{(by symmetry)}$$

and

$$\overline{\mathcal{N}}(l,N):=\sum_{r\leq -l}\mathcal{N}(r,N)=\sum_{r\geq l}\mathcal{N}(r,N)\quad\text{(by symmetry)}.$$

Then Bringmann and Mahlburg tested with MAPLE for $1\leq N\leq 100$ and found that, for $l>0$,

$$\overline{\mathcal{N}}(l,N)\leq\overline{\mathcal{M}}(l,N)\leq\overline{\mathcal{N}}(l-1,N).$$

By definition, we have $\mathcal{N}(l-1,N)=\overline{\mathcal{N}}(l-1,N)-\overline{\mathcal{N}}(l,N)$. Hence, the above inequalities give

$$0\leq\overline{\mathcal{M}}(l,N)-\overline{\mathcal{N}}(l,N)\leq\mathcal{N}(l-1,N), \tag{1.2}$$

for $l\geq 1$ and $1\leq N\leq 100$. Noting that $p(N)=\overline{\mathcal{M}}(0,N)+\overline{\mathcal{M}}(1,N)=\overline{\mathcal{N}}(0,N)+\overline{\mathcal{N}}(1,N)$, by (1.2), we find that

$$0\leq\overline{\mathcal{N}}(0,N)-\overline{\mathcal{M}}(0,N)\leq\mathcal{N}(0,N), \tag{1.3}$$

for $l\geq 1$ and $1\leq N\leq 100$.

In our paper, we obtained generalizations of asymptotic versions of (1.2) and (1.3) for k-ranks and their *cumulation functions*. Generalizing the definitions by Bringmann and Mahlburg, we define the k-rank *cumulation functions* by

$$\overline{\mathcal{N}}_k(l,N):=\sum_{r\geq l}\mathcal{N}_k(r,N).$$

By (1.1), when $l\geq 0$, we have the following generating function for $\overline{\mathcal{N}}_k(l,N)$.

$$\sum_{N\geq 0}\overline{\mathcal{N}}_k(l,N)q^N=\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2+ln}. \tag{1.4}$$

Remark. Equality (1.4) with $k=1$ (resp. $k=2$) can be deduced from [15, Theorem 7.19] (resp. [5, Eq. (2.12)]).

For convenience, let

$$S_{k,l}(q):=\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2+ln}$$

and

$$F_{k,l}(q):=\frac{1}{(q;q)_\infty}S_{k,l}(q).$$

Then we have $\sum_{N\geq 0}\overline{\mathcal{N}}_k(l,N)q^N=F_{k,l}(q)$. To generalize (1.2) and (1.3), we first estimate the magnitude of the *cumulation functions* $\overline{\mathcal{N}}_k(l,N)$ as follows.

Theorem 1.1. *Suppose that $k\in\mathbb{N}^+$ and $l\in\mathbb{N}$.*

(i) *As $N\to\infty$, we have*

$$\overline{\mathcal{N}}_k(l,N)\sim\frac{1}{8\sqrt{3}N}e^{2\pi\sqrt{\frac{N}{6}}}\sim\frac{1}{2}p(N), \tag{1.5}$$

where $p(N)$ is the number of partitions of N.

(ii) *As $N \to \infty$, we have*

$$\overline{\mathcal{N}}_k(l,N) - \overline{\mathcal{N}}_{k+1}(l,N) \sim \frac{(2l-1)\pi^2}{384\sqrt{3}N^2} e^{2\pi\sqrt{\frac{N}{6}}} \sim \frac{(2l-1)\pi^2}{96N} p(N) \tag{1.6}$$

The following corollary follows immediately from Theorem 1.1.

Corollary 1.2. *For fixed $k \in \mathbb{N}^+$ and sufficiently large N, we have*

$$\overline{\mathcal{N}}_k(0,N) < \overline{\mathcal{N}}_{k+1}(0,N)$$

and for $l \geq 1$,

$$\overline{\mathcal{N}}_k(l,N) > \overline{\mathcal{N}}_{k+1}(l,N).$$

In particular, for sufficiently large N, both of the first inequalities of (1.2) and (1.3) are true.

Special case of Corollary 1.2 with $k = 1$ can also be deduced from [3, Theorem 1.3]. Indeed, the unconditional inequality for all N (but "$>$" should be replaced by "$\geq$") is equivalent to [3, Conjecture 1.1] (see [3, Theorem 1.2]) which was proved by William Y.C. Chen, Kathy Q. Ji and Wenston J. T. Zang in [12].

The proof of Theorem 1.1 depends on the following representation of the main terms in the asymptotic expansion of $\overline{\mathcal{N}}_k(l,N)$ in terms of the modified Bessel functions.

Theorem 1.3. *As $N \to \infty$, we have*

$$\begin{aligned}\overline{\mathcal{N}}_k(l,N) = {} & \frac{\pi}{2\sqrt{2}} \times \left(\frac{1}{\sqrt{6}}\right)^{3/2} N^{-3/4} I_{-3/2}\left(\pi\sqrt{\frac{2N}{3}}\right) - \frac{\pi^2}{\sqrt{2}}\left(\frac{l}{4} - \frac{5}{48}\right) \times \left(\frac{1}{\sqrt{6}}\right)^{5/2} N^{-5/4} I_{-5/2}\left(\pi\sqrt{\frac{2N}{3}}\right) \\ & - \frac{\pi^3 \xi_{k,l}}{\sqrt{2}} \left(\frac{1}{\sqrt{6}}\right)^{7/2} N^{-7/4} I_{-7/2}\left(\pi\sqrt{\frac{2N}{3}}\right) + O\left(N^{-5/2} e^{2\pi\sqrt{\frac{N}{6}}}\right),\end{aligned} \tag{1.7}$$

where $\xi_{k,l} = \frac{(2l-1)(2k-1)}{32} - \frac{l}{96} + \frac{11}{2304}$ and $I_\nu(x)$ is the modified Bessel function.

Our proof of Theorem 1.3 is motivated by the work in [9] and depends on a variant of the Hardy–Ramanujan Circle Method due to E. Wright. We will discuss this in Section 4.

To find generalizations of asymptotic versions of the second inequalities in (1.2) and (1.3), we need to establish an asymptotic formula for $\mathcal{N}_k(l,N)$. Before stating our results, we recall an asymptotic formula for the cranks of partitions conjectured by F. J. Dyson [14, Eq. (1.24)] (we modify Dyson's notation so that the formula agrees with our previous definitions). As $N \to \infty$,

$$\mathcal{M}(l,N) \sim \frac{1}{4}\beta \operatorname{sech}^2\left(\frac{1}{2}\beta l\right) p(N), \tag{1.8}$$

where

$$\beta = \left(\frac{\pi^2}{6N}\right)^{1/2}.$$

Noting that $\operatorname{sech} x = 1 + O(x)$ (as $x \to 0$), Eq. (1.8) gives

$$\mathcal{M}(l,N) \sim \frac{\pi}{4\sqrt{6N}} p(N) \quad (\text{as } N \to \infty). \tag{1.9}$$

Although we have no idea to prove (1.8), a generalization of (1.9) is obtained with the aid of Theorem 1.3.

Theorem 1.4. *Suppose that $k \in \mathbb{N}^+$ and $l \in \mathbb{N}$.*

(i) *As $N \to \infty$, we have*

$$\mathcal{N}_k(l,N) \sim \frac{\pi}{48\sqrt{2}N^{3/2}} e^{2\pi\sqrt{\frac{N}{6}}} \sim \frac{\pi}{4\sqrt{6N}} p(N). \tag{1.10}$$

In particular, Eq. (1.9) is true.

(ii) *As $N \to \infty$, we have*

$$\mathcal{N}_k(l,N) - \mathcal{N}_{k+1}(l,N) \sim \frac{\pi^2}{192\sqrt{3}N^2} e^{2\pi\sqrt{\frac{N}{6}}} \sim -\frac{\pi^2}{48N} p(N). \tag{1.11}$$

Now, by (1.6) and (1.10), we see that, for fixed $k \in \mathbb{N}^+$, $l \in \mathbb{N}$, and sufficiently large N,

$$\left|\overline{\mathcal{N}}_k(l,N) - \overline{\mathcal{N}}_{k+1}(l,N)\right| < \mathcal{N}_k(l,N),$$

which generalizes both of the second inequalities of (1.2) and (1.3). As a corollary of Theorem 1.4, we give some interesting inequalities between the k-ranks.

Corollary 1.5. *For fixed $k \in \mathbb{N}^+$, $l \in \mathbb{N}$, and sufficiently large N, we have*

$$\mathcal{N}_k(l,N) < \mathcal{N}_{k+1}(l,N).$$

In particular, we have

$$\mathcal{M}(l,N) < \mathcal{N}(l,N). \tag{1.12}$$

It was conjectured by S. J. Kaavya in [20] that $\mathcal{M}(0,N) \le \mathcal{N}(0,N)$ for all positive integers N. Thus our inequality (1.12) implies Kaavya's conjecture asymptotically.

The paper is organized as follows. We prove Theorems 1.1 and 1.4 in Section 2. In Section 3 we study the asymptotic behavior of $S_{k,l}(q)$ and $F_{k,l}(q)$ when q is near their singularities. We need these when we apply the Circle Method in Section 4 where we complete the proof of Theorem 1.3. In the Appendix, we discuss the asymptotic expansions of certain partial theta functions and prove a result on $S_{k,l}(q)$.

2. Proof of Theorems 1.1 and 1.4

In this section, we apply Theorem 1.3 to prove Theorems 1.1 and 1.4. First, we prove Theorem 1.1.

Proof of Theorem 1.1. By [1, Eq (4.12.7)], we know that, as $x \to \infty$ (which holds for any index ν),

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right).$$

Replacing x by $\pi\sqrt{\frac{2N}{3}}$, the above equation gives

$$I_\nu\left(\pi\sqrt{\frac{2N}{3}}\right) = \frac{3^{1/4}N^{-1/4}e^{2\pi\sqrt{\frac{N}{6}}}}{2^{3/4}\pi} + O\left(N^{-3/4}e^{2\pi\sqrt{\frac{N}{6}}}\right) \quad (\text{as } N \to \infty). \tag{2.1}$$

Substituting the above equation into (1.7), we find that

$$\overline{\mathcal{N}}_k(l,N) = \frac{1}{8\sqrt{3}N}e^{2\pi\sqrt{\frac{N}{6}}} + O\left(N^{-3/2}e^{2\pi\sqrt{\frac{N}{6}}}\right).$$

Recalling the famous asymptotic formula for $p(N)$ by G. H. Hardy and S. Ramanujan [19]:

$$p(N) \sim \frac{1}{4\sqrt{3}N}e^{2\pi\sqrt{\frac{N}{6}}} \quad (\text{as } N \to \infty),$$

we complete the proof of (1.5).

Next, we prove (1.6). Applying Theorem 1.3, we find that

$$\overline{\mathcal{N}}_k(l,N) - \overline{\mathcal{N}}_{k+1}(l,N) = (\xi_{k+1,l} - \xi_{k,l})\frac{\pi^3}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}\right)^{7/2} N^{-7/4} I_{-7/2}\left(\pi\sqrt{\frac{2N}{3}}\right) + O\left(N^{-5/2}e^{2\pi\sqrt{\frac{N}{6}}}\right).$$

Noting that

$$\xi_{k+1,l} - \xi_{k,l} = \frac{2l-1}{16}$$

and

$$N^{-7/4}I_{-7/2}\left(\pi\sqrt{\frac{2N}{3}}\right) = \frac{3^{1/4}N^{-2}e^{2\pi\sqrt{\frac{N}{6}}}}{2^{3/4}\pi} + O\left(N^{-5/2}e^{2\pi\sqrt{\frac{N}{6}}}\right),$$

we have

$$\overline{\mathcal{N}}_k(l,N) - \overline{\mathcal{N}}_{k+1}(l,N) = \frac{(2l-1)\pi^2}{384\sqrt{3}N^2}e^{2\pi\sqrt{\frac{N}{6}}} + O\left(N^{-5/2}e^{2\pi\sqrt{\frac{N}{6}}}\right).$$

This completes the proof of (1.6). $\square$

Now, we prove Theorem 1.4.

Proof of Theorem 1.4. Since $\mathcal{N}_k(l,N) = \overline{\mathcal{N}}_k(l,N) - \overline{\mathcal{N}}_k(l+1,N)$, by Theorem 1.3 and Eq. (2.1), we have

$$\begin{aligned}\mathcal{N}_k(l,N) &= \frac{\pi^2}{4\sqrt{2}}\left(\frac{1}{\sqrt{6}}\right)^{5/2} N^{-5/4} I_{-5/2}\left(\pi\sqrt{\frac{2N}{3}}\right) + O\left(N^{-2}e^{2\pi\sqrt{\frac{N}{6}}}\right)\\ &= \frac{\pi}{48\sqrt{2}N^{3/2}}e^{2\pi\sqrt{\frac{N}{6}}} + O\left(N^{-2}e^{2\pi\sqrt{\frac{N}{6}}}\right).\end{aligned}$$

Thus Eq. (1.10) follows.

Next, we prove (1.11). We have

$$\begin{aligned}\mathcal{N}_k(l,N) - \mathcal{N}_{k+1}(l,N) &= \left\{\overline{\mathcal{N}}_k(l,N) - \overline{\mathcal{N}}_k(l+1,N)\right\} - \left\{\overline{\mathcal{N}}_{k+1}(l,N) - \overline{\mathcal{N}}_{k+1}(l+1,N)\right\}\\ &= \left\{\overline{\mathcal{N}}_k(l,N) - \overline{\mathcal{N}}_{k+1}(l,N)\right\} - \left\{\overline{\mathcal{N}}_k(l+1,N) - \overline{\mathcal{N}}_{k+1}(l+1,N)\right\}\\ &\sim \frac{(2l-1)\pi^2}{384\sqrt{3}N^2}e^{2\pi\sqrt{\frac{N}{6}}} - \frac{(2l+1)\pi^2}{384\sqrt{3}N^2}e^{2\pi\sqrt{\frac{N}{6}}} \quad \text{(by Eq. (1.6))}\\ &= -\frac{\pi^2}{192\sqrt{3}N^2}e^{2\pi\sqrt{\frac{N}{6}}}.\end{aligned}$$

This completes the proof of Theorem 1.4. □

3. Asymptotic behavior of generating functions

In this section, we study the asymptotic behavior of the generating function $F_{k,l}(q)$ when q is near its essential singularities on the unit circle. We set $q = e^{2\pi i\tau}$, where $\tau = x + iy$ and $y > 0$. Since the asymptotic behavior is largely controlled by the exponential singularities of $(q;q)_\infty^{-1}$, our dominant pole is at $q = 1$. The main task is to understand the asymptotic behavior of the partial theta function $S_{k,l}(q)$ near this point.

3.1. Asymptotic behavior of $F_{k,l}(q)$ near the dominant pole

First, we need the following lemma on the asymptotic behavior of $S_{k,l}(q)$ near $q = 1$.

Proposition 3.1. *For $y = \frac{1}{2\sqrt{6N}}$ and $|x| \le y$, as $N \to \infty$, we have*

$$S_{k,l}(q) = \frac{1}{2} + \frac{2l-1}{4}(\pi i\tau) - \frac{(2l-1)(2k-1)}{8}(\pi i\tau)^2 + \zeta^*\tau^3 + O\left(N^{-7/4}\right), \tag{3.1}$$

where ζ^* is a constant (which depends on k and l).

We will prove the above proposition in the Appendix. where a more general result on asymptotic expansions of partial theta functions will be discussed. The constant ζ^* can be replaced by an explicit formula with k and l. We do not do this because the asymptotic contribution of the term containing ζ^* will be absorbed into the error term. However, we emphasize that, for convenience, in the rest of this article, we will repeatedly use ζ^* to denote constants even though it represents different values in different equations.

Corollary 3.2. *For $y = \frac{1}{2\sqrt{6N}}$ and $|x| \le y$, as $N \to \infty$, we have*

$$\begin{aligned}F_{k,l}(q) &= \frac{1}{2\sqrt{2\pi}}(-2\pi i\tau)^{1/2}e^{\frac{\pi i}{12\tau}} - \frac{1}{\sqrt{2\pi}}\left(\frac{2l-1}{8} + \frac{1}{48}\right)(-2\pi i\tau)^{3/2}e^{\frac{\pi i}{12\tau}}\\ &\quad - \frac{\xi_{k,l}}{\sqrt{2\pi}}(-2\pi i\tau)^{5/2}e^{\frac{\pi i}{12\tau}} + \zeta^*(-2\pi i\tau)^{7/2}e^{\frac{\pi i}{12\tau}} + O\left(N^{-2}e^{\pi\sqrt{\frac{N}{6}}}\right),\end{aligned} \tag{3.2}$$

where $\xi_{k,l} = \frac{(2l-1)(2k-1)}{32} - \frac{l}{96} + \frac{11}{2304}$ and we take the principal branches of $\sqrt{\tau}$.

Proof. By the transformation formula of $\eta(\tau)$, $\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)$, or following directly from [9, Eq. (3.8)], as $N \to \infty$, we have

$$\begin{aligned}\frac{1}{(q;q)_\infty} &= \frac{q^{\frac{1}{24}}\sqrt{-i\tau}}{\eta\left(-\frac{1}{\tau}\right)} = \sqrt{-i\tau}\,e^{\frac{2\pi i}{24}(\tau+1/\tau)}\left(1 + O\left(e^{-2\pi\sqrt{6N}}\right)\right)\\ &= \sqrt{-i\tau}\,e^{\frac{\pi i}{12\tau}}\left(1 + \frac{2\pi i\tau}{24} + \frac{(2\pi i\tau)^2}{1152} + \frac{(2\pi i\tau)^3}{3!24^3} + O\left(N^{-2}\right)\right).\end{aligned} \tag{3.3}$$

Multiplying the above equation on both sides of (3.1), we find that

$$F_{k,l}(q) = \left\{\sqrt{-i\tau}e^{\frac{\pi i}{12\tau}}\left(1+\frac{2\pi i\tau}{24}+\frac{(2\pi i\tau)^2}{1152}+\frac{(2\pi i\tau)^3}{3!24^3}+O\left(N^{-2}\right)\right)\right\}$$
$$\times\left\{\frac{1}{2}+\frac{2l-1}{4}(\pi i\tau)-\frac{(2l-1)(2k-1)}{8}(\pi i\tau)^2+\zeta^*\tau^3+O\left(N^{-7/4}\right)\right\}.$$

Expanding the above equation and noting that $\sqrt{-i\tau}e^{\frac{\pi i}{12\tau}} = O(N^{-1/4}e^{\pi\sqrt{\frac{N}{6}}})$, we get (3.2). □

3.2. Bounds away from the dominant pole

First, we consider the asymptotic behavior of $S_{k,l}(q)$ when q is not near 1.

Proposition 3.3. *If $y = \frac{1}{2\sqrt{6N}}$, then, as $N \to \infty$, we have $|S_{k,l}(q)| = O\left(\sqrt{N}\right)$.*

Proof. For $q = e^{2\pi i\tau}$, where $\tau = x + \frac{1}{2\sqrt{6N}}i$, as $N \to \infty$, we have

$$\begin{aligned}|S_{k,l}(q)| &= \left|\sum_{n=1}^{\infty}(-1)^{n-1}q^{n((2k-1)n-1)/2+ln}\right| \\ &\le \sum_{n=1}^{\infty}|q^{n((2k-1)n-1)/2+ln}| \\ &\le \sum_{n=1}^{\infty}|q^n| \le \frac{1}{1-|q|} = \frac{1}{1-e^{-\frac{\pi}{\sqrt{6N}}}} = O\left(\sqrt{N}\right). \quad \square\end{aligned}$$

By the above proposition, we get a bound for $F_{k,l}(q)$ in the region away from 1. This bound is exponentially smaller than the asymptotic discussed in Section 3.1.

Corollary 3.4. *If $y = \frac{1}{2\sqrt{6N}}$ and $y \le |x| \le \frac{1}{2}$, then, as $N \to \infty$, we have*

$$|F_{k,l}(q)| = O\left(\sqrt{N}e^{\frac{\pi}{2}\sqrt{\frac{N}{6}}}\right). \tag{3.4}$$

Proof. By Eq. (3.3), as $N \to \infty$, we have

$$\left|\frac{1}{(q;q)_\infty}\right| \sim \sqrt{|\tau|}\left|e^{\frac{2\pi i}{24\tau}}\right| \le e^{\frac{\pi y}{12(x^2+y^2)}} \le e^{\frac{\pi}{24y}} = e^{\frac{\pi}{2}\sqrt{\frac{N}{6}}}. \tag{3.5}$$

This together with Proposition 3.3 implies (3.4). □

4. The circle method

In this section, by an argument analogous to that in [9, Section 4], we apply the Circle Method to complete the proof of Theorem 1.3. By Cauchy's residue theorem, we have the following representation of the coefficients of $F_{k,l}(q)$.

$$\overline{\mathcal{N}}_k(l,N) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{F_{k,l}(q)}{q^{N+1}}dq = \int_{-\frac{1}{2}}^{\frac{1}{2}}F_{k,l}\left(e^{-\frac{\pi}{\sqrt{6N}}+2\pi ix}\right)e^{\pi\sqrt{\frac{N}{6}}-2\pi iNx}dx, \tag{4.1}$$

where the contour is the counterclockwise traversal of the circle $\mathcal{C} := \left\{|q| = e^{-\frac{\pi}{\sqrt{6N}}}\right\}$. We separate the integral in (4.1) into two ranges, writing $\overline{\mathcal{N}}_k(l,N) = I' + I''$, with

$$I' := \int_{|x|\le\frac{1}{2\sqrt{6N}}}F_{k,l}\left(e^{-\frac{\pi}{\sqrt{6N}}+2\pi ix}\right)e^{\pi\sqrt{\frac{N}{6}}-2\pi iNx}dx$$

and

$$I'' := \int_{|x|\le\frac{1}{2\sqrt{6N}}\le\frac{1}{2}}F_{k,l}\left(e^{-\frac{\pi}{\sqrt{6N}}+2\pi ix}\right)e^{\pi\sqrt{\frac{N}{6}}-2\pi iNx}dx.$$

We will show later that the main term in the asymptotic expansion of $\overline{\mathcal{N}}_k(l,N)$ in Theorem 1.3 comes only from I', however, the integral I'' will be absorbed into the error term.

4.1. Main arc

We introduce the auxiliary function P_s which is originally due to Wright before examining the integral I'. For $s \in \mathcal{R}$, we define

$$P_s := \frac{1}{2\pi i}\int_{1-i}^{1+i} v^s e^{\pi\sqrt{\frac{N}{6}}(v+\frac{1}{v})}dv.$$

Then, by [9, Lemma 4.2], as $N \to \infty$, we have

$$P_s - I_{-s-1}\left(\pi\sqrt{\frac{2N}{3}}\right) = O\left(e^{\frac{3\pi}{2}\sqrt{\frac{N}{6}}}\right).$$

We evaluate I' by the modified Bessel functions up to an allowable error.

Proposition 4.1. *As $N \to \infty$, we have*

$$\begin{aligned} I' &= \frac{\pi}{2\sqrt{2}} \times \left(\frac{1}{\sqrt{6}}\right)^{3/2} N^{-3/4} I_{-3/2}\left(\pi\sqrt{\frac{2N}{3}}\right) - \frac{\pi^2}{\sqrt{2}}\left(\frac{2l-1}{8} + \frac{1}{48}\right) \times \left(\frac{1}{\sqrt{6}}\right)^{5/2} N^{-5/4} I_{-5/2}\left(\pi\sqrt{\frac{2N}{3}}\right) \\ &\quad - \frac{\pi^3 \xi_{k,l}}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}\right)^{7/2} N^{-7/4} I_{-7/2}\left(\pi\sqrt{\frac{2N}{3}}\right) + O\left(N^{-5/2} e^{2\pi\sqrt{\frac{N}{6}}}\right). \end{aligned}$$

Proof. First, writing $\tau = \frac{1}{2\sqrt{6N}}(u+i)$, i.e., replacing x by $\frac{u}{2\sqrt{6N}}$ we find that

$$\begin{aligned} I' &= \int_{|x| \le \frac{1}{2\sqrt{6N}}} F_{k,l}\left(e^{-\frac{\pi}{\sqrt{6N}} + 2\pi i x}\right) e^{\pi\sqrt{\frac{N}{6}} - 2\pi i N x} dx \\ &= \frac{1}{2\sqrt{6N}}\int_{-1}^{1} F_{k,l}\left(e^{\frac{\pi}{\sqrt{6N}}(-1+iu)}\right) e^{\pi\sqrt{\frac{N}{6}}(1-iu)} du. \end{aligned} \tag{4.2}$$

Next, replacing τ by $\frac{1}{2\sqrt{6N}}(u+i)$ in (3.2) and noting that $-2\pi i\tau = \frac{\pi(1-iu)}{\sqrt{6N}}$ and $e^{\frac{\pi i}{12\tau}} = e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}\right)}$, we have

$$\begin{aligned} F_{k,l}\left(e^{\frac{\pi}{\sqrt{6N}}(-1+iu)}\right) &= \frac{1}{2\sqrt{2\pi}}\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{1/2} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}\right)} - \frac{1}{\sqrt{2\pi}}\left(\frac{2l-1}{8} + \frac{1}{48}\right)\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{3/2} \\ &\quad \times e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}\right)} - \frac{\xi_{k,l}}{\sqrt{2\pi}}\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{5/2} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}\right)} \\ &\quad + \zeta^*\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{7/2} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}\right)} + O\left(N^{-2} e^{\pi\sqrt{\frac{N}{6}}}\right) \quad (\text{as } N \to \infty). \end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2), we get

$$\begin{aligned} I' &= \frac{1}{2\sqrt{2\pi}} \times \frac{1}{2\sqrt{6N}}\int_{-1}^{1}\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{1/2} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}+(1-iu)\right)} du \\ &\quad - \frac{1}{\sqrt{2\pi}}\left(\frac{2l-1}{8} + \frac{1}{48}\right) \times \frac{1}{2\sqrt{6N}}\int_{-1}^{1}\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{3/2} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}+(1-iu)\right)} du \\ &\quad - \frac{\xi_{k,l}}{\sqrt{2\pi}}\frac{1}{2\sqrt{6N}}\int_{-1}^{1}\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{5/2} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}+(1-iu)\right)} du \\ &\quad + \zeta^*\frac{1}{2\sqrt{6N}}\int_{-1}^{1}\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{7/2} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}+(1-iu)\right)} du \\ &\quad + \frac{1}{2\sqrt{6N}}\int_{-1}^{1} O\left(N^{-2} e^{\pi\sqrt{\frac{N}{6}}}\right) e^{\pi\sqrt{\frac{N}{6}}(1-iu)} du \quad (\text{as } N \to \infty). \end{aligned}$$

Making the change of variables $u = i(v-1)$, we find that

$$\begin{aligned}
&\frac{1}{2\sqrt{6N}}\int_{-1}^{1}\left(\frac{\pi(1-iu)}{\sqrt{6N}}\right)^{s} e^{\pi\sqrt{\frac{N}{6}}\left(\frac{1}{1-iu}+(1-iu)\right)}du \\
&\quad = \frac{i}{2\sqrt{6N}}\int_{1+i}^{1-i}\left(\frac{\pi v}{\sqrt{6N}}\right)^{s} e^{\pi\sqrt{\frac{N}{6}}(v+\frac{1}{v})}dv \\
&\quad = \frac{-i}{2\sqrt{6N}}\left(\frac{\pi}{\sqrt{6N}}\right)^{s}\int_{1-i}^{1+i} v^{s} e^{\pi\sqrt{\frac{N}{6}}(v+\frac{1}{v})}dv \\
&\quad = \left(\frac{\pi}{\sqrt{6N}}\right)^{s+1} P_s \\
&\quad = \left(\frac{\pi}{\sqrt{6N}}\right)^{s+1} I_{-s-1}\left(\pi\sqrt{\frac{2N}{3}}\right) + O\left(e^{\frac{3\pi}{2}\sqrt{\frac{N}{6}}}\right) \quad (\text{as } N\to\infty).
\end{aligned}$$

From this, we see that

$$\begin{aligned}
I' &= \frac{1}{2\sqrt{2\pi}}\times\left(\frac{\pi}{\sqrt{6N}}\right)^{3/2} I_{-3/2}\left(\pi\sqrt{\frac{2N}{3}}\right) \\
&\quad - \frac{1}{\sqrt{2\pi}}\left(\frac{2l-1}{8}+\frac{1}{48}\right)\times\left(\frac{\pi}{\sqrt{6N}}\right)^{5/2} I_{-5/2}\left(\pi\sqrt{\frac{2N}{3}}\right) \\
&\quad - \frac{\xi_{k,l}}{\sqrt{2\pi}}\left(\frac{\pi}{\sqrt{6N}}\right)^{7/2} I_{-7/2}\left(\pi\sqrt{\frac{2N}{3}}\right) \\
&\quad + \zeta^{*}\left(\frac{\pi}{\sqrt{6N}}\right)^{9/2} I_{-9/2}\left(\pi\sqrt{\frac{2N}{3}}\right) + O\left(N^{-5/2}e^{2\pi\sqrt{\frac{N}{6}}}\right) \quad (\text{as } N\to\infty).
\end{aligned} \tag{4.4}$$

Recall Eq. (2.1):

$$I_{\nu}\left(\pi\sqrt{\frac{2N}{3}}\right) = \frac{3^{1/4}N^{-1/4}e^{2\pi\sqrt{\frac{N}{6}}}}{2^{3/4}\pi} + O\left(N^{-3/4}e^{2\pi\sqrt{\frac{N}{6}}}\right) \quad (\text{as } N\to\infty), \tag{4.5}$$

setting $\nu = -9/2$, we find that

$$\zeta^{*}\left(\frac{\pi}{\sqrt{6N}}\right)^{9/2} I_{-9/2}\left(\pi\sqrt{\frac{2N}{3}}\right) = O\left(N^{-5/2}e^{2\pi\sqrt{\frac{N}{6}}}\right) \quad (\text{as } N\to\infty).$$

Substituting the above equation into (4.4) and simplifying, we complete our proof of the proposition. □

4.2. Error arc

We give a bound for I'' which is exponentially smaller than the error term of I'.

Proposition 4.2. *As $N \to \infty$,*

$$I'' = O\left(\sqrt{N}e^{\frac{3\pi}{2}\sqrt{\frac{N}{6}}}\right).$$

Proof. By Corollary 3.4, as $N \to \infty$, we have

$$\begin{aligned}
|I''| &= \left|\int_{|x|\le\frac{1}{2\sqrt{6N}}\le\frac{1}{2}} F_{k,l}\left(e^{-\frac{\pi}{\sqrt{6N}}+2\pi ix}\right)e^{\pi\sqrt{\frac{N}{6}}-2\pi iNx}dx\right| \\
&\le \sqrt{N}e^{\frac{\pi}{2}\sqrt{\frac{N}{6}}}\left|\int_{|x|\le\frac{1}{2\sqrt{6N}}\le\frac{1}{2}} e^{\pi\sqrt{\frac{N}{6}}-2\pi iNx}dx\right| \\
&= O\left(\sqrt{N}e^{\frac{3\pi}{2}\sqrt{\frac{N}{6}}}\right). \quad \square
\end{aligned}$$

4.3. Proof of Theorem 1.3

Invoking Propositions 4.1 and 4.2 in Eq. (4.1), we find that, as $N \to \infty$,

$$\begin{aligned}\overline{\mathcal{N}}_k(l,N) &= I' + I'' \\ &= \frac{\pi}{2\sqrt{2}} \times \left(\frac{1}{\sqrt{6}}\right)^{3/2} N^{-3/4} I_{-3/2}\left(\pi\sqrt{\frac{2N}{3}}\right) - \frac{\pi^2}{\sqrt{2}}\left(\frac{l}{4} - \frac{5}{48}\right) \times \left(\frac{1}{\sqrt{6}}\right)^{5/2} N^{-5/4} I_{-5/2}\left(\pi\sqrt{\frac{2N}{3}}\right) \\ &\quad - \frac{\pi^3 \xi_{k,l}}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}\right)^{7/2} N^{-7/4} I_{-7/2}\left(\pi\sqrt{\frac{2N}{3}}\right) + O\left(N^{-5/2} e^{2\pi\sqrt{\frac{N}{6}}}\right).\end{aligned}$$

This completes the proof of Theorem 1.3.

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Appendix. Asymptotic expansions of partial theta function

In this section, we establish an asymptotic expansion of a class of partial theta functions which generalizes a result in [7]. As its application, we will prove Proposition 3.1.

First, we recall an asymptotic expansion by S. Ramanujan [6, p. 545],

$$2\sum_{n=0}^{\infty}(-1)^n\left(\frac{1-t}{1+t}\right)^{n^2+n} \sim 1 + t + t^2 + 2t^3 + 5t^5 + \cdots,$$

where $t \to 0^+$. Recently, this result was generalized by B. C. Berndt and B. Kim in [7]. For real numbers b, β and $\gamma > 0$, we define

$$F_1(\theta) := 2\sum_{n=0}^{\infty}(-1)^n e^{-(n^2+bn)\theta}, \tag{A.1}$$

where $\theta = \gamma + \beta i$. Note that we abandon the notation "$F_1(q)$" in [7, Eq. (2.4)] to avoid misunderstanding. Then, by [7, Theorem 1.1] or [7, Eq. (2.9)], for any non-negative integer M and $\beta = 0$, i.e., $\theta > 0$, as $\theta \to 0^+$, we have

$$F_1(\theta) = e^{(2b-1)\theta/4}\sum_{n=0}^{M}\frac{E_{2n}\theta^n}{2^{2n}(2n)!}H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) + O(\theta^{M+1/2}), \tag{A.2}$$

where E_n, $n \geq 0$, is the n-th Euler number, and $H_n(x)$, $n \geq 0$, is the n-th Hermite polynomial.

By an argument analogous to that in [7], we prove a generalization of (A.2).

Theorem A.1. *For $\theta = \gamma + \beta i$ satisfying $|\beta| \leq \gamma$ and any non-negative integer M, as $\gamma \to 0^+$, we have*

$$F_1(\theta) = e^{(2b-1)\theta/4}\sum_{n=0}^{M}\frac{E_{2n}\theta^n}{2^{2n}(2n)!}H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) + O(|\theta|^{M+1/2}), \tag{A.3}$$

where we take the principal branches of $\sqrt{\theta}$.

To prove Theorem A.1, we need two lemmas.

Lemma A.2. *Let $\gamma > 0$ and β, a, b be real. If $H_n(x)$, $n \geq 0$, denotes the n-th Hermite polynomial, then, for $\theta = \gamma + \beta i$ satisfying $|\beta| \leq \gamma$, we have*

$$\frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} z^{2n} e^{biz} e^{-z^2/\theta}\, dz = \frac{(-1)^n\theta^n}{2^{2n}} e^{-b^2\theta/4} H_{2n}\left(\frac{b\sqrt{\theta}}{2}\right), \tag{A.4}$$

where we take the principal branches of $\sqrt{\theta}$ and $\sqrt{\pi\theta}$.

Proof. Let $D := \{\theta \in \mathbb{C} \mid \theta = \gamma + \beta i \text{ with } \gamma > 0 \text{ and } |\beta| \leq \gamma\}$. First, we examine the integral on the left side of (A.4) with $a = 0$ and $\theta \in D$. Let T be any non-negative real number. Since $\frac{1}{\theta} = (\gamma - \beta i)/|\theta|^2$, we have

$$\begin{aligned}\left|\int_{|z|\geq T} z^{2n} e^{biz} e^{-z^2/\theta} dz\right| &\leq \int_{|z|\geq T} z^{2n} \left|e^{-z^2/\theta}\right| dz \\ &= \int_{|z|\geq T} z^{2n} \left|e^{-z^2(\gamma-\beta i)/|\theta|^2}\right| dz \\ &= \int_{|z|\geq T} z^{2n} e^{-z^2\gamma/|\theta|^2} dz \\ &= \left(\frac{|\theta|}{\sqrt{\gamma}}\right)^{2n+1} \int_{|u|\geq \frac{\sqrt{\gamma}T}{|\theta|}} u^{2n} e^{-u^2} du \quad \left(u = \frac{\sqrt{\gamma}z}{|\theta|}\right). \end{aligned} \tag{A.5}$$

By $|\beta| \leq \gamma$, we know that, for all $\theta \in D$, $|\theta|^2 \leq 2\gamma^2$. Hence, we have $\frac{|\theta|}{\sqrt{\gamma}} \leq 2^{\frac{1}{4}}\sqrt{|\theta|}$. This together with (A.5) implies

$$\left|\int_{|z|\geq T} z^{2n} e^{biz} e^{-z^2/\theta} dz\right| \leq 2^{\frac{2n+1}{4}} |\theta|^{n+1/2} \int_{|u|\geq \frac{\sqrt{\gamma}T}{|\theta|}} u^{2n} e^{-u^2} du. \tag{A.6}$$

Since $\gamma > 0$ and $\int_{-\infty}^{\infty} u^{2n} e^{-u^2} du$ converges for all non-negative integers n, $\forall \varepsilon > 0$, there exists a positive number T_0, such that, for all $T \geq T_0$,

$$\int_{|u|\geq \frac{\sqrt{\gamma}T}{|\theta|}} u^{2n} e^{-u^2} du \leq \frac{\varepsilon}{2^{\frac{2n+1}{4}} C^{n+1/2}},$$

where C is any fixed positive real number. By (A.6), $\forall \theta \in D$ with $|\theta| \leq C$ and $T \geq T_0$, we have

$$\begin{aligned}\left|\int_{|z|\geq T} z^{2n} e^{biz} e^{-z^2/\theta} dz\right| &\leq 2^{\frac{2n+1}{4}} |\theta|^{n+1/2} \int_{|u|\geq \frac{\sqrt{\gamma}T}{|\theta|}} u^{2n} e^{-u^2} du \\ &\leq 2^{\frac{2n+1}{4}} |\theta|^{n+1/2} \frac{\varepsilon}{2^{\frac{2n+1}{4}} C^{n+1/2}} \\ &\leq \varepsilon. \end{aligned}$$

Now, we see that $\int_{-\infty}^{\infty} z^{2n} e^{biz} e^{-z^2/\theta} dz$ converges uniformly for all θ in any compact subset of D, thus defines a function of θ which is continuous on D and analytic at all of its interior points.

Next, we show that the integral on the left side of (A.4) is independent of the parameter a. For $L \geq |a|$, we consider now

$$I_L = \int_{\mathcal{C}_L} z^{2n} e^{biz} e^{-z^2/\theta} dz,$$

where the contour $\mathcal{C}_L$ is the positive oriented rectangle with vertices $\pm L$ and $\pm L + ai$. Since the integrand is an analytic function of z on the whole complex plane, by the Cauchy integral theorem, we have $I_L = 0$. For the integral on the two vertical edges of $\mathcal{C}_L$, we have the following estimate,

$$\begin{aligned}\left|\int_0^a (\pm L + yi)^{2n} e^{bi(\pm L + yi)} e^{-(\pm L + yi)^2/\theta} dy\right| &\leq \int_0^{|a|} \left|(\pm L + yi)^{2n} e^{bi(\pm L + yi)} e^{-(\pm L + yi)^2/\theta}\right| dy \\ &\leq e^{b|a|} \int_0^{|a|} (L^2 + y^2)^n \left|e^{-(\pm L + yi)^2(\gamma - \beta i)/|\theta|^2}\right| dy \\ &\leq e^{b|a| - L^2\gamma/|\theta|^2} \int_0^{|a|} (L^2 + y^2)^n e^{(y^2\gamma + 2Ly|\beta|)/|\theta|^2} dy \\ &\leq (L^2 + |a|^2)^n e^{b|a| + (|a|^2 - L^2)\gamma/|\theta|^2} \int_0^{|a|} e^{2Ly/|\theta|} dy \quad (\text{by } |\beta| \leq |\theta|) \\ &\leq (L^2 + |a|^2)^n e^{b|a| + (|a|^2 - L^2)\gamma/|\theta|^2} |a| e^{2|a|L/|\theta|} \\ &\leq 2^n e^{b|a| + |a|^2\gamma/|\theta|^2} L^{2n+1} e^{2|a|L/|\theta| - L^2\gamma/|\theta|^2} \quad (\text{by } |a| \leq L). \end{aligned} \tag{A.7}$$

Since $\gamma > 0$, we have $L^{2n+1}e^{2|a|L/|\theta|-L^2\gamma/|\theta|^2} = O(e^{-L})$, as $L \to \infty$. Hence, we see that, as $L \to \infty$, Eq. (A.7) together with $I_L = 0$ implies that

$$\int_{-\infty+ai}^{\infty+ai} z^{2n}e^{biz}e^{-z^2/\theta}dz = \int_{-\infty}^{\infty} z^{2n}e^{biz}e^{-z^2/\theta}dz,$$

for any real number a.

From the above proof, we know that the integral on the left side of (A.4) defines a function of θ which is continuous on D and analytic at all of its interior points. Since $\frac{1}{\sqrt{\pi\theta}}$ is analytic on D, the function on the left side of (A.4) has the same analytic property as the integral. On the other hand, it is clear that the function on the right side of (A.4) is analytic on the complex plane. Thus, by analytic continuation, it suffices to show that (A.4) is true for all real positive θ which is stated in [7, Lemma 2.2]. □

Lemma A.3. *Let $\gamma > 0$ and β, a, b be real. Then, for $\theta = \gamma + \beta i$ satisfying $|\beta| \le \gamma$, we have*

$$\frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(2n+b)iz}dz = e^{-(n+b/2)^2\theta} \tag{A.8}$$

where we take the principal branches of $\sqrt{\theta}$ and $\sqrt{\pi\theta}$.

Proof. By an argument analogous to that in the proof of Lemma A.2, we can show that the function (independent of a) on the left side of (A.8) is continuous at any θ in D and analytic at all of its interior points. Clearly, $e^{-(n+b/2)^2\theta}$ is analytic on the whole complex plane. Then, by analytic continuation, it suffices to show that (A.8) is true for all real positive θ which is stated in [7, Lemma 2.4]. □

Now we are in a position to prove Theorem A.1. We follow the steps in the proof [7, Theorem 1.1] except the estimate of the error term R_M.

Proof of Theorem A.1. Write

$$F_1(\theta) = 2\sum_{n=0}^{\infty}(-1)^n e^{-(n^2+bn)\theta} = 2e^{b^2\theta/4}\sum_{n=0}^{\infty}(-1)^n e^{-(n+b/2)^2\theta} \tag{A.9}$$

and let $G_1(\theta) := 2\sum_{n=0}^{\infty}(-1)^n e^{-(n+b/2)^2\theta}$. By (A.8) (we require $a > 0$), we have

$$e^{-(n+b/2)^2\theta} = \frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(2n+b)iz}dz.$$

Multiply both sides of the above equation by $2(-1)^n$ and sum on n, $0 \le n < \infty$, to obtain

$$\begin{aligned} G_1(\theta) &= \frac{2}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta}\sum_{n=0}^{\infty}(-1)^n e^{(2n+b)iz}dz \\ &= \frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(b-1)iz}\frac{1}{\cos z}dz, \end{aligned} \tag{A.10}$$

where we interchanged the order of summation and integration by using the absolute and uniform convergence of the series on the path of integration, as $a > 0$. Using the generating function

$$\frac{1}{\cos x} = \sec x = \sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}x^{2n}, \quad |x| < \pi/2,$$

for the Euler numbers E_{2n}, by (A.10), we write

$$G_1(\theta) = \frac{1}{\sqrt{\pi\theta}}\sum_{n=0}^{M}\frac{(-1)^n E_{2n}}{(2n)!}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(b-1)iz}z^{2n}dz + R_M, \tag{A.11}$$

where

$$R_M = \frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(b-1)iz}\left(\sec z - \sum_{n=0}^{M}\frac{(-1)^n E_{2n}}{(2n)!}z^{2n}\right)dz. \tag{A.12}$$

Multiplying by $e^{b^2\theta/4}$ on both sides and invoking equation (A.4), Eq. (A.11) gives

$$F_1(\theta) = e^{(2b-1)\theta/4}\sum_{n=0}^{M}\frac{E_{2n}\theta^n}{2^{2n}(2n)!}H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) + R_M.$$

We need to examine the error term R_M. If $0 < a \le 1$, for all points z on the contour $(-\infty + ai, \infty + ai)$, by [6, Eq. (16.3)], there exists a positive constant C_2 which is dependent only on N but not on z or a, such that

$$\left|\frac{a}{z^{2M+2}}\left(\sec z - \sum_{n=0}^{M}\frac{(-1)^n E_{2n}}{(2n)!}z^{2n}\right)\right| \le C_2.$$

Substituting the above inequality into (A.12), we find that

$$\begin{aligned}
|R_M| &= \left|\frac{1}{\sqrt{\pi\theta}}\int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta+(b-1)iz}\left(\sec z - \sum_{n=0}^{M}\frac{(-1)^n E_{2n}}{(2n)!}z^{2n}\right)dz\right| \\
&\le \left|\frac{C_2}{a\sqrt{\pi\theta}}\right|\int_{-\infty+ai}^{\infty+ai}\left|e^{-z^2/\theta}z^{2M+2}\right|dz \\
&\le \left|\frac{C_2}{a\sqrt{\pi\theta}}\right|\int_{-\infty}^{\infty}\left|e^{-(x+ai)^2/\theta}(x+ai)^{2M+2}\right|dx \\
&\le \left|\frac{C_2}{a\sqrt{\pi\theta}}\right|\int_{-\infty}^{\infty}\left|e^{-(x^2-a^2+2axi)/\theta}\right|(x^2+a^2)^{M+1}dx \\
&\le \left|\frac{C_2}{a\sqrt{\pi\theta}}\right|\int_{0}^{\infty}\left|e^{-(x^2-a^2+2axi)/\theta}\right|(x^2+a^2)^{M+1}dx \\
&\quad + \left|\frac{C_2}{a\sqrt{\pi\theta}}\right|\int_{-\infty}^{0}\left|e^{-(x^2-a^2+2axi)/\theta}\right|(x^2+a^2)^{M+1}dx.
\end{aligned} \tag{A.13}$$

We denote the first (resp. second) integral in the last inequality above by I_1 (resp. I_2). Substituting $\theta = \gamma + \beta i$ into I_1 and noting that $\frac{1}{\theta} = \frac{\gamma-\beta i}{|\theta|^2}$, we find that

$$\begin{aligned}
I_1 &= \int_0^{\infty}\left|e^{-(x^2-a^2+2axi)/\theta}\right|(x^2+a^2)^{M+1}dx \\
&= e^{\gamma a^2/|\theta|^2}\int_0^{\infty} e^{-(\gamma x^2+2ax\beta)/|\theta|^2}(x^2+a^2)^{M+1}dx.
\end{aligned} \tag{A.14}$$

Since $|\beta| \le \gamma$ (by assumption), $a > 0$ and $x > 0$, we have $x^2 + a^2 \le (x+a)^2$ and $e^{-(\gamma x^2+2ax\beta)/|\theta|^2} \le e^{-(\gamma x^2-2ax\gamma)/|\theta|^2}$. Substituting these two inequalities into (A.14), we find that

$$\begin{aligned}
I_1 &\le e^{\gamma a^2/|\theta|^2}\int_0^{\infty} e^{-(\gamma x^2-2ax\gamma)/|\theta|^2}(x+a)^{2M+2}dx \\
&= e^{2\gamma a^2/|\theta|^2}\int_0^{\infty} e^{-\gamma(x-a)^2/|\theta|^2}(x+a)^{2M+2}dx \\
&= \frac{|\theta|}{\sqrt{\gamma}}e^{2\gamma a^2/|\theta|^2}\int_{-\frac{a\sqrt{\gamma}}{|\theta|}}^{\infty} e^{-u^2}\left(\frac{|\theta|}{\sqrt{\gamma}}u + 2a\right)^{2M+2}du \quad \left(u = \frac{\sqrt{\gamma}(x-a)}{|\theta|}\right).
\end{aligned} \tag{A.15}$$

Since $|\theta|^2 = \gamma^2 + \beta^2 \le 2\gamma^2$, we have $\frac{|\theta|}{\gamma\sqrt{2}} \le 1$ which implies $\frac{|\theta|}{\sqrt{2\gamma}} \le \sqrt{\gamma} \ll 1$, as $\gamma \to 0^+$. This allows us to set $a = \frac{|\theta|}{\sqrt{2\gamma}}$ in (A.15). Hence, we arrive at

$$\begin{aligned}
I_1 &\le \frac{|\theta|}{\sqrt{\gamma}}e\int_{-\frac{\sqrt{2}}{2}}^{\infty} e^{-u^2}\left(\frac{|\theta|}{\sqrt{\gamma}}u + \frac{\sqrt{2}|\theta|}{\sqrt{\gamma}}\right)^{2M+2}du \\
&= e\left(\frac{|\theta|}{\sqrt{\gamma}}\right)^{2M+3}\int_{-\frac{\sqrt{2}}{2}}^{\infty} e^{-u^2}(u+\sqrt{2})^{2M+2}du \\
&\le e\gamma^{M+3/2}2^{M+3/2}\int_{-\frac{\sqrt{2}}{2}}^{\infty} e^{-u^2}(u+\sqrt{2})^{2M+2}du \\
&\le e|\theta|^{M+3/2}2^{M+3/2}\int_{-\frac{\sqrt{2}}{2}}^{\infty} e^{-u^2}(u+\sqrt{2})^{2M+2}du.
\end{aligned}$$

The convergence of $\int_{-\frac{\sqrt{2}}{2}}^{\infty} e^{-u^2}(u+\sqrt{2})^{2M+2}du$ (for any non-negative integer M) implies $I_1 = O(|\theta|^{M+3/2})$, as $|\theta| \to 0^+$. Similarly, we can prove that, for any non-negative integer M, $I_2 = O(|\theta|^{M+3/2})$, as $|\theta| \to 0^+$. Substituting these bounds into

(A.13) with a replacing by $\frac{|\theta|}{\sqrt{2\gamma}}$ and noting that $\left|\frac{C_2}{a\sqrt{\pi\theta}}\right| = \left|\frac{C_2\sqrt{2\gamma}}{|\theta|\sqrt{\pi\theta}}\right| = O(|\theta|^{-1})$, as $|\theta| \to 0^+$, we find that $R_M = O(|\theta|^{M+1/2})$, as $|\theta| \to 0^+$, or $\gamma \to 0^+$. This completes the proof of Theorem A.1. □

Now we prove Proposition 3.1 with a special case of Theorem A.1. Setting $M = 3$ in (A.3) and noting that $E_0 = 1, E_2 = -1, E_4 = 5, H_0 = 1, H_2(x) = 4x^2 - 2$ and $H_4(x) = 16x^4 - 48x^2 + 12$, we find that, for $\theta = \gamma + \beta i$ with $\gamma > 0$ and $|\beta| \le \gamma$,

$$\begin{aligned} F_1(\theta) &= e^{(2b-1)\theta/4} \sum_{n=0}^{3} \frac{E_{2n}\theta^n}{2^{2n}(2n)!} H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) + O(|\theta|^{7/2}) \\ &= \left(\sum_{n=0}^{3} \frac{(2b-1)^n\theta^n}{4^n n!} + O(|\theta|^4)\right) \sum_{n=0}^{3} \frac{E_{2n}\theta^n}{2^{2n}(2n)!} H_{2n}\left(\frac{(b-1)\sqrt{\theta}}{2}\right) + O(|\theta|^{7/2}) \\ &= 1 + \frac{b\theta}{2} + \frac{b\theta^2}{4} + \zeta^*\theta^3 + O(|\theta|^{7/2}), \end{aligned} \tag{A.16}$$

as $\gamma \to 0^+$.

Proof of Proposition 3.1. Since $q = e^{2\pi i\tau}$, where $\tau = x + yi$, we have

$$\begin{aligned} 2(S_{k,l}(q) - 1) &= 2\sum_{n=0}^{\infty}(-1)^{n-1} q^{n((2k-1)n-1)/2+ln} \\ &= -2\sum_{n=0}^{\infty}(-1)^n \left(q^{\frac{2k-1}{2}}\right)^{n^2+\left(\frac{2l}{2k-1}-\frac{1}{2k-1}\right)n} \\ &= -2\sum_{n=0}^{\infty}(-1)^n \left(e^{\pi i\tau(2k-1)}\right)^{n^2+\left(\frac{2l}{2k-1}-\frac{1}{2k-1}\right)n}. \end{aligned}$$

Next, let $\theta = -\pi i\tau(2k-1) = (2k-1)\pi y - (2k-1)\pi x i$. By assumption, we have $y > 0$ and $|x| \le y$ which imply $(2k-1)\pi y > 0$ and $|(2k-1)\pi x| < (2k-1)\pi y$. Hence, applying (A.16) with θ and b replaced by $-\pi i\tau(2k-1)$ and $\frac{2l}{2k-1} - \frac{1}{2k-1}$, respectively, we find that, as $|\tau| \to 0^+$,

$$-2(S_{k,l}(q) - 1) = 1 - \frac{(2l-1)\pi i}{2}\tau - \frac{(2l-1)(2k-1)\pi^2}{4}\tau^2 + \zeta^*\tau^3 + O(|\tau|^{7/2}). \tag{A.17}$$

Since $\tau = x + yi$ with $|x| \le y$ and $y = \frac{1}{2\sqrt{6N}}$, we have $|\tau|^2 \le \frac{1}{12N}$. This together with (A.17) implies

$$S_{k,l}(q) = \frac{1}{2} + \frac{(2l-1)\pi i}{4}\tau + \frac{(2l-1)(2k-1)\pi^2}{8}\tau^2 + \zeta^*\tau^3 + O(N^{-7/4}).$$

This completes the proof of Proposition 3.1. □

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