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Asymptotic expansions of the moments of extremes from general error distribution

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ABSTRACT

In this paper, we derive the asymptotic expansions of the moments of normalized partial maxima for general error distribution. A byproduct is to deduce the convergence rates of the moments of normalized maxima to the moments of the corresponding extreme value distribution.

1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with marginal cumulative distribution function (cdf) $F_v \sim \mathrm{GED}(v)$, the general error distribution with shape parameter $v > 0$. The probability density function (pdf) of $\mathrm{GED}(v)$ is defined by

$$f_v(x) = \frac{v \exp(-(1/2)|x/\lambda|^v)}{\lambda 2^{1+1/v} \Gamma(1/v)}, \quad x \in \mathbb{R},$$

where $\lambda = [2^{-2/v}\Gamma(1/v)/\Gamma(3/v)]^{1/2}$ and $\Gamma(\cdot)$ denotes the gamma function [10]. For $v = 2$, $\mathrm{GED}(2)$ reduces to the standard normal distribution.

Recently, asymptotic behaviors related to $\mathrm{GED}(v)$ have been studied in the literature. Peng et al. [11] established the Mills ratio and distributional tail representation of $\mathrm{GED}(v)$, and showed that $F_v \in D(\Lambda)$, i.e., there exist norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\mathbb{P}(M_n \leq a_n x + b_n) - \Lambda(x)\right| = 0, \tag{1.1}$$

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where $M_n = \max_{1\le k\le n} X_k$ denotes the partial maximum of $\{X_n,\ n \ge 1\}$ and $\Lambda(x) = \exp(-e^{-x})$, the Gumble extreme value distribution. Uniform convergence rate of $\mathbb{P}(M_n \le a_n x + b_n) - \Lambda(x)$ has been established by Peng et al. [12] which extended the work of Hall [1] for the case of GED(2). For higher-order expansion of $\mathbb{P}(M_n \le a_n x + b_n)$, see Nair [9] for GED(2) and Jia and Li [3] for general GED(v) with shape parameter $v > 0$.

Moments convergence of extremes was studied by McCord [8], Pickands [13] and Ramachandran [14], see Section 2.1 in Resnick [15]. The relationship between weak convergence and moment convergence of order statistics was considered by Hill and Spruill [2]. The objective of this paper is to establish the higher-order expansions of the moments of M_n for GED(v). Nair [9] derived the higher-order expansions of moments of extremes for standard normal distribution GED(2). Recall that Peng et al. [12] showed that (1.1) holds with norming constants a_n and b_n satisfying the following equations:

$$1 - F_v(b_n) = n^{-1} \quad \text{and} \quad a_n = 2v^{-1}\lambda^v b_n^{1-v}. \tag{1.2}$$

Noting that for $v = 1$ the norming constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}\log n/2$. By Proposition 2.1(iii) in Resnick [15], we have

$$\Delta_r(n) = \mathbf{E}\left(\frac{M_n - b_n}{a_n}\right)^r - \int\limits_{x\in\mathbb{R}} x^r\, d\Lambda(x) \to 0 \tag{1.3}$$

as $n \to \infty$ for all nonnegative integers r. The following work is to establish the asymptotic expansions of $\mathbf{E}((M_n - b_n)/a_n)^r$, from which we can derive the convergence rates of $\Delta_r(n)$. For more related work on asymptotic expansions of distributions and moments of extremes for given distributions, see Peng et al. [12], Liao and Peng [4] and Liao et al. [5–7].

This paper is organized as follows. Section 2 provides the main results, and some necessary auxiliary lemmas and their proofs are given in Section 3. The proofs of the main results are given in Section 4.

2. Main results

In this section, we provide the main results. In the sequel, for nonnegative integers r let

$$m_r(n) = \mathbf{E}\left(\frac{M_n - b_n}{a_n}\right)^r = \int\limits_{x\in\mathbb{R}} x^r dF_v^n(a_n x + b_n)$$

and

$$m_r = \mathbf{E}\xi^r = \int\limits_{x\in\mathbb{R}} x^r d\Lambda(x)$$

respectively denote the rth moments of $(M_n - b_n)/a_n$ and $\xi \sim \Lambda(x)$, and the norming constants a_n and b_n are given by (1.2). The main results are stated as follows.

Theorem 2.1. *Let $\{X_n,\ n \ge 1\}$ be an iid sequence with marginal distribution $F_v \sim$ GED(v). Then,*

(i) *for $v \ne 1$, with norming constants a_n and b_n given by (1.2), we have*

$$b_n^v\left[b_n^v\left(m_r(n) - m_r\right) + \left(1 - v^{-1}\right)\lambda^v r(m_{r+1} + 2m_r)\right]$$

$$\to 2r\lambda^{2v}(1-v^{-1})\Big[\big((1-v^{-1})(r+1)+2\big)m_r+\big((1-v^{-1})(r+1)+1\big)m_{r+1} + \left(\frac{1}{4}(1-v^{-1})(r-1)+\frac{1}{3}(2-v^{-1})\right)m_{r+2}\Big] \tag{2.1}$$

as $n \to \infty$;

(ii) *for $v=1$, with norming constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}\log n/2$, we have*

$$n\left[n\big(m_r(n)-m_r\big)+(-1)^r\frac{r}{2}\Gamma^{(r-1)}(2)\right] \to (-1)^{r-1}\frac{r}{24}\big[8\Gamma^{(r-1)}(3)-3\Gamma^{(r-1)}(4)\big] \tag{2.2}$$

as $n\to\infty$, where $\Gamma^{(r-1)}(s)$ denotes the $(r-1)$th derivative of the gamma function at $x=s$.

Remark 2.1. For the case of $v=2$, i.e., the standard normal case, the result of Theorem 2.1(i) is agreement with Theorem 3.1 in Nair [9].

Noting that $b_n^v \sim 2\lambda^v \log n$ due to (1.2), by Theorem 2.1 we can derive the rates of convergence of moments which is stated as follows.

Theorem 2.2. *For the moments of normalized maxima of* GED(v), *for large n we have*

$$\Delta_r(n) \sim -\frac{(1-v^{-1})r}{2\log n}(m_{r+1}+2m_r)$$

if $v\neq 1$, and

$$\Delta_r(n) \sim (-1)^{r+1}\frac{r}{2n}\Gamma^{(r-1)}(2)$$

if $v=1$.

3. Auxiliary lemmas

In order to derive the main results, we need some auxiliary results. The first one is about the Mills-type inequalities of the GED(v).

Lemma 3.1. *Let $F_v(x)$ and $f_v(x)$ denote the cdf and pdf of* GED(v), *respectively. Then,*

(i) *for $v>1$ we have*

$$\frac{2\lambda^v}{v}x^{1-v}\left(1+\frac{2(v-1)\lambda^v}{v}x^{-v}\right)^{-1} < \frac{1-F_v(x)}{f_v(x)} < \frac{2\lambda^v}{v}x^{1-v} \tag{3.1}$$

for all $x>0$, where $\lambda = [2^{-2/v}\Gamma(1/v)/\Gamma(3/v)]^{1/2}$;

(ii) *for $0<v<1$ we have*

$$\frac{2\lambda^v}{v}x^{1-v} < \frac{1-F_v(x)}{f_v(x)} < \frac{2\lambda^v}{v}x^{1-v}\left(1+\frac{2(v-1)\lambda^v}{v}x^{-v}\right)^{-1} \tag{3.2}$$

for all $x>\lambda[2(1/v-1)]^{1/v}$.

Proof. Note that Lemma 2.1 in Peng et al. [11] shows that (3.1) holds as $v>1$. The rest is to prove that (3.2) holds if $0<v<1$.

For $t > [2(1/v-1)]^{1/v}$,

$$\frac{1}{t^v}\int_t^\infty \exp\left(-\frac{y^v}{2}\right)dy > \int_t^\infty y^{-v}\exp\left(-\frac{y^v}{2}\right)dy$$
$$= \frac{\exp(-t^v/2)}{(v-1)t^{v-1}} - \frac{v}{2(v-1)}\int_t^\infty \exp\left(-\frac{y^v}{2}\right)dy,$$

which implies that

$$\left(\frac{vt^{v-1}}{2}\right)^{-1}\exp\left(-\frac{t^v}{2}\right) < \int_t^\infty \exp\left(-\frac{y^v}{2}\right)dy < \left(\frac{v-1}{t}+\frac{vt^{v-1}}{2}\right)^{-1}\exp\left(-\frac{t^v}{2}\right)$$

holds if $0 < v < 1$. So, by the definition of the GED(v) with $0 < v < 1$, we can get

$$1-F_v(x) = \frac{v}{2^{1+1/v}\Gamma(1/v)}\int_{x/\lambda}^\infty \exp\left(-\frac{y^v}{2}\right)dy$$
$$< f_v(x)\frac{2\lambda^v}{v}x^{1-v}\left(1+\frac{2(v-1)\lambda^v}{v}x^{-v}\right)^{-1}$$

and

$$1-F_v(x) > f_v(x)\frac{2\lambda^v}{v}x^{1-v}$$

as $x > \lambda[2(1/v-1)]^{1/v}$. The desired result follows. □

The following distributional expansion of maximum for GED(v) is due to Jia and Li [3].

Lemma 3.2. *Let F_v denote the cdf of* GED(v). *Then,*

(i) *for $v \neq 1$, with norming constants a_n and b_n given by (1.2) we have*

$$b_n^v\left[b_n^v\left(F_v^n(a_nx+b_n)-\Lambda(x)\right)-k_v(x)\Lambda(x)\right] \to \left(w_v(x)+\frac{k_v^2(x)}{2}\right)\Lambda(x) \tag{3.3}$$

as $n\to\infty$, where $k_v(x)$ and $w_v(x)$ are respectively given by

$$k_v(x) = \left(1-v^{-1}\right)\lambda^v\left(x^2+2x\right)e^{-x}$$

and

$$w_v(x) = \left(v^{-1}-1\right)\lambda^{2v}\left[4x+2x^2+\frac{2}{3}\left(2-v^{-1}\right)x^3+\frac{1}{2}\left(1-v^{-1}\right)x^4\right]e^{-x};$$

(ii) *for $v=1$, with norming constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}\log n/2$, we have*

$$n\left[n\left(F_1^n(a_nx+b_n)-\Lambda(x)\right)+\frac{1}{2}e^{-2x}\Lambda(x)\right] \to \left(\frac{1}{8}e^{-4x}-\frac{1}{3}e^{-3x}\right)\Lambda(x) \tag{3.4}$$

as $n\to\infty$.

Lemma 3.3. *Let the norming constant b_n be given by (1.2). For any constant $0 < c < 1$ and arbitrary nonnegative real numbers i and j, we have*

$$\lim_{n\to\infty} b_n^i \int_{cb_n^{v/3}}^{\infty} x^j \big(1 - \Lambda(x)\big) dx = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n^i \int_{cb_n^{v/3}}^{\infty} x^j d\Lambda(x) = 0. \tag{3.5}$$

Proof. It follows from the fact $1 - x < e^{-x} < 1$ for $x > 0$ that

$$\begin{aligned} b_n^i \int_{cb_n^{v/3}}^{\infty} x^j \big(1 - \exp(-e^{-x})\big) dx &\le b_n^i \int_{cb_n^{v/3}}^{\infty} x^j e^{-x} dx \\ &\le b_n^i \exp\left(-\frac{c}{2} b_n^{\frac{v}{3}}\right) \int_{cb_n^{v/3}}^{\infty} x^j \exp\left(-\frac{x}{2}\right) dx \to 0 \end{aligned}$$

as $n \to \infty$. Similarly,

$$b_n^i \int_{cb_n^{v/3}}^{\infty} x^j d\Lambda(x) \le b_n^i \int_{cb_n^{v/3}}^{\infty} x^j e^{-x} dx \to 0$$

as $n \to \infty$. The desired result follows. □

Lemma 3.4. *Assume that the shape parameter $v \neq 1$, then for any constant $0 < d < 1$ and arbitrary nonnegative real numbers i and j, we have*

$$\lim_{n\to\infty} b_n^i \int_{-\infty}^{-d \log b_n} |x|^j \Lambda(x) dx = 0, \qquad \lim_{n\to\infty} b_n^i \int_{-\infty}^{-d \log b_n} |x|^j d\Lambda(x) = 0 \tag{3.6}$$

and

$$\lim_{n\to\infty} b_n^i \int_{-\infty}^{-d \log b_n} |x|^j F_v^n(a_n x + b_n) dx = 0. \tag{3.7}$$

Proof. Note that $b_n \to \infty$ as $n \to \infty$ since $1 - F_v(b_n) = n^{-1}$. For $0 < d < 1$, we have

$$\begin{aligned} b_n^i \int_{-\infty}^{-d \log b_n} |x|^j \Lambda(x) dx &\le b_n^i \exp\left(-\frac{b_n^d}{2}\right) \int_{-\infty}^{-1} |x|^j \exp\left(-\frac{e^{-x}}{2}\right) dx \\ &= b_n^i \exp\left(-\frac{b_n^d}{2}\right) \int_{1}^{\infty} x^j \exp\left(-\frac{e^{x}}{2}\right) dx \to 0 \end{aligned}$$

as $n \to \infty$, and

$$b_n^i \int_{-\infty}^{-d \log b_n} |x|^j d\Lambda(x) \le b_n^i \exp\left(-\frac{b_n^d}{2}\right) \int_{1}^{\infty} x^j e^x \exp\left(-\frac{e^x}{2}\right) dx \to 0$$

as $n \to \infty$ since $\int_1^\infty x^j e^x \exp(-\frac{e^x}{2}) dx < \infty$.

Note that $a_n = 2v^{-1}\lambda^v b_n^{1-v}$ implies that $b_n - da_n \log b_n \to \infty$ as $n \to \infty$. By the following facts

$$(1-y)^v < 1 - vy + v(v-1)y^2 \quad \text{for } 0 < y < \frac{1}{4},\ v > 1$$

and

$$(1-y)^v < 1 - vy \quad \text{for } 0 < y < 1,\ 0 < v < 1$$

and Lemma 3.1, we can derive that

$$b_n^k F_v^n(b_n - da_n \log b_n) \to 0$$

as $n \to \infty$. So,

$$\begin{aligned}
b_n^i \int\limits_{-\infty}^{-d\log b_n} |x|^j F_v^n(a_n x + b_n)dx &\le b_n^i a_n^{-j-1} F_v^{n-1}(b_n - da_n \log b_n) \int\limits_{-\infty}^{b_n - da_n \log b_n} |y - b_n|^j F_v(y)dy \\
&\le b_n^i a_n^{-j-1} F_v^{n-1}(b_n - da_n \log b_n) \int\limits_{-\infty}^{0} |y - b_n|^j F_v(y)dy \\
&\quad + b_n^{i+j+1} a_n^{-j-1} F_v^n(b_n - da_n \log b_n) \int\limits_{0}^{1 - da_n b_n^{-1} \log b_n} |y-1|^j dy \\
&\le \sum_{s=0}^{j} \binom{j}{s} b_n^{i+s} a_n^{-j-1} F_v^{n-1}(b_n - da_n \log b_n) \int\limits_0^\infty y^{j-s} F_v(-y)dy \\
&\quad + b_n^{i+j+1} a_n^{-j-1} F_v^n(b_n - da_n \log b_n) \int\limits_0^1 (1-y)^j dy \\
&\to 0
\end{aligned}$$

as $n \to \infty$ since $\int_0^\infty y^r F_v(-y)dy$ is finite for all $r > 0$. The desired result follows. □

Lemma 3.5. *For any constant $0 < c < 1$ and arbitrary nonnegative real numbers i and j, we have*

$$\lim_{n\to\infty} b_n^i \int\limits_{cb_n^{v/3}}^{\infty} x^j \big(1 - F_v^n(a_n x + b_n)\big)dx = 0 \tag{3.8}$$

and

$$\lim_{x\to\infty} x^i \big(1 - F_v^n(a_n x + b_n)\big) = 0. \tag{3.9}$$

Proof. By Corollary 3 in Peng et al. [11] and Lemma 1 in Jia and Li [3], we have the following distributional tail representation of GED(v) with $v > 0$:

$$1 - F_v(x) = c(x) \exp\left(- \int\limits_\lambda^x \frac{g(t)}{f(t)} dt \right)$$

for large $x > 0$, where $c(x) \to c > 0$, $g(x) \to 1$ as $x \to \infty$, and the auxiliary function $f(x) = 2v^{-1}\lambda^v x^{1-v} > 0$ on (λ, ∞) with $\lim_{x\to\infty} f'(x) = 0$. Recall that $1 - F_v(b_n) = n^{-1}$ and $a_n = f(b_n)$. By arguments similar to Lemma 2.2(a) in Resnick [15], we have

$$1 - F_v^n(a_n x + b_n) \le (1+\varepsilon)^2(1+\varepsilon x)^{-\varepsilon^{-1}+1} \tag{3.10}$$

for $x > 0$, arbitrary $\varepsilon > 0$ and large n. So, for $0 < \varepsilon < \frac{v}{3i+vj+3v}$, we have

$$\begin{aligned}
0 &\le \lim_{n\to\infty} b_n^i \int_{cb_n^{v/3}}^{\infty} x^j\big(1 - F_v^n(a_n x + b_n)\big)dx \\
&\le \lim_{n\to\infty} (1+\varepsilon)^2\big(c\varepsilon + b_n^{-v/3}\big)^{-3i/v} \int_{cb_n^{v/3}}^{\infty} x^j(1+\varepsilon x)^{-j-2}dx \\
&= 0
\end{aligned}$$

by (3.10) since $\int_1^\infty x^j(1+\varepsilon x)^{-j-2}dx$ is finite for all $j \ge 0$.

From (3.10), for $0 < \varepsilon < \frac{1}{i+2}$, we have

$$0 \le \lim_{x\to\infty} \sup x^i\big(1 - F_v^n(a_n x + b_n)\big) \le \lim_{x\to\infty} (1+\varepsilon)^2 x^i (1+\varepsilon x)^{-i-1} = 0.$$

The proof is complete. □

Lemma 3.6. *Let $h_n(x) = n\log F_v(a_n x + b_n) + e^{-x}$, where a_n and b_n are given by (1.2), i.e.,*

$$1 - F_v(b_n) = n^{-1}, \qquad a_n = 2v^{-1}\lambda^v b_n^{1-v}.$$

Then, for $v \ne 1$ and sufficiently large n we have

$$\big|h_n(x)\big| < 3$$

uniformly for all $-d\log b_n < x < cb_n^{v/3}$ with $0 < c < 1$ and $0 < d < \alpha = \min(1, v)$.

Proof. We only prove the case of $v > 1$. For the case of $0 < v < 1$, the proofs are similar and details are omitted here.

By integration by parts, we have

$$\begin{aligned}
1 - F_v(x) &= 2v^{-1}\lambda^v f_v(x)x^{1-v} - r(x) \\
&= 2v^{-1}\lambda^v f_v(x)x^{1-v}\big(1 - 2\big(1 - v^{-1}\big)\lambda^v x^{-v}\big) + s(x)
\end{aligned} \tag{3.11}$$

for large $x > 0$ and for $v > 1$, where

$$0 < r(x) < 4v^{-1}\big(1 - v^{-1}\big)f_v(x)x^{1-2v} \quad \text{and} \quad s(x) > 0. \tag{3.12}$$

Let $\varphi_n(x) = 1 - F_v(a_n x + b_n)$ and

$$n\log F_v(a_n x + b_n) = -n\varphi_n(x) - R_n(x),$$

where

$$0 < R_n(x) < \frac{n\varphi_n^2(x)}{2(1-\varphi_n(x))}$$

since $-y - \frac{y^2}{2(1-y)} < \log(1-y) < -y$ for $0 < y < 1$. Hence,

$$\left|h_n(x)\right| = \left|-n\varphi_n(x) + e^{-x} - R_n(x)\right| \le \left|-n\varphi_n(x) + e^{-x}\right| + R_n(x). \tag{3.13}$$

For large n and $-d\log b_n < x < cb_n^{v/3}$, we have

$$\varphi_n(x) < \varphi_n(-d\log b_n) = 1 - F_v(b_n - da_n\log b_n) < c_0 < 1$$

and

$$\begin{aligned}
0 < R_n(x) &< \frac{1}{1-c_0}\frac{(1-F_v(a_nx+b_n))^2}{1-F_v(b_n)} \\
&< \frac{\lambda^{v-1}}{(1-c_0)2^{1/v}\Gamma(1/v)}\frac{1+2(1-v^{-1})\lambda^v b_n^{-v}}{b_n^{v-1}(1-2dv^{-1}\lambda^v b_n^{-v}\log b_n)^{2v-2}}\exp\left(-\frac{b_n^v}{2\lambda^v} + 2d\log b_n\right) \\
&< 1
\end{aligned} \tag{3.14}$$

due to (3.1) and the inequality $1 + vy \le (1+y)^v$ as $-1 < y < 1$ for $v > 1$. For $x \ge 0$, we have

$$\left|-n\varphi_n(x) + e^{-x}\right| \le n\varphi_n(x) + e^{-x} \le n\bigl(1 - F_v(b_n)\bigr) + 1 = 2. \tag{3.15}$$

It follows from (3.14) and (3.15) that $|h_n(x)| < 3$ holds for $0 \le x < cb_n^{v/3}$.

Next, we consider the case of $-d\log b_n < x < 0$. By (3.11) and (3.12), we have

$$\begin{aligned}
-n\varphi_n(x) + e^{-x} &= -\frac{1-F_v(a_nx+b_n)}{1-F_v(b_n)} + e^{-x} \\
&= -\frac{1-2^{1/v}\Gamma(1/v)\lambda^{1-v}r(a_nx+b_n)(a_nx+b_n)^{v-1}\exp(\frac{1}{2\lambda^v}(a_nx+b_n)^v)}{(1+2v^{-1}\lambda^v b_n^{-v}x)^{v-1}[1-2^{1/v}\Gamma(1/v)\lambda^{1-v}r(b_n)b_n^{v-1}\exp(\frac{1}{2\lambda^v}b_n^v)]} \\
&\quad\times\exp\left[-\frac{1}{2\lambda^v}\bigl((a_nx+b_n)^v - b_n^v\bigr)\right] + e^{-x} \\
&= e^{-x}\bigl(1+2v^{-1}\lambda^v b_n^{-v}x\bigr)^{1-v}C_n(x),
\end{aligned}$$

where

$$C_n(x) = \bigl(1+2v^{-1}\lambda^v b_n^{-v}x\bigr)^{v-1} - \exp\left(-\frac{b_n^v}{2\lambda^v}\sum_{k=2}^{\infty}\binom{v}{k}\bigl(2v^{-1}\lambda^v b_n^{-v}x\bigr)^k\right)\frac{1-\mu(a_nx+b_n)}{1-\mu(b_n)}$$

with $\mu(x) = 2^{1/v}\Gamma(1/v)\lambda^{1-v}r(x)x^{v-1}\exp(\frac{1}{2\lambda^v}x^v)$ satisfying $0 < \mu(x) < 2(1-v^{-1})\lambda^v x^{-v}$ for large $x > 0$, where

$$\binom{v}{k} = v(v-1)\cdots(v-k+1)/k!, \quad k! = k(k-1)\cdots 2\cdot 1$$

for $v \in \mathbb{R}$ and positive integer k.

For $-d\log b_n < x < 0$, let

$$T_n(x) = \sum_{k=2}^{\infty}\binom{v}{k}\bigl(2v^{-1}\lambda^v b_n^{-v}x\bigr)^k \quad\text{and}\quad Q_n = \sum_{k=1}^{\infty}\binom{v-1}{k}\bigl(2v^{-1}\lambda^v b_n^{-v}x\bigr)^k,$$

we have $T_n(x) > 0$ and $Q_n(x) < 0$ since $1 + vy < (1 + y)^v < 1$ as $v > 1$ and $-1 < y < 0$. Noting that $1 - y < e^{-y} < 1$ for $y > 0$, we have

$$\begin{aligned}
C_n(x) &< \big(1 + 2v^{-1}\lambda^v b_n^{-v} x\big)^{v-1} - \left(1 - \frac{b_n^v}{2\lambda^v} T_n(x)\right) \frac{1 - \mu(a_n x + b_n)}{1 - \mu(b_n)} \\
&< 1 - \left(1 - \frac{b_n^v}{2\lambda^v} T_n(x)\right)\big(1 - \mu(a_n x + b_n)\big) \\
&< \frac{b_n^v}{2\lambda^v} T_n(x) + 2\big(1 - v^{-1}\big)\lambda^v (a_n x + b_n)^{-v}
\end{aligned}$$

and

$$\begin{aligned}
C_n(x) &> \big(1 + 2v^{-1}\lambda^v b_n^{-v} x\big)^{v-1} - \frac{1 - \mu(a_n x + b_n)}{1 - \mu(b_n)} \\
&> 1 + Q_n(x) - \frac{1}{1 - \mu(b_n)} \\
&> 2Q_n(x) - 4\big(1 - v^{-1}\big)\lambda^v b_n^{-v}.
\end{aligned}$$

Hence for large n we have

$$\big|C_n(x)\big| < \left|\frac{b_n^v}{2\lambda^v} T_n(x)\right| + 2\big|Q_n(x)\big| + 6\big(1 - v^{-1}\big)\lambda^v (a_n x + b_n)^{-v}$$

if $-d \log b_n < x < 0$. Note that for large n

$$\left|\frac{b_n^v}{2\lambda^v} T_n(x)\right| \le 2d^2\big(1 - v^{-1}\big)\lambda^v b_n^{-v} (\log b_n)^2$$

and

$$\big|Q_n(x)\big| \le 4d\big(1 - v^{-1}\big)\lambda^v b_n^{-v} \log b_n$$

hold uniformly for all $-d \log b_n < x < 0$, and

$$(a_n x + b_n)^{-v} \le b_n^{-v}\big(1 - 2dv^{-1}\lambda^v b_n^{-v} \log b_n\big)^{-v},$$

so that there exists a constant $c_1 > 0$ satisfying

$$\big|C_n(x)\big| < c_1 b_n^{-v} (\log b_n)^2.$$

Hence, for sufficiently large n,

$$\begin{aligned}
\big|-n\varphi_n(x) + e^{-x}\big| &= e^{-x}\big(1 + 2v^{-1}\lambda^v b_n^{-v} x\big)^{1-v}\big|C_n(x)\big| \\
&< c_1\big(1 - 2dv^{-1}\lambda^v b_n^{-v} \log b_n\big)^{1-v} b_n^{d-v} (\log b_n)^2 \\
&< 1
\end{aligned} \tag{3.16}$$

uniformly for all $-d \log b_n < x < 0$. Combining with (3.14) and (3.16), we have $|h_n(x)| < 3$ uniformly for $-d \log b_n < x < 0$. The desired result follows. $\square$

Lemma 3.7. *For $v \neq 1$, let $\alpha = \min(1, v)$. For large n and $-d \log b_n < x < cb_n^{v/3}$,*

$$x^r b_n^v \left[b_n^v \left(F_v^n(a_n x + b_n) - \Lambda(x)\right) - k_v(x)\Lambda(x)\right]$$

is bounded by integrable functions independent of n, with $r > 0$, $0 < c < 1$ and $0 < d < \alpha$, where a_n and b_n are given by (1.2), and $k_v(x)$ is given by $(1 - v^{-1})\lambda^v(x^2 + 2x)e^{-x}$.

Proof. We only consider the case of $v > 1$. For the case of $0 < v < 1$, the proofs are similar and details are omitted here.

By Lemma 3.6, for large n we have

$$\begin{aligned}
& b_n^v \left[b_n^v \left(F_v^n(a_n x + b_n) - \Lambda(x)\right) - k_v(x)\Lambda(x)\right] \\
& \quad < b_n^v \left[b_n^v h_n(x) - k_v(x)\right]\Lambda(x) + b_n^{2v} h_n^2(x)\left(\frac{1}{2} + \exp\left(\left|h_n(x)\right|\right)\right)\Lambda(x) \\
& \quad < b_n^v \left[b_n^v h_n(x) - k_v(x)\right]\Lambda(x) + b_n^{2v} h_n^2(x)\left(\frac{1}{2} + e^3\right)\Lambda(x),
\end{aligned}$$

where $h_n(x) = n \log F_v(a_n x + b_n) + e^{-x}$.

Note that $\int_{-\infty}^{\infty} x^k e^{-tx} \exp(-e^{-x})dx = (-1)^k \Gamma^{(k)}(t)$ is finite for $t > 0$ and nonnegative integers k. In the following we will show that both $|b_n^v(b_n^v h_n(x) - k_v(x))|$ and $|b_n^v h_n(x)|$ are bounded by $m(x)e^{-x}$, where $m(x)$ is a polynomial on x. Since proofs of the above two cases are similar, we only prove that $|b_n^v(b_n^v h_n(x) - k_v(x))|$ is bounded by $m(x)e^{-x}$ here. Rewrite

$$b_n^v \left(b_n^v h_n(x) - k_v(x)\right) = b_n^{2v}\left(-n\varphi_n(x) + e^{-x} - b_n^{-v} k_v(x)\right) - b_n^{2v} R_n(x). \tag{3.17}$$

By (3.14), for large n we have

$$\begin{aligned}
b_n^{2v} R_n(x) &< \frac{\lambda^{v-1}}{(1 - c_0)2^{1/v}\Gamma(1/v)} \frac{(1 + 2(1 - v^{-1})\lambda^v b_n^{-v})b_n^{v+1}}{(1 + 2v^{-1}\lambda^v b_n^{-v} x)^{2v-2}} e^{-x} \exp\left(-\frac{b_n^v}{2\lambda^v} + d \log b_n\right) \\
&< \frac{\lambda^{v-1}}{(1 - c_0)2^{1/v}\Gamma(1/v)} \frac{(1 + 2(1 - v^{-1})\lambda^v b_n^{-v})b_n^{v+d+1}}{(1 - 2dv^{-1}\lambda^v b_n^{-v} \log b_n)^{2v-2}} e^{-x} \exp\left(-\frac{b_n^v}{2\lambda^v}\right) \\
&< e^{-x}
\end{aligned} \tag{3.18}$$

for $-d \log b_n < x < cb_n^{v/3}$. Obviously, $a_n x + b_n > 0$ if $-d \log b_n < x < cb_n^{v/3}$ with large n. It follows from (3.1) that

$$\begin{aligned}
\frac{1 - F_v(a_n x + b_n)}{1 - F_v(b_n)} &< \frac{(a_n x + b_n)^{1-v} f_v(a_n x + b_n)}{(1 + 2(1 - v^{-1})\lambda^v b_n^{-v})^{-1} b_n^{1-v} f_v(b_n)} \\
&= \frac{1 + 2(1 - v^{-1})\lambda^v b_n^{-v}}{(1 + 2v^{-1}\lambda^v b_n^{-v} x)^{v-1}} \exp\left[-\frac{b_n^v}{2\lambda^v}\left(\left(1 + 2v^{-1}\lambda^v b_n^{-v} x\right)^v - 1\right)\right] \\
&< 2e^{-x}
\end{aligned} \tag{3.19}$$

holds for all $-d \log b_n < x < cb_n^{v/3}$ since $1 + vy \leq (1 + y)^v$ for $v > 1$ and $-1 < y < 1$. By Lemma 2 of Jia and Li [3], we have

$$\frac{1 - F_v(b_n)}{1 - F_v(a_n x + b_n)} e^{-x} = A_v(n, x) \exp\left[\int_0^x \left(\frac{(v - 1)a_n}{b_n + a_n t} + \frac{v a_n (b_n + a_n t)^{v-1}}{2\lambda^v} - 1\right)dt\right],$$

where

$$A_v(n,x)=\frac{1+2(v^{-1}-1)\lambda^v b_n^{-v}+4(v^{-1}-1)(v^{-1}-2)\lambda^{2v}b_n^{-2v}+O(b_n^{-3v})}{1+2(v^{-1}-1)\lambda^v(a_nx+b_n)^{-v}+4(v^{-1}-1)(v^{-1}-2)\lambda^{2v}(a_nx+b_n)^{-2v}+O(b_n^{-3v})}$$

with $\lim_{n\to\infty}A_v(n,x)=1$ uniformly for all $-d\log b_n<x<cb_n^{v/3}$. Rewrite

$$\begin{aligned}b_n^{2v}\big(-n\varphi_n(x)+e^{-x}-b_n^{-v}k_v(x)\big)&=\frac{1-F_v(a_nx+b_n)}{1-F_v(b_n)}b_n^{2v}\left(-1+\frac{1-F_v(b_n)}{1-F_v(a_nx+b_n)}e^{-x}\big(1-k_v(x)e^xb_n^{-v}\big)\right)\\&=\frac{1-F_v(a_nx+b_n)}{1-F_v(b_n)}\big[G_n(x)+H_n(x)+I_n(x)+J_n(x)\big],\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}G_n(x)&=b_n^{2v}\big(A_v(n,x)-1\big),\\H_n(x)&=b_n^{2v}A_v(n,x)\left[\int_0^x\left(\frac{(v-1)a_n}{b_n+a_nt}+\frac{va_n(b_n+a_nt)^{v-1}}{2\lambda^v}-1\right)dt-\big(1-v^{-1}\big)\lambda^v\big(x^2+2x\big)b_n^{-v}\right],\\I_n(x)&=b_n^vA_v(n,x)\big(1-v^{-1}\big)\lambda^v\big(x^2+2x\big)\int_0^x\left(\frac{(v-1)a_n}{b_n+a_nt}+\frac{va_n(b_n+a_nt)^{v-1}}{2\lambda^v}-1\right)dt,\\J_n(x)&=b_n^{2v}A_v(n,x)\big(1-\big(1-v^{-1}\big)\lambda^v\big(x^2+2x\big)b_n^{-v}\big)\sum_{k=2}^\infty\frac{[\int_0^x(\frac{(v-1)a_n}{b_n+a_nt}+\frac{va_n(b_n+a_nt)^{v-1}}{2\lambda^v}-1)dt]^k}{k!}.\end{aligned}$$

First, we consider the bound of $G_n(x)$. For the case of $0\le x<cb_n^{v/3}$, we have

$$\begin{aligned}\big|G_n(x)\big|&<\big(1-2\big(1-v^{-1}\big)\lambda^v(a_nx+b_n)^{-v}\big)^{-1}b_n^{2v}\\&\quad\times\big|2\big(v^{-1}-1\big)\lambda^vb_n^{-v}\big(1-\big(1+2v^{-1}\lambda^vb_n^{-v}x\big)^{-v}\big)\\&\quad+4\big(v^{-1}-1\big)\big(v^{-1}-2\big)\lambda^{2v}b_n^{-2v}\big(1-\big(1+2v^{-1}\lambda^vb_n^{-v}x\big)^{-2v}\big)+O\big(b_n^{-3v}\big)\big|\\&<\big(1-2\big(1-v^{-1}\big)\lambda^vb_n^{-v}\big)^{-1}\big[4\big(1-v^{-1}\big)\lambda^{2v}x+16\big(1-v^{-1}\big)\big(2-v^{-1}\big)\lambda^{3v}b_n^{-v}x+O\big(b_n^{-v}\big)\big]\\&<1+8\lambda^{2v}\big(1+8\lambda^v\big)x\end{aligned}\tag{3.21}$$

due to $1-vy<(1+y)^{-v}<1$ for $v>0$ and $y>0$. Next consider the case of $-d\log b_n<x<0$. Noting that $1+vy<(1+y)^v<1$ for $v>1$ and $-1<y<0$, we have

$$\begin{aligned}\big|G_n(x)\big|&<\big(1-2\big(1-v^{-1}\big)\lambda^v(b_n-da_n\log b_n)^{-v}\big)^{-1}b_n^{2v}\\&\quad\times\big|2\big(v^{-1}-1\big)\lambda^vb_n^{-v}\big(1-2dv^{-1}\lambda^vb_n^{-v}\log b_n\big)^{-v}\big(\big(1+2v^{-1}\lambda^vb_n^{-v}x\big)^v-1\big)\\&\quad+4\big(v^{-1}-1\big)\big(v^{-1}-2\big)\lambda^{2v}b_n^{-2v}\big(1-2dv^{-1}\lambda^vb_n^{-v}\log b_n\big)^{-2v}\big(\big(1+2v^{-1}\lambda^vb_n^{-v}x\big)^{2v}-1\big)\\&\quad+O\big(b_n^{-3v}\big)\big|\\&<1+16\lambda^{2v}\big(1+8\lambda^v\big)|x|\end{aligned}\tag{3.22}$$

for large n. Similarly, for the bounds of $H_n(x)$, $I_n(x)$ and $J_n(x)$, as $-d\log b_n<x<cb_n^{v/3}$ we have

$$\big|H_n(x)\big|<2\big(1-v^{-1}\big)\big|2^{-1}v\lambda^{-v}-d\big|^{-1}\lambda^vx^2+\frac{4}{3}\big(1-v^{-1}\big)\big|1-2v^{-1}\big|\lambda^{2v}|x|^3,\tag{3.23}$$

$$\begin{aligned}\left|I_n(x)\right| &< 2\left(1-v^{-1}\right)\lambda^v\left(x^2+2|x|\right)\left[\frac{v-1}{|2^{-1}v\lambda^{-v}-d|}|x|\right.\\ &\quad\left.+\left(1-v^{-1}\right)\lambda^v x^2+\frac{4}{3}\left(1-v^{-1}\right)\left|1-2v^{-1}\right|\lambda^{2v}|x|^3\right]\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\left|J_n(x)\right| &< 2\exp\left(\frac{2(v-1)d}{1-d}\right)\left(1+\left(1-v^{-1}\right)\lambda^v\left(x^2+2|x|\right)\right)\\ &\quad\times\left[\frac{v-1}{|2^{-1}v\lambda^{-v}-d|}|x|+\left(1-v^{-1}\right)\lambda^v x^2+\frac{4}{3}\left(1-v^{-1}\right)\left|1-2v^{-1}\right|\lambda^{2v}|x|^3\right]^2\end{aligned} \tag{3.25}$$

for large n. Hence, we derive the desired result by combining (3.17)–(3.25) together. □

For $v=1$, noting that the GED(1) is the Laplace distribution with pdf given by

$$f_1(x)=2^{-1/2}\exp\left(-2^{1/2}|x|\right),\quad x\in\mathbb{R}, \tag{3.26}$$

the distributional tail can be written by

$$1-F_1(x)=2^{-1/2}f_1(x)=2^{-1}\exp\left(-2^{1/2}\right)\exp\left(-\int_1^x\frac{1}{f(t)}dt\right),\quad x>0 \tag{3.27}$$

with $f(t)=2^{-1/2}$. For the Laplace distribution, we have the following results. Details are omitted here since the arguments are similar to that in the case of $v>1$. Recall that the norming constants $a_n=2^{-1/2}$ and $b_n=2^{-1/2}\log n/2$ as $v=1$.

Lemma 3.8. *For $0<d<1$ and arbitrary nonnegative real number j, we have*

$$\lim_{n\to\infty} n^2\int_{-\infty}^{-db_n^{1/2}}|x|^j\Lambda(x)dx=0,\qquad \lim_{n\to\infty} n\int_{-\infty}^{-db_n^{1/2}}|x|^je^{-2x}\Lambda(x)dx=0 \tag{3.28}$$

and

$$\lim_{n\to\infty} n^2\int_{-\infty}^{-db_n^{1/2}}|x|^jF_1^n(a_nx+b_n)dx=0. \tag{3.29}$$

Lemma 3.9. *For $v=1$, let $h_n(x)=n\log F_1(a_nx+b_n)+e^{-x}$ with $a_n=2^{-1/2}$, $b_n=2^{-1/2}\log n/2$. For sufficiently large n, we have $|h_n(x)|<2$ uniformly for all $x>-db_n^{1/2}$ with $0<d<1$. Furthermore, for $x>-db_n^{1/2}$, $x^rn[n(F_1^n(a_nx+b_n)-\Lambda(x))+\frac{1}{2}e^{-2x}\Lambda(x)]$ is bounded by integrable functions independent of n, where $r>0$ and $0<d<1$.*

4. Proofs

By Proposition 2.1(iii) in Resnick [15], we have

$$\lim_{n\to\infty}m_r(n)=\lim_{n\to\infty}\mathbf{E}\left(\frac{M_n-b_n}{a_n}\right)^r=m_r=\int_{-\infty}^{\infty}x^rd\Lambda(x)=(-1)^r\Gamma^{(r)}(1),$$

since $\int_{-\infty}^{0}|x|^r f_v(x)dx < \infty$ for all positive integers r, where $\Gamma^{(r)}(1)$ denotes the rth derivative of the gamma function at $x = 1$. Hence, for large n, $m_r(n)$ is finite and

$$\begin{aligned} m_r(n) - m_r &= \int_{-\infty}^{\infty} x^r \big(F_v^n(a_n x + b_n) - \Lambda(x)\big)' dx \\ &= \int_{-\infty}^{\infty} x^r d\big(F_v^n(a_n x + b_n) - \Lambda(x)\big). \end{aligned}$$

Noting that $\int_{-\infty}^{0}|x|^r f_v(x)dx < \infty$ implies that $\lim_{x\to-\infty}|x|^r F_v(x) = 0$, and by the C_r-inequality, we have

$$0 \le \lim_{x\to-\infty} \sup|x|^r F_v^n(a_n x + b_n) \le \lim_{y\to-\infty} \frac{2^{r-1}(|y|^r + |b_n|^r)}{a_n^r} F_v^n(y) = 0,$$

which implies

$$\lim_{x\to-\infty} x^r F_v^n(a_n x + b_n) = 0. \tag{4.1}$$

Hence, by (3.9) and (4.1), we have

$$\lim_{x\to\infty} x^r \big(F_v^n(a_n x + b_n) - \Lambda(x)\big) = \lim_{x\to\infty} x^r \big(1 - \Lambda(x)\big) - \lim_{x\to\infty} x^r \big(1 - F_v^n(a_n x + b_n)\big) = 0$$

and

$$\lim_{x\to-\infty} x^r \big(F_v^n(a_n x + b_n) - \Lambda(x)\big) = \lim_{x\to-\infty} x^r F_v^n(a_n x + b_n) - \lim_{x\to-\infty} x^r \Lambda(x) = 0.$$

So, by integration by parts, we have

$$m_r(n) - m_r = -r \int_{-\infty}^{\infty} x^{r-1}\big(F_v^n(a_n x + b_n) - \Lambda(x)\big)dx \tag{4.2}$$

and

$$\int_{-\infty}^{\infty} x^{r+1} e^{-2x} \Lambda(x)dx = -(r+1)m_r + m_{r+1}. \tag{4.3}$$

For $v \ne 1$, by (3.3), (4.2) and (4.3), Lemmas 3.3–3.7, and the dominated convergence theorem, we have

$$\begin{aligned} & b_n^v\big[b_n^v\big(m_r(n) - m_r\big) + \big(1 - v^{-1}\big)\lambda^v r(m_{r+1} + 2m_r)\big] \\ &= -r \int_{-\infty}^{\infty} b_n^v\big[b_n^v x^{r-1}\big(F_v^n(a_n x + b_n) - \Lambda(x)\big) - x^{r-1}k_v(x)\Lambda(x)\big]dx \\ &= -r \int_{cb_n^{v/3}}^{\infty} b_n^{2v} x^{r-1}\big[\big(1 - \Lambda(x)\big) - \big(1 - F_v^n(a_n x + b_n)\big)\big]dx + r \int_{cb_n^{v/3}}^{\infty} b_n^v x^{r-1} k_v(x)\Lambda(x)dx \end{aligned}$$

$$
\begin{aligned}
&- r \int_{-d\log b_n}^{c b_n^{v/3}} x^{r-1} b_n^v \big[b_n^v \big(F_v^n(a_n x + b_n) - \Lambda(x) \big) - k_v(x)\Lambda(x) \big] dx \\
&- r \int_{-\infty}^{-d\log b_n} b_n^{2v} x^{r-1} \big(F_v^n(a_n x + b_n) - \Lambda(x) \big) dx + r \int_{-\infty}^{-d\log b_n} b_n^v x^{r-1} k_v(x)\Lambda(x) dx \\
\to & -r \int_{-\infty}^{\infty} \left(w_v(x) + \frac{k_v^2(x)}{2} \right) x^{r-1}\Lambda(x) dx \\
= & \, 2r\lambda^{2v}\big(1 - v^{-1}\big) \Big[\big(\big(1 - v^{-1}\big)(r+1) + 2 \big) m_r + \big(\big(1 - v^{-1}\big)(r+1) + 1 \big) m_{r+1} \\
&+ \left(\frac{1}{4}\big(1 - v^{-1}\big)(r-1) + \frac{1}{3}\big(2 - v^{-1}\big) \right) m_{r+2} \Big]
\end{aligned}
$$

as $n \to \infty$, which is (2.1).

For the case of $v = 1$, first note that

$$
\int_{-\infty}^{\infty} x^k e^{-mx} \Lambda(x) dx = (-1)^k \Gamma^{(k)}(m)
$$

for all nonnegative integers k and m, where $\Gamma^{(r)}(m)$ denotes the kth derivative of the gamma function at $x = m$. By (3.4) and (4.2), Lemmas 3.8–3.9, and the dominated convergence theorem, we have

$$
\begin{aligned}
& n\left[n\big(m_r(n) - m_r\big) + (-1)^r \frac{r}{2} \Gamma^{(r-1)}(2) \right] \\
&= -r \int_{-\infty}^{\infty} n \left[x^{r-1} n\big(F_1^n(a_n x + b_n) - \Lambda(x) \big) + \frac{1}{2} x^{r-1} e^{-2x} \Lambda(x) \right] dx \\
&= -r \int_{-d b_n^{1/2}}^{\infty} n \left[x^{r-1} n\big(F_1^n(a_n x + b_n) - \Lambda(x) \big) + \frac{1}{2} x^{r-1} e^{-2x} \Lambda(x) \right] dx \\
&\quad - r \int_{-\infty}^{-d b_n^{1/2}} n \left[x^{r-1} n\big(F_1^n(a_n x + b_n) - \Lambda(x) \big) + \frac{1}{2} x^{r-1} e^{-2x} \Lambda(x) \right] dx \\
&\to r \int_{-\infty}^{\infty} x^{r-1} \left(\frac{1}{3} e^{-3x} - \frac{1}{8} e^{-4x} \right) \Lambda(x) dx \\
&= (-1)^{r-1} \frac{r}{24} \big[8\Gamma^{(r-1)}(3) - 3\Gamma^{(r-1)}(4) \big]
\end{aligned}
$$

as $n \to \infty$, which is (2.2).

The proofs are complete. □

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