Toeplitzness of products of composition operators and their adjoints ☆

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ABSTRACT

We examine the Toeplitzness of products of composition operators and their adjoints. We show, among other things, that $C_\phi^* C_\phi$ is strongly asymptotically Toeplitz for all analytic self-maps ϕ of the unit disk, and that $C_\phi C_\phi^*$ is Toeplitz if and only if ϕ is the identity or a rotation. Also, we see that $C_\phi C_\phi^*$ can exhibit varying degrees of asymptotic Toeplitzness.

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1. Introduction

In recent years it has been seen that there are several ways in which composition operators and Toeplitz operators are closely related. In this paper we will explore the Toeplitzness of certain products of composition operators and their adjoints.

We will use as our setting the Hilbert space H^2 of analytic functions on the unit disk with square-summable Taylor coefficients. We will use the standard inner product on H^2: for $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$ in H^2 (and identifying functions in H^2 with their boundary functions),

$$\langle f, g \rangle = \sum_{n=0}^\infty a_n \overline{b_n}$$
$$= \int_{\partial \mathbb{D}} f(w) \overline{g(w)} \, dm(w), \quad \text{where } m \text{ is normalized Lebesgue measure}$$
$$= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(z) \overline{g(z)} \frac{dz}{z}.$$

Important to us will be the kernel functions $K_a(z) = \frac{1}{1 - \bar{a}z}$ for each $a \in \mathbb{D}$. These are functions in H^2 with the property that, for any $f \in H^2$,

$$\langle f, K_a \rangle = f(a).$$

We will use $k_a(z) = \frac{(1 - |a|^2)^{1/2}}{1 - \bar{a}z}$ for the normalized reproducing kernels.

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For analytic maps $\phi : \mathbb{D} \to \mathbb{D}$, we will be interested in the composition operators $C_\phi : H^2 \to H^2$ defined by

$$C_\phi f = f \circ \phi.$$

It is well-known that such composition operators are bounded linear operators (see [8] or [5]). We will also be interested in Toeplitz operators defined on H^2 in the normal way: For a function $\psi \in L^\infty(\partial\mathbb{D})$, we define $T_\psi : H^2 \to H^2$ by $T_\psi f = P(\psi f)$, where P is the projection operator from L^2 to H^2. In the special case where ψ is an analytic function on the disk (or its boundary function – we will identify the two), T_ψ can be thought of as a multiplication operator. Throughout this paper we will be making use of the shift operator, $S = T_z$, and its adjoint, $S^* = T_{\bar{z}}$, the backward shift.

When we talk about the Toeplitzness of an operator, we are referring to whether or not an operator is a Toeplitz operator, or how asymptotically close to a Toeplitz operator it is, in the sense of Barría and Halmos in [1]. This notion of asymptotic Toeplitzness has been developed in, among other papers, [7] and [10]. It has been noted that an operator $T \in \mathcal{L}(H^2)$ (the set of bounded linear operators from H^2 to itself) is a Toeplitz operator if and only if $S^*TS = T$.

We will say that T is *Uniformly Asymptotically Toeplitz* if there is a bounded operator A, necessarily Toeplitz, such that

$$\lim_{n \to \infty} \left\| S^{*n} T S^n - A \right\| = 0.$$

It is shown in [7] that a bounded linear operator T is uniformly asymptotically Toeplitz if and only if it is the sum of a Toeplitz operator and a compact operator. The only composition operator which is Toeplitz is the identity operator, and it is shown in [10, Theorem 1.1] that the only composition operators which are uniformly asymptotically Toeplitz are the identity operator and those which are compact.

We will say that an operator $T \in \mathcal{L}(H^2)$ is *Strongly Asymptotically Toeplitz* (SAT) if there is a bounded operator A such that for each $f \in H^2$,

$$\left(S^{*n} T S^n - A \right) f \to 0,$$

and that an operator $T \in \mathcal{L}(H^2)$ is *Weakly Asymptotically Toeplitz* (WAT) if there is a bounded operator A such that for each $f, g \in H^2$,

$$\left\langle S^{*n} T S^n f, g \right\rangle \to \langle Af, g \rangle.$$

2. The Toeplitzness of $C_\phi^* C_\phi$

We will begin with a simple but important intertwining relationship between composition operators and Toeplitz operators which, along with more complicated variants, can be found in a number of papers (see [6,3], or [9]).

Lemma 1. *For an analytic self-map of the disk ϕ, $C_\phi S = T_\phi C_\phi$ and $C_\phi^* T_\phi^* = S^* C_\phi^*$.*

Proof. For any $h \in H^2$, $(C_\phi S)h(z) = C_\phi(zh(z)) = \phi(z)h(\phi(z)) = \phi(z)C_\phi h(z) = (T_\phi C_\phi)h(z)$. This proves the first part of the lemma. The second part of the lemma follows by taking adjoints of both sides. $\quad\square$

For which self-maps of the disk ϕ is $C_\phi^* C_\phi$ a Toeplitz operator? This has been answered in [2] and generalized in interesting ways in [9]: It is precisely the inner functions ϕ which make $C_\phi^* C_\phi$ a Toeplitz operator. We present this result here, with a slightly different proof from earlier ones, since the proof leads us to some of our later results.

Proposition 1. $C_\phi^* C_\phi$ *is a Toeplitz operator if and only if ϕ is an inner function, and in this case*

$$C_\phi^* C_\phi = T_\psi \quad \text{where } \psi(z) = \frac{1 - |\phi(0)z|^2}{|1 - \overline{\phi(0)}z|^2}.$$

Proof. $C_\phi^* C_\phi$ is a Toeplitz operator if and only if $S^* C_\phi^* C_\phi S = C_\phi^* C_\phi$ But

$$S^* C_\phi^* C_\phi S = C_\phi^* T_\phi^* T_\phi C_\phi = C_\phi^* T_{|\phi|^2} C_\phi.$$

So $C_\phi^* C_\phi$ is a Toeplitz operator if and only if $C_\phi^* T_{1-|\phi|^2} C_\phi = 0$. Thus it is clear that if ϕ is inner, $T_{1-|\phi|^2} = 0$ and $C_\phi^* C_\phi$ is a Toeplitz operator. In this case, set $C_\phi^* C_\phi = T_\psi$. Then

$$C_\phi^* C_\phi 1 = C_\phi^* 1 = C_\phi^* K_0 = K_{\phi(0)} = T_\psi 1 = P(\psi).$$

Since $T_{\overline{\psi}} = T_\psi^* = (C_\phi^* C_\phi)^* = C_\phi^* C_\phi = T_\psi$, thus ψ is real-valued. Hence

$$\psi = K_{\phi(0)} + \overline{K_{\phi(0)}} - 1 = \frac{1 - |\phi(0)z|^2}{|1 - \overline{\phi(0)}z|^2}.$$

Now if $C_\phi^* C_\phi$ is a Toeplitz operator, from the above discussion, $C_\phi^* T_{1-|\phi|^2} C_\phi = 0$. Since $T_{1-|\phi|^2}$ is positive, $C_\phi^* T_{1-|\phi|^2} C_\phi = 0$ if and only if $T_{1-|\phi|^2} C_\phi = 0$. In particular, $T_{1-|\phi|^2} C_\phi 1 = P(1 - |\phi|^2) = 0$. This implies that $1 - |\phi|^2 = 0$ (a.e.) on the circle since $1 - |\phi|^2$ is a real-valued function. □

In [10] it was shown that many composition operators are not strongly asymptotically Toeplitz, including many known to be weakly asymptotically Toeplitz. It is conjectured that all composition operators, except those whose symbols are the identity or rotations, are weakly asymptotically Toeplitz – this is the WAT conjecture in [10]. In [9, Theorem 5] it is shown that the operator $C_\phi^* C_\phi$ is always weakly asymptotically Toeplitz. We will show that $C_\phi^* C_\phi$ is, in fact, always strongly asymptotically Toeplitz.

Lemma 2. *Assume $f_n \in L^\infty$ and $\|f_n\|_\infty \leqslant C$ for some constant C and for all $n \geqslant 1$ and $f_n \to 0$ pointwise a.e. on $\partial\mathbb{D}$. Then $T_{f_n} \to 0$ strongly as $n \to \infty$.*

Proof. Let $h \in H^2$, then

$$\|T_{f_n}h\|^2 = \|P[f_n h]\|^2 \leqslant \|f_n h\|^2 = \int_{\partial\mathbb{D}} |f_n(w)|^2 |h(w)|^2 \, dm(w) \to 0 \quad \text{as } n \to \infty,$$

where the limit follows from the assumption and the Lebesgue Dominated Convergence Theorem.

Theorem 1. *$C_\phi^* C_\phi$ is strongly asymptotically Toeplitz. That is*

$$S^{*n} C_\phi^* C_\phi S^n \to C_\phi^* T_{\chi(E)} C_\phi \quad \text{strongly},$$

where $\chi(E)$ is the characteristic function of the set $E = \{w \in \partial\mathbb{D}: |\phi(w)| = 1\}$. Furthermore,

$$C_\phi^* T_{\chi(E)} C_\phi = T_\psi, \quad \text{where } \psi(z) = \int_E \frac{(1 - |z|^2)}{|1 - \bar{z}\phi(w)|^2} \, dm(w). \quad \square$$

Proof. Note that by repeatedly using Lemma 1,

$$S^{*n} C_\phi^* C_\phi S^n = C_\phi^* T_\phi^{*n} T_\phi^n C_\phi = C_\phi^* T_{|\phi|^{2n}} C_\phi.$$

It is clear that the infinity norm of $|\phi|^{2n}$ is at most 1 and $|\phi(w)|^{2n} - \chi(E) \to 0$ pointwise on $\partial\mathbb{D}$. Therefore by the above Lemma 2, $T_{|\phi|^{2n}} \to T_{\chi(E)}$ strongly. Set $C_\phi^* T_{\chi(E)} C_\phi = T_\psi$, then for $z \in \mathbb{D}$,

$$\psi(z) = \langle T_\psi k_z, k_z \rangle = \langle C_\phi^* T_{\chi(E)} C_\phi k_z, k_z \rangle = \langle T_{\chi(E)} C_\phi k_z, C_\phi k_z \rangle = (1 - |z|^2) \int_E |(C_\phi k_z)(w)|^2 \, dm(w)$$

$$= \int_E \frac{(1 - |z|^2)}{|1 - \bar{z}\phi(w)|^2} \, dm(w). \quad \square$$

As was noted before, we know that if $C_\phi^* C_\phi$ is uniformly asymptotically Toeplitz, then $C_\phi^* C_\phi = T_f + K$ for some Toeplitz operator T_f and compact operator K. If this is the case, then the Toeplitz operator T_f must equal $C_\phi^* T_{\chi(E)} C_\phi$ as in the proof above, so $C_\phi^* C_\phi = C_\phi^* T_{\chi(E)} C_\phi + K$. Equivalently $C_\phi^* C_\phi - C_\phi^* T_{\chi(E)} C_\phi = C_\phi^* T_{1-\chi(E)} C_\phi = K$. Since $T_{1-\chi(E)}$ is a positive operator, $C_\phi^* T_{1-\chi(E)} C_\phi$ is compact if and only if $T_{1-\chi(E)} C_\phi$ is compact. Therefore we have the following result:

Theorem 2. *$C_\phi^* C_\phi$ is uniformly asymptotically Toeplitz if and only if $T_{1-\chi(E)} C_\phi$ is compact.*

The following theorem generalizes Theorem 2.2 in [10] with an elementary proof.

Theorem 3. *If ϕ_1 and ϕ_2 are analytic self-maps of the disk, then $C_{\phi_1}^* C_{\phi_2}$ is Mean Strongly Asymptotically Toeplitz (MSAT). Further, if $\phi_1 \neq \phi_2$, then*

$$\frac{1}{N+1} \sum_{n=0}^{n=N} S^{*n} C_{\phi_1}^* C_{\phi_2} S^n \to 0 \quad \text{strongly}.$$

Proof. If $\phi_1 = \phi_2$, we know that $C_{\phi_1}^* C_{\phi_2}$ is SAT, thus MSAT. Now assume $\phi_1 \neq \phi_2$. Note that

$$\frac{1}{N+1} \sum_{n=0}^{n=N} S^{*n} C_{\phi_1}^* C_{\phi_2} S^n = \frac{1}{N+1} \sum_{n=0}^{n=N} C_{\phi_1}^* T_{\phi_1}^{*n} T_{\phi_2}^n C_{\phi_2} = C_{\phi_1}^* T_{g_N} C_{\phi_2},$$

where

$$g_N = \frac{1}{N+1} \sum_{n=0}^{n=N} \overline{\phi_1^n} \phi_2^n = \frac{1}{N+1} \frac{1 - \overline{\phi_1^{N+1}} \phi_2^{N+1}}{1 - \overline{\phi_1} \phi_2}.$$

Since $\phi_1 \neq \phi_2$, $1 - \overline{\phi_1} \phi_2 \neq 0$ a.e. on $\partial \mathbb{D}$. Note also $|g_N| \leqslant 1$ on $\partial \mathbb{D}$. Furthermore

$$|g_N| \leqslant \frac{1}{N+1} \frac{2}{|1 - \overline{\phi_1} \phi_2|} \quad \text{on } \partial \mathbb{D},$$

so $g_N \to 0$ pointwise on $\partial \mathbb{D}$ as $N \to \infty$. By Lemma 2, $T_{g_N} \to 0$ strongly, hence $C_{\phi_1}^* T_{g_N} C_{\phi_2} \to 0$ strongly. The proof is complete. $\square$

Remark 1. From above theorem we know that if $\phi_1 \neq \phi_2$ and $C_{\phi_1}^* C_{\phi_2} = T_f + K$, then $f = 0$. This proves that $C_{\phi_1}^* C_{\phi_2}$ is not uniformly Toeplitz unless $C_{\phi_1}^* C_{\phi_2}$ is compact.

3. The Toeplitzness of $C_\phi C_\phi^*$

We will begin by asking what we can say about the Toeplitzness of $C_{\phi_1} C_{\phi_2}^*$ for analytic self-maps of the unit disk ϕ_1 and ϕ_2.

When is it true that $C_{\phi_1} C_{\phi_2}^*$ is a Toeplitz operator, i.e., when do we have $C_{\phi_1} C_{\phi_2}^* = S^* C_{\phi_1} C_{\phi_2}^* S$? We compute the action of each side of this equation on the reproducing kernels, $K_a(z) = \frac{1}{1-\bar{a}z}$ for each $a \in \mathbb{D}$. $C_{\phi_2}^* K_a(z) = K_{\phi_2(a)}(z) = \frac{1}{1-\overline{\phi_2(a)}z}$, so we have

$$C_{\phi_1} C_{\phi_2}^* K_a(z) = \frac{1}{1 - \overline{\phi_2(a)} \phi_1(z)}. \tag{3.1}$$

To compute $S^* C_{\phi_1} C_{\phi_2}^* S K_a(z)$, we note that $S K_a(z) = \frac{z}{1-\bar{a}z} = \frac{1}{\bar{a}}(\frac{1}{1-\bar{a}z} - 1) = \frac{1}{\bar{a}}(K_a(z) - K_0(z))$ (for $a \neq 0$), so $C_{\phi_2}^* S K_a(z) = \frac{1}{\bar{a}}(K_{\phi_2(a)}(z) - K_{\phi_2(0)}(z)) = \frac{1}{\bar{a}}(\frac{1}{1-\overline{\phi_2(a)}z} - \frac{1}{1-\overline{\phi_2(0)}z})$. We thus have

$$C_{\phi_1} C_{\phi_2}^* S K_a(z) = \frac{1}{\bar{a}} \left(\frac{1}{1 - \overline{\phi_2(a)} \phi_1(z)} - \frac{1}{1 - \overline{\phi_2(0)} \phi_1(z)} \right)$$

and

$$S^* C_{\phi_1} C_{\phi_2}^* S K_a(z) = \frac{1}{\bar{a}z} \left[\frac{1}{1 - \overline{\phi_2(a)} \phi_1(z)} - \frac{1}{1 - \overline{\phi_2(a)} \phi_1(0)} - \frac{1}{1 - \overline{\phi_2(0)} \phi_1(z)} + \frac{1}{1 - \overline{\phi_2(0)} \phi_1(0)} \right].$$

We now have:

Theorem 4. *For analytic self-maps of the disk ϕ_1 and ϕ_2, $C_{\phi_1} C_{\phi_2}^*$ is a Toeplitz operator if and only if*

$$\frac{1 - \bar{a}z}{1 - \overline{\phi_2(a)} \phi_1(z)} = \frac{1}{1 - \overline{\phi_2(a)} \phi_1(0)} + \frac{1}{1 - \overline{\phi_2(0)} \phi_1(z)} - \frac{1}{1 - \overline{\phi_2(0)} \phi_1(0)} \tag{3.2}$$

for all $a, z \in \mathbb{D}$.

Proof. Equating $C_{\phi_1} C_{\phi_2}^* K_a(z)$ and $S^* C_{\phi_1} C_{\phi_2}^* S K_a(z)$ by putting Eq. (3.1) together with the equation above, we obtain Eq. (3.2). We note that $C_{\phi_1} C_{\phi_2}^* = S^* C_{\phi_1} C_{\phi_2}^* S$ if and only if their actions are equal on each kernel function $K_a(z)$, and it is enough, by continuity of the operators, to consider those kernels with $a \neq 0$. Note that Eq. (3.2) is easily seen to hold when $a = 0$. $\square$

We next ask the question, for analytic self-maps of the unit disk ϕ_1 and ϕ_2: If $C_{\phi_1} C_{\phi_2}^*$ were a Toeplitz operator, what Toeplitz operator would it be? I.e., what would the symbol g have to be in order to have $C_{\phi_1} C_{\phi_2}^* = T_g$? It is easy, from the definition of a Toeplitz operator, to see that if we write $g(z) = h(z) + \overline{k(z)}$ where h and k are analytic and $k(0) = 0$ (breaking g into its analytic and co-analytic parts), then we know that $h(z) = T_g 1(z)$ and, since $T_g^* = T_{\bar{g}}$, and $\overline{g(z)} = \overline{h(z)} + k(z)$, we know $k(z) = T_g^* 1(z) - T_g^* 1(0)$. If we are to have $C_{\phi_1} C_{\phi_2}^* = T_g$, then

$$h(z) = T_g 1(z) = C_{\phi_1} C_{\phi_2}^* 1(z) = C_{\phi_1} C_{\phi_2}^* K_0(z) = \frac{1}{1 - \overline{\phi_2(0)}\phi_1(z)}$$

as before, and

$$T_g^* 1(z) = C_{\phi_2} C_{\phi_1}^* 1(z) = C_{\phi_2} C_{\phi_1}^* K_0(z) = \frac{1}{1 - \overline{\phi_1(0)}\phi_2(z)},$$

so $k(z) = \frac{1}{1 - \overline{\phi_1(0)}\phi_2(z)} - \frac{1}{1 - \overline{\phi_1(0)}\phi_2(0)}$, and

$$g(z) = \frac{1}{1 - \overline{\phi_2(0)}\phi_1(z)} + \frac{1}{1 - \overline{\phi_1(0)}\phi_2(z)} - \frac{1}{1 - \overline{\phi_1(0)}\phi_2(0)}. \tag{3.3}$$

If we let $a = z$ in Eq. (3.2), we see that if $C_{\phi_1} C_{\phi_2}^*$ is a Toeplitz operator, then

$$\frac{1 - |z|^2}{1 - \overline{\phi_2(z)}\phi_1(z)} = \frac{1}{1 - \overline{\phi_2(z)}\phi_1(0)} + \frac{1}{1 - \overline{\phi_2(0)}\phi_1(z)} - \frac{1}{1 - \overline{\phi_2(0)}\phi_1(0)}. \tag{3.4}$$

This then tells us

Corollary 1. *If $\phi_1 \neq \phi_2$, then $C_{\phi_1} C_{\phi_2}^*$ is not Toeplitz.*

Proof. Notice that the right hand sides of Eqs. (3.3) and (3.4) are equal. This tells us that if $C_{\phi_1} C_{\phi_2}^*$ is Toeplitz, its symbol must be $\frac{1 - |z|^2}{1 - \overline{\phi_2(z)}\phi_1(z)}$. Since $\phi_1 \neq \phi_2$ are analytic in the unit disk, the boundary uniqueness property tells us that $\overline{\phi_2}\phi_1$ cannot be 1 on a set of positive measure on the boundary. Thus $\frac{1 - |z|^2}{1 - \overline{\phi_2(z)}\phi_1(z)}$ has boundary function which is zero a.e., i.e., the Toeplitz operator would be the zero operator. This is impossible if it is equal to $C_{\phi_1} C_{\phi_2}^*$. □

What can we say when in the special case where $\phi_1 = \phi_2$ ($= \phi$, say)? From Theorem 4 we get:

Corollary 2. *For an analytic self-map of the disk ϕ, $C_\phi C_\phi^*$ is a Toeplitz operator if and only if*

$$\frac{1 - \bar{a}z}{1 - \overline{\phi(a)}\phi(z)} = \frac{1}{1 - \overline{\phi(a)}\phi(0)} + \frac{1}{1 - \overline{\phi(0)}\phi(z)} - \frac{1}{1 - |\phi(0)|^2}$$

for all $a, z \in \mathbb{D}$.

As before, if we let $a = z$, we see that if $C_\phi C_\phi^*$ is a Toeplitz operator, then from Corollary 2 it must be true that

$$\frac{1 - |z|^2}{1 - |\phi(z)|^2} = \frac{1}{1 - \overline{\phi(z)}\phi(0)} + \frac{1}{1 - \overline{\phi(0)}\phi(z)} - \frac{1}{1 - |\phi(0)|^2}. \tag{3.5}$$

From lines (3.3) and (3.5), we get

Corollary 3. *If $C_\phi C_\phi^*$ is a Toeplitz operator, say T_g, then*

$$g(z) = \frac{1}{1 - \overline{\phi(z)}\phi(0)} + \frac{1}{1 - \overline{\phi(0)}\phi(z)} - \frac{1}{1 - |\phi(0)|^2} = \frac{1 - |z|^2}{1 - |\phi(z)|^2}.$$

It is clear that when $\phi(z) = \lambda z$ for some λ with $|\lambda| = 1$, C_ϕ will be a unitary operator. This can be seen in a number of ways, for example, look at the matrix for C_ϕ – it is diagonal, with entries along the diagonal which are just successive powers of λ. $C_\phi C_\phi^*$ is the identity operator, which is Toeplitz. Are there other self-maps of the disk ϕ for which $C_\phi C_\phi^*$ is Toeplitz?

Lemma 3. *For an analytic self-map of the disk ϕ with $\phi(0) = 0$, $C_\phi C_\phi^*$ is Toeplitz if and only if $\phi(z) = \lambda z$ for some λ with $|\lambda| = 1$, i.e., ϕ is the identity function or a rotation.*

Proof. The only part to prove is the "only if" part. By Corollary 3 above, if $C_\phi C_\phi^*$ is Toeplitz and $\phi(0) = 0$, the symbol for the Toeplitz operator is the boundary function of $\frac{1 - |z|^2}{1 - |\phi(z)|^2} = 1$ a.e. on the unit circle. This tells us that $|\phi(z)|^2 = |z|^2$ in the disk, which implies that $\phi(z) = \lambda z$ for some constant λ with $|\lambda| = 1$. □

We would like to know if these simple functions, the identity and rotations, are the only self-maps of the disk for which $C_\phi C_\phi^*$ is Toeplitz. It turns out that, indeed, these are the only ones.

Theorem 5. *If ϕ is an analytic self-map of the disk, then $C_\phi C_\phi^*$ is Toeplitz if and only if $\phi(z) = \lambda z$ for some λ with $|\lambda| = 1$.*

Proof. Again, we need only prove the "only if" part. Because of Lemma 3 we need only consider the case where $\phi(0) \neq 0$, since if $\phi(0) = 0$, then we already know exactly which ϕ make $C_\phi C_\phi^*$ Toeplitz. We will show that no such ϕ with $\phi(0) \neq 0$ can make $C_\phi C_\phi^*$ a Toeplitz operator. For purposes of contradiction, assume there is a self-map of the disk ϕ with $\phi(0) \neq 0$ and such that $C_\phi C_\phi^*$ is Toeplitz.

Since ϕ certainly cannot be a constant function, we can pick some $a \in \mathbb{D}$, $a \neq 0$ at which $\phi(a) \neq 0$ and also $\phi(a) \neq \phi(0)$, and use this in Corollary 2 to get

$$\frac{1 - \bar{a}z}{1 - \overline{\phi(a)}\phi(z)} = \frac{1}{1 - \overline{\phi(a)}\phi(0)} + \frac{1}{1 - \overline{\phi(0)}\phi(z)} - \frac{1}{1 - |\phi(0)|^2}$$

for all $z \in \mathbb{D}$. To simplify the notation (a little), we can let $b = \overline{\phi(a)}$, $d = \overline{\phi(0)}$ and $c = \frac{1}{1 - \overline{\phi(a)}\phi(0)} - \frac{1}{1 - |\phi(0)|^2}$. By our choice of a, none of these new constants are zero. The equation above becomes

$$\frac{1 - \bar{a}z}{1 - b\phi(z)} = \frac{1}{1 - d\phi(z)} + c.$$

Clearing the denominators and gathering terms, we see that we must have

$$bcd\big(\phi(z)\big)^2 + (b + d - bc - cd - \bar{a}dz)\phi(z) + c + \bar{a}z = 0.$$

The coefficient on $\phi(z)^2$ is nonzero, so we can divide through by it, complete the square, and get

$$\big(\phi(z) + (rz + s)\big)^2 = tz^2 + uz + v$$

for some constants $r, s, t, u,$ and v. Now the left side is the square of an analytic function (on $\mathbb{D}$), and for it to equal a quadratic, it must be a linear function, from which we see that ϕ itself must be a linear function, i.e., $\phi(z) = \alpha z + \beta$ for some constants α and β. Also, we know that β cannot be zero, since $\phi(0) \neq 0$. But now we see that $\phi(z)$ can have modulus 1 on only a set of measure zero on $\partial\mathbb{D}$ (at most one point). By Corollary 3, if $C_\phi C_\phi^*$ were Toeplitz, its symbol would have to be $\frac{1 - |z|^2}{1 - |\phi(z)|^2}$, which would be zero a.e. on $\partial\mathbb{D}$, i.e., the Toeplitz operator would be the zero operator, and this is impossible. This completes the proof. $\quad\square$

The question of when $C_\phi C_\phi^*$ is uniformly Toeplitz seems to be more difficult. But we have the following result which generalizes Theorem 1.1 in [10] with a shorter proof.

Theorem 6. *If $\phi_1 \neq \phi_2$, then $C_{\phi_1} C_{\phi_2}^*$ is not uniformly Toeplitz unless $C_{\phi_1} C_{\phi_2}^*$ is compact. Similarly, if $\phi_1 \neq \phi_2$, then $C_{\phi_1}^* C_{\phi_2}$ is not uniformly Toeplitz unless $C_{\phi_1}^* C_{\phi_2}$ is compact.*

Proof. Assume $C_{\phi_1} C_{\phi_2}^* = T_f + K$ for some nonzero Toeplitz operator T_f and compact operator K. Note that

$$C_{\phi_1} C_{\phi_2}^* - C_{\phi_1} SS^* C_{\phi_2}^* = C_{\phi_1}\big(I - SS^*\big)C_{\phi_2}^* = C_{\phi_1}(e_0 \otimes e_0)C_{\phi_2}^* = e_0 \otimes e_0.$$

Here $e_0 \otimes e_0$ represents the rank 1 operator of projection onto the subspace of constant functions. On the other hand,

$$C_{\phi_1} C_{\phi_2}^* - C_{\phi_1} SS^* C_{\phi_2}^* = C_{\phi_1} C_{\phi_2}^* - T_{\phi_1} C_{\phi_1} C_{\phi_2}^* T_{\phi_2}^* = T_f + K - T_{\phi_1}(T_f + K)T_{\phi_2}^* = T_f - T_{\phi_1} T_f T_{\phi_2}^* + K + T_{\phi_1} K T_{\phi_2}^*.$$

Therefore $T_f - T_{\phi_1} T_f T_{\phi_2}^*$ is compact which implies that $f = \phi_1 f \overline{\phi_2}$, or $(1 - \phi_1\overline{\phi_2})f = 0$ a.e. on $\partial\mathbb{D}$. Again, we note that since $\phi_1 \neq \phi_2$, $1 - \phi_1\overline{\phi_2} \neq 0$ a.e. on $\partial\mathbb{D}$. Thus we conclude $f = 0$. For the proof that if $\phi_1 \neq \phi_2$, then $C_{\phi_1}^* C_{\phi_2}$ is not uniformly Toeplitz unless $C_{\phi_1}^* C_{\phi_2}$ is compact, see Remark 1 after Theorem 3. $\quad\square$

From the calculation above we get the following result:

Corollary 4. *If $C_\phi C_\phi^* = T_f + K$, for some compact operator K, then $(1 - |\phi|^2)f = 0$ a.e. on $\partial\mathbb{D}$.*

3.1. Examples of the Toeplitzness of $C_\phi C_\phi^*$

In contrast to the earlier theorem which told us that $C_\phi^* C_\phi$ is always Toeplitz if ϕ is an inner function and, in fact, strongly asymptotically Toeplitz for all ϕ, we will see by means of some examples that we get less Toeplitzness for $C_\phi C_\phi^*$.

Consider first the Möbius functions $\phi_a(z) = \frac{a - z}{1 - \bar{a}z}$ for $a \in \mathbb{D}$. These are automorphisms of the disk. The result below can be obtained in a different form in [2, Corollary 3]. We present it here with a more direct, computational proof:

Example 1. For each nonzero $a \in \mathbb{D}$, $C_{\phi_a} C_{\phi_a}^*$ is a Toeplitz operator plus a rank 1 operator.

Proof. In fact, what we will show is that we can compute the matrix explicitly for $C_{\phi_a} C_{\phi_a}^*$ (with respect to the standard basis for H^2, $\{z^n: n = 0, 1, 2, \ldots\}$). We will show that $C_{\phi_a} C_{\phi_a}^*$ has matrix $\begin{bmatrix} \alpha & \bar{\gamma} & 0 & 0 & \cdots \\ \gamma & \beta & \bar{\gamma} & 0 & \ddots \\ 0 & \gamma & \beta & \bar{\gamma} & \ddots \\ 0 & 0 & \gamma & \beta & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$, where $\alpha = \frac{1}{1-|a|^2}$, $\beta = \frac{1+|a|^2}{1-|a|^2}$,

and $\gamma = -\frac{\bar{a}}{1-|a|^2}$. Note that if $a = 0$, this is the identity matrix. For any other $a \in \mathbb{D}$, the matrix is a tri-diagonal matrix except for the upper-left entry, which is a different number from those in the rest of the main diagonal. This is then the matrix for the operator $T_\psi - \frac{|a|^2}{1-|a|^2} e_0 \otimes e_0$, where the symbol ψ for the Toeplitz operator is given by $\psi(z) = \bar{\gamma} \frac{1}{z} + \beta + \gamma z = -\frac{a}{1-|a|^2} \frac{1}{z} + \frac{1+|a|^2}{1-|a|^2} - \frac{\bar{a}}{1-|a|^2} z$. Note that the constant $-\frac{|a|^2}{1-|a|^2}$ which multiplies the rank one operator $e_0 \otimes e_0$ is just $\alpha - \beta$.

To see that this is the correct matrix, we can compute the entries in the matrix directly using the formula from Cowen and MacCluer [5] for the adjoint of a composition operator with linear fractional symbol. Here, this formula tells us that $C_{\phi_a}^* = T_{\frac{1}{1-\bar{a}z}} C_{\phi_a} T_{1-\bar{a}z}^*$. We can use this formula to compute the m,nth entry in the matrix ($m, n = 0, 1, 2, \ldots$), which is given by $\langle C_{\phi_a} C_{\phi_a}^* z^n, z^m \rangle = \langle C_{\phi_a}^* z^n, C_{\phi_a}^* z^m \rangle$. We use $T_{1-\bar{a}z}^* = T_{1-a\bar{z}}$ to compute, when $n \geq 1$,

$$C_{\phi_a}^* z^n = T_{\frac{1}{1-\bar{a}z}} C_{\phi_a} T_{1-a\bar{z}} z^n = \frac{1}{1-\bar{a}z} C_{\phi_a}(z^n - az^{n-1}) = \frac{z(|a|^2-1)}{(1-\bar{a}z)(a-z)} \left(\frac{a-z}{1-\bar{a}z}\right)^n = \frac{z(|a|^2-1)}{(1-\bar{a}z)(a-z)} \phi_a^n(z). \quad (3.6)$$

There are several cases to consider. First, when m and n are both zero, we have $\langle C_{\phi_a}^* z^n, C_{\phi_a}^* z^m \rangle = \langle C_{\phi_a}^* 1, C_{\phi_a}^* 1 \rangle = \langle C_{\phi_a}^* K_0, C_{\phi_a}^* K_0 \rangle = \langle K_{\phi_a(0)}, K_{\phi_a(0)} \rangle = \langle K_a, K_a \rangle = \frac{1}{1-|a|^2} = \alpha$. When $m = 0$ and $n \geq 1$, we use

$$\langle C_{\phi_a}^* z^n, C_{\phi_a}^* z^0 \rangle = \langle C_{\phi_a}^* z^n, K_a \rangle = \left\langle \frac{z(|a|^2-1)}{(1-\bar{a}z)(a-z)} \left(\frac{a-z}{1-\bar{a}z}\right)^n, K_a \right\rangle.$$

This last quantity can easily be seen to be $-\frac{a}{1-|a|^2} = \bar{\gamma}$ when $n = 1$ and 0 when $n > 1$. The formula on line (3.6) then gives us, for $n \geq m \geq 1$ (and using $|\phi_a(z)| = 1$),

$$\langle C_{\phi_a}^* z^n, C_{\phi_a}^* z^m \rangle = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{(|a|^2-1)^2 z \phi_a^n(z) \overline{z \phi_a^m(z)}}{(1-\bar{a}z)(a-z)\overline{(1-\bar{a}z)(a-z)}} \frac{dz}{z} = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{z(1-|a|^2)^2 \phi_a^{n-m}(z)}{(1-\bar{a}z)^2 (z-a)^2} dz. \quad (3.7)$$

We can evaluate this using residues, giving us, when $n = m$, $\frac{1+|a|^2}{1-|a|^2} = \beta$, and when $n - m = 1$, we again compute using residues that this quantity is $-\frac{a}{1-|a|^2} = \bar{\gamma}$. Finally, when $m > n$, we get entries which are just the conjugates of those with the row and column reversed. Together, these give us the matrix as claimed above. $\square$

The examples discussed above show that for some self-maps of the disk ϕ, $C_\phi C_\phi^*$ is either Toeplitz (when ϕ is just a rotation) or a rank one perturbation of a Toeplitz operator (when ϕ is a single Blaschke factor, other than a rotation). Is it true that for other functions, even for inner functions or even Blaschke products with more than one factor that $C_\phi C_\phi^*$ is similarly close to being a Toeplitz operator? The answer is no.

Example 2. For $\phi(z) = z^2$, the matrix for C_ϕ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & & & & \ddots \end{bmatrix}$$

and the matrix for $C_\phi C_\phi^*$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & & & & \ddots \end{bmatrix}.$$

That is, $C_\phi C_\phi^*$ is the projection operator onto the subspace of H^2 generated by the even powers of z. Similarly one can easily see that when $\phi(z) = z^n$ (for any $n \geqslant 0$), $C_\phi C_\phi^*$ is the projection operator onto the subspace of H^2 generated by the powers of z which are multiples of n. For $n \geqslant 2$, the entries on the diagonal certainly have no limit, so the operators are not even weakly asymptotically Toeplitz.

Remark 2. It shouldn't be surprising that in these last examples $C_\phi C_\phi^*$ turned out to be a projection operator. Whenever ϕ is inner with $\phi(0) = 0$, the composition operator C_ϕ is an isometry, and for any isometry A, AA^* is a projection operator.

4. Questions

We saw in Theorem 2 that the operator $C_\phi^* C_\phi$ is uniformly Toeplitz if an only if $T_{1-\chi(E)} C_\phi$ is compact. For which self-maps of the disk ϕ is $T_{1-\chi(E)} C_\phi$ compact? This certainly includes all inner functions, for which $1 - \chi(E)$ is (a.e.) zero, so $T_{1-\chi(E)} C_\phi$ is the zero operator (and for which it is known, in any case, that $C_\phi^* C_\phi$ is a Toeplitz operator). It also includes all ϕ for which C_ϕ is compact – these have been characterized in [11] and [4]. Are there any others?

Can we characterize those self-maps of the disk ϕ for which $C_\phi C_\phi^*$ is uniformly Toeplitz? These certainly include rotations and Möbius functions (Example 1) but exclude certain other simple inner functions (Example 2).

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