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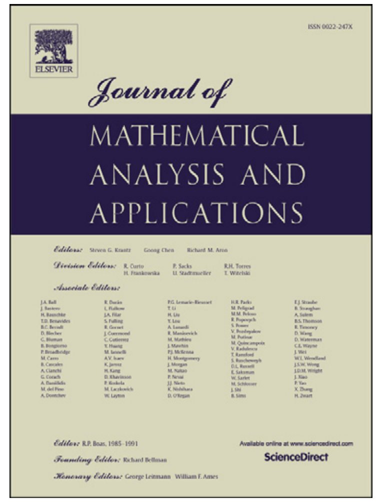

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Deforming convex curves with fixed elastic energy *

Laiyuan Gao† · **Yiling Wang**‡

Abstract This paper presents a curve flow which preserves the elastic energy of the evolving curve. If the initial curve is a planar, simple and smooth curve with positive curvature then the local and global existence of the flow is proved. Under this flow, the evolving curve will converge to a finite circle in the C^∞ metric as time goes to infinity.

Keywords convex curves, nonlocal flow, elastic energy
Mathematics Subject Classification (2000) 35K15, 35K55, 53A04

1 Introduction

In many fields, curve evolution problems are proposed to smooth curves from noisy input data, such as image processing ([8]), phase transitions ([12]), reverse engineering, image registration and mesh optimization ([14]), etc. Because of these various applications, the curve flows have been studied extensively. Many of the curve evolutions focus on the famous curve-shortening flow and its generalizations (see [4], [7], [9], [1], [6], [17], [3], etc.). If one needs to smooth the input data with some global quality preserved then he demands non-local flows, such as the area-preserving flows ([5], [18], [20]) and the length-preserving flows ([19], [22]), etc. Since the elastic energy of a closed curve is a kind of interesting geometric quality and arouses some research interest (see [13], [23], [16]), we pose a flow in this paper to deform a given curve with its elastic energy preserved in order to obtain some more geometric properties of convex curves, such as geometric inequalities involving elastic energy. As far as we know, to construct such a kind of flow is the first try in this field.

Given a simple, closed and C^2 curve $X(\cdot)$ in the plane, one can define a kind of energy in the following form

$$E_n \triangleq \int_0^L (\kappa(s))^n ds,$$

where n is a positive integer and κ is the relative curvature of the curve. If $n = 1$ then $E_1 = 2\pi$ (see Theorem 7 on page 20 of [24]). In the case of $n = 2$, E_2 is so called the elastic energy (or bending energy) of the curve (see the definition on page 1 of [23]).

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†The corresponding author. Email: hytcmath031g@163.com Address: Department of Mathematics, Shanghai University, Shanghai, 200444, China.

‡Email: ylwang@math.ecnu.edu.cn Address: Department of Mathematics, East China Normal University, Shanghai, 200062, P. R. China.

Let $X_0 : S^1 \to \mathbb{R}^2$ be a smooth curve in the plane. We say X_0 is convex if it is a simple curve with positive curvature. Denote by $T(\varphi)$ the unit tangential vector of the curve at $X(\varphi)$. For each φ, let $N(\varphi)$ be the unit normal such that $\{T(\varphi), N(\varphi)\}$ gives a positive orientation of the plane $\mathbb{R}^2$. The support function of X_0 is defined by $p = -\langle X_0, N\rangle$. In this paper, the following evolution problem for convex curves will be investigated:

$$\begin{cases} \frac{\partial X}{\partial t}(\varphi,t) = \left(p(\varphi,t) - \frac{\int_0^{L(t)} \kappa^2(s,t)ds}{\int_0^{L(t)} \kappa^3(s,t)ds} \right) N(\varphi,t), \\ \qquad\qquad (\varphi,t) \in S^1 \times [0,\omega), \\ X(\varphi,0) = X_0(\varphi), \quad \varphi \in S^1, \end{cases} \tag{1.1}$$

where $X(\varphi,t)$ is the evolving curve with its curvature, perimeter and support function denoted by $\kappa(\varphi,t)$, $L(t)$ and $p(\varphi,t)$, respectively. Under the flow (1.1), the elastic energy of the evolving curve is invariable, which makes this flow differ from all the previous ones.

Let θ be the oriented angle from the positive x-axis to the unit tangential vector of the curve. By the definition of the curvature κ, one has

$$\frac{d\theta}{ds} = \kappa.$$

So we can use $\theta \in [0, 2\pi]$ as a parameter for convex curves. From now on, we choose a convex curve to be the initial curve X_0 of our flow (1.1).

If the flow (1.1) has a solution on $S^1 \times [0,\omega)$ then the curvature of the evolving curve satisfies the following Cauchy problem (see Equation (1.15) on page 20 of [3]):

$$\begin{cases} \frac{\partial \kappa}{\partial t}(\theta,t) = \kappa(\theta,t) - \frac{\int_0^{2\pi} \kappa(\theta,t)d\theta}{\int_0^{2\pi} \kappa^2(\theta,t)d\theta}\kappa^2(\theta,t), \quad (\theta,t) \in [0,2\pi] \times [0,\omega), \\ \kappa(\theta,0) = \kappa_0(\theta), \quad \theta \in [0,2\pi], \end{cases} \tag{1.2}$$

where $\kappa_0(\theta)$ is the curvature of a convex curve in the plane and it satisfies the closing condition:

$$\int_0^{2\pi} \frac{e^{i\theta}}{\kappa_0(\theta)} d\theta = 0. \tag{1.3}$$

Although the support function p appears in the evolution equation (1.1), the shape of the evolving curve $X(\cdot,t)$ does not rely on the choice of the original point of the plane because $X(\cdot,t)$ is in fact uniquely determined by its curvature, the unique solution of (1.2)-(1.3).

In comparison with the previous work, the difficulty to investigate the flow (1.1) is first to prove the existence of the positive and uniformly bounded solution for the integro-differential equation (1.2)-(1.3) on time interval $[0,+\infty)$. And then we also need to study the asymptotic behavior of κ as time goes to infinity. To settle the first problem, we reduce the existence problem to find a fixed point in a closed set of a Banach space and this goal can be efficiently achieved by using Banach's fixed point theorem. Then we use comparison principles to prove the positivity and boundedness of κ in any finite time interval. In order to give a uniform bound of κ, we introduce a new trick to establish its Hanarck estimate. To deal with the second problem, we use some famous or important geometric inequalities, such as Bonnesen's inequality ([2] and

[21]), Gage's inequality ([4]) and Lin-Tai's inequality ([15]) to obtain the convergence of our flow. Our main theorem of the flow (1.1) is given as follows.

Main Theorem *If a convex curve $X(\varphi, 0)$ evolves according to (1.1) then one obtains a family of convex curves $\{X(\varphi,t)|(\varphi,t) \in S^1 \times [0,\infty)\}$ with fixed elastic energy $E_2(0) = \int_0^L (\kappa_0(s))^2 ds$, where $\kappa_0(s)$ is the curvature of the initial curve. If t tends to infinity then the evolving curve $X(\cdot,t)$ converges to a circle in the C^∞ metric. The radius of the limiting circle is $\frac{2\pi}{E_2(0)}$ and the center of this circle is the original point of the plane.*

This paper is organized as follows. In section 2, it is proved that the evolution problem (1.1) is equivalent to the Cauchy problem (1.2)-(1.3) for some $T > 0$, if the initial curve is smooth and convex. It is then shown that the Cauchy problem (1.2)-(1.3) has a positive and bounded solution on $[0, 2\pi] \times [0, T)$, for any $T > 0$. In section 3, the asymptotic behavior of evolving curves $X(\cdot, t)$ is considered and the Main Theorem is proved. In the end of this paper, a new geometric inequality is proved for convex curves in the plane by using the flow (1.1).

2 Long Time Existence

In this section, the equivalence of the curve evolution problem (1.1) and the Cauchy problem (1.2)-(1.3) will be proved. And then the positive solution of (1.2) will be obtained on the domain $[0, 2\pi] \times [0, +\infty)$.

Given a convex curve $X(\varphi)$ in the plane for $\varphi \in S^1$. Denoted by θ its tangential angle, i.e. the angle between the positive direction of x-axis and the tangential vector T. The convexity of the curve X implies that θ can be chosen as a parameter of this curve. Noticing that

$$T = (\cos\theta, \sin\theta), \quad N = (-\sin\theta, \cos\theta),$$

one obtains the structural equation of the curve:

$$\frac{dX}{d\theta}(\theta) = \frac{1}{\kappa(\theta)} T(\theta), \quad \frac{dT}{d\theta} = N(\theta), \quad \frac{dN}{d\theta}(\theta) = -T(\theta).$$

Now, the flow (1.1) can be rewritten as follows,

$$\begin{cases} \frac{\partial X}{\partial t}(\theta,t) = \left(p(\theta,t) - \frac{E_2(t)}{E_3(t)}\right) N(\theta,t), \\ \qquad\qquad (\theta,t) \in [0,2\pi] \times [0,\omega), \\ X(\theta,0) = X_0(\theta), \quad \theta \in [0,2\pi]. \end{cases} \tag{2.1}$$

Since θ depends on κ and the arc length s, it is in fact a function with respect to the time t. In order to make θ be independent of t, one can add a tangential component to the flow (2.1) to obtain a new one:

$$\begin{cases} \frac{\partial \widetilde{X}}{\partial t}(\theta,t) = \alpha(\theta,t) T(\theta,t) + \left(p(\theta,t) - \frac{E_2(t)}{E_3(t)}\right) N(\theta,t), \\ \widetilde{X}(\theta,0) = X_0(\theta), \end{cases} \tag{2.2}$$

where $\alpha(\theta,t)$ is to be determined later. Under the flow (2.2), the tangential angle θ and the Frenet frame satisfy the following equations (see page 20 of [3]),

$$\frac{\partial\theta}{\partial t}=\alpha k+\frac{\partial\beta}{\partial s}=\left(\alpha+\frac{\partial\beta}{\partial\theta}\right)k,$$
$$\frac{\partial T}{\partial t}=\left(\alpha k+\frac{\partial\beta}{\partial s}\right)N,\quad \frac{\partial N}{\partial t}=-\left(\alpha k+\frac{\partial\beta}{\partial s}\right)T,$$

where $\beta=p(\theta,t)-\frac{E_2(t)}{E_3(t)}$. If one sets $\alpha=-\frac{\partial p}{\partial\theta}$ then $\frac{\partial\theta}{\partial t}=0$, i.e., θ is a variable independent of time t. And one also gets $\frac{\partial T}{\partial t}=\frac{\partial N}{\partial t}=0$. The Proposition 1.1 on page 6 of [3] tells us that the solution $\widetilde{X}$ of (2.2) differs from the solution of (2.1) only by altering the parametrization. So one can consider the flow (2.2) instead of (2.1). Now it is time to show that the new evolution problem (2.2) is equivalent to the Cauchy problem (1.2)-(1.3).

Theorem 2.1 *If the initial curve $X_0(\varphi)$ is smooth and strictly convex then the flow (2.2) is equivalent to the Cauchy problem (1.2) for some $T>0$.*

Proof. Suppose the initial curve is strictly convex and $\widetilde{X}(\theta,t)$ is a solution of (2.2) for $(\theta,t)\in[0,2\pi]\times[0,T)$. By Equation (1.16) on page 20 of [3], the curvature of $\widetilde{X}$ evolves according to

$$\frac{\partial\kappa}{\partial t}=\kappa^2\left(\frac{\partial^2 p}{\partial\theta^2}+p-\frac{E_2}{E_3}\right).$$

Denote by ρ the radius of curvature, i.e., $\rho=\frac{1}{\kappa}$. Since $\rho=\frac{\partial^2 p}{\partial\theta^2}+p$ (c.f. Proposition 1.6 of [11]), one obtains that

$$\frac{\partial\kappa}{\partial t}=\kappa-\kappa^2\frac{E_2}{E_3}.$$

So one obtains a solution of the Cauchy problem (1.2) from the flow (2.2). By the closed condition of $\widetilde{X}(\cdot,0)$, one gets (1.3) immediately.

Conversely, suppose the Cauchy problem (1.2)-(1.3) has a positive solution κ in $S^1\times[0,T)$. Noticing that

$$\frac{d}{dt}\int_0^{2\pi}\frac{e^{i\theta}}{\kappa}d\theta=\int_0^{2\pi}-\frac{e^{i\theta}}{\kappa^2}\left(\kappa-\kappa^2\frac{E_2}{E_3}\right)d\theta=-\int_0^{2\pi}\frac{e^{i\theta}}{\kappa}d\theta,$$

one obtains the closing condition of κ:

$$\int_0^{2\pi}\frac{e^{i\theta}}{\kappa(\theta,t)}d\theta=\int_0^{2\pi}\frac{e^{i\theta}}{\kappa(\theta,0)}d\theta\cdot e^{-t}=0.$$

If one sets $\rho(\theta,t)\triangleq\frac{1}{\kappa(\theta,t)}$ then he obtains a integrability condition from the above closing condition

$$\int_0^{2\pi}\rho(\theta,t)\sin\theta d\theta=\int_0^{2\pi}\rho(\theta,t)\cos\theta d\theta=0, \tag{2.3}$$

for the following second order o.d.e.

$$\frac{\partial^2 p}{\partial \theta^2}(\theta,t) + p(\theta,t) = \rho(\theta,t), \quad (\theta,t) \in [0,2\pi] \times [0,T), \tag{2.4}$$

In fact one can define $p(\theta,t)$ as follows and check (2.4) directly by using (2.3).

$$p(\theta,t) = \sin\theta \int_0^\theta \rho(\varphi,t)\cos\varphi d\varphi - \cos\theta \int_0^\theta \rho(\varphi,t)\sin\varphi d\varphi.$$

Now let us construct a family of curves $\widetilde{X}(\theta,t) = (\widetilde{x}(\theta,t), \widetilde{y}(\theta,t)) + (C_1(t), C_2(t))$ by setting

$$\widetilde{x}(\theta,t) = \int_0^\theta \rho(\phi,t)\cos\phi \; d\phi, \qquad \widetilde{y}(\theta,t) = \int_0^\theta \rho(\phi,t)\sin\phi \; d\phi,$$
$$C_1(t) = -\int_0^t \frac{\partial p}{\partial \theta}(0,\tau)d\tau, \quad C_2(t) = \int_0^t \left(p(0,\tau) - \frac{E_2(\tau)}{E_3(\tau)}\right) d\tau.$$

Direct computations can give us the following

$$\begin{aligned}
\frac{\partial \widetilde{x}}{\partial t}(\theta,t) &= \int_0^\theta \frac{\partial \rho}{\partial t}(\phi,t)\cos\phi \; d\phi, = \int_0^\theta \left(\frac{E_2(t)}{E_3(t)} - \rho(\theta,t)\right)\cos\phi \; d\phi, \\
&= \frac{E_2(t)}{E_3(t)}\sin\theta - \int_0^\theta \left(\frac{\partial^2 p}{\partial \theta^2}(\phi,t) + p(\phi,t)\right)\cos\phi d\phi \\
&= \frac{E_2(t)}{E_3(t)}\sin\theta - \frac{\partial p}{\partial \theta}(\phi,t)\cos\phi\Big|_0^\theta - \int_0^\theta \frac{\partial p}{\partial \theta}(\phi,t)\sin\phi d\theta - \int_0^\theta p(\theta,t)\cos\phi \; d\phi, \\
&= \frac{E_2(t)}{E_3(t)}\sin\theta - \frac{\partial p}{\partial \theta}(\theta,t)\cos\theta + \frac{\partial p}{\partial \theta}(0,t) - p(\theta,t)\sin\theta \\
&= -\frac{\partial p}{\partial \theta}(\theta,t)\cos\theta + \left(p - \frac{E_2(t)}{E_3(t)}\right)(-\sin\theta) + \frac{\partial p}{\partial \theta}(0,t).
\end{aligned}$$

Similarly, one has

$$\frac{\partial \widetilde{y}}{\partial t}(\theta,t) = -\frac{\partial p}{\partial \theta}(\theta,t)\sin\theta + \left(p - \frac{E_2(t)}{E_3(t)}\right)\cos\theta + \frac{E_2(t)}{E_3(t)} - p(0,t).$$

So the curves $\widetilde{X}(\cdot,t)$ satisfy the evolution equation (2.2). Q. E. D.

In the next, we will show that the Cauchy problem (1.2)-(1.3) has a unique positive smooth solution κ. Now let us set the constants

$$m = \min\{\kappa_0(\theta) | \theta \in S^1\}, \qquad M = \max\{\kappa_0(\theta) | \theta \in S^1\},$$

and denote

$$\widetilde{m} = \frac{1}{\lambda}m, \quad \widetilde{M} = \lambda M, \quad Q_T = S^1 \times [0,T),$$

where $\lambda > 1$ and $T > 0$.

Theorem 2.2 *The Cauchy problem (1.2)-(1.3) has a unique positive smooth solution κ on Q_T for some $T > 0$.*

Proof. First, let us define

$$T = \min\left\{\frac{(1-\frac{1}{\lambda})m^3}{\lambda^5 M^3},\ 1-\frac{1}{\lambda},\ \frac{1}{2}\frac{\widetilde{m}^4}{\widetilde{m}^4 + 2\widetilde{M}^3 + 3\widetilde{M}^2}\right\}$$

and consider an equation as follows

$$\begin{cases} \frac{\partial u}{\partial t}(\theta,t) = v(\theta,t) - \frac{\int_0^{2\pi} v(\theta,t)d\theta}{\int_0^{2\pi} v^2(\theta,t)d\theta} v^2(\theta,t), & (\theta,t) \in [0,2\pi]\times[0,T), \\ u(\theta,0) = \kappa_0(\theta), & \theta \in [0,2\pi], \end{cases} \tag{2.5}$$

where $v \in C(\overline{Q}_T)$ and $\widetilde{m} \le v \le \widetilde{M}$. By the choice of T, one obtains

$$\begin{aligned} u(\theta,t) &\ge \kappa_0(\theta) + \left(\widetilde{m} - \widetilde{M}^2 \frac{\widetilde{M}}{\widetilde{m}^2}\right) t \ \ge\ m - \frac{\lambda^3 M^3}{\frac{1}{\lambda^2} m^2} t \\ &= m - \frac{\lambda^5 M^3}{m^2} t \ \ge\ \frac{m}{\lambda} \ =\ \widetilde{m}, \end{aligned}$$

and

$$u(\theta,t) \ \le\ \kappa_0(\theta) + \widetilde{M}t \le M + \lambda M t \le \lambda M \ =\ \widetilde{M}.$$

So $u \in C(\overline{Q}_T)$ and $\widetilde{m} \le u \le \widetilde{M}$. In order to simplify the statement, one can introduce a space as follows

$$V = \{f \in C(\overline{Q}_T) | \widetilde{m} \le f(\theta,t) \le \widetilde{M}\}.$$

V is a closed subset of the Banach space $C(\overline{Q}_T)$ endowed with the norm

$$\|f\|_{C(\overline{Q}_T)} = \max\{|f(\theta,t)|\ |(\theta,t) \in \overline{Q}_T\}.$$

Therefore the solution of the (2.5) defines an operator $\mathbb{T}$ from V to itself. In the next, we will show that the operator $\mathbb{T}$ is a contract mapping. Let v_1, v_2 be two functions in V. Let us define $u_i = \mathbb{T}v_i, i = 1,2$. Since

$$u = \int_0^t v dt - \frac{\int_0^{2\pi} v d\theta}{\int_0^{2\pi} v^2 d\theta} \int_0^t v^2 dt + \kappa_0(\theta),$$

one obtains

$$u_1 - u_2 = \int_0^t (v_1 - v_2) dt + \frac{\int_0^{2\pi} v_2 d\theta \int_0^{2\pi} v_1^2 d\theta \int_0^t v_2^2 dt - \int_0^{2\pi} v_1 d\theta \int_0^{2\pi} v_2^2 d\theta \int_0^t v_1^2 dt}{\int_0^{2\pi} v_1^2 d\theta \int_0^{2\pi} v_2^2 d\theta}.$$

Noticing that

$$
\begin{aligned}
&\left|\int_0^{2\pi} v_2 d\theta \int_0^{2\pi} v_1^2 d\theta \int_0^t v_2^2 dt - \int_0^{2\pi} v_1 d\theta \int_0^{2\pi} v_2^2 d\theta \int_0^t v_1^2 dt\right| \\
= & \left| \begin{array}{l} \int_0^{2\pi} v_2 d\theta \int_0^{2\pi} v_1^2 d\theta \int_0^t (v_2^2 - v_1^2) dt \\ + \left(\int_0^{2\pi} v_2 d\theta \int_0^{2\pi} v_1^2 d\theta - \int_0^{2\pi} v_1 d\theta \int_0^{2\pi} v_2^2 d\theta\right) \int_0^t v_1^2 dt \end{array} \right| \\
\leq & \; 4\pi^2 \widetilde{M}^3 \int_0^t 2\widetilde{M} |v_1 - v_2| dt \\
& + \left(\int_0^{2\pi} (v_2 - v_1) d\theta \int_0^{2\pi} v_1^2 d\theta + \int_0^{2\pi} v_1 d\theta \int_0^{2\pi} (v_1^2 - v_2^2) d\theta\right) \int_0^t v_1^2 dt \\
\leq & \; 8\pi^2 \widetilde{M}^3 \|v_1 - v_2\|_{C(\overline{Q}_T)} t \\
& + \left(2\pi \|v_1 - v_2\|_{C(\overline{Q}_T)} \cdot 2\pi \widetilde{M}^2 + 2\pi \widetilde{M} \cdot 2\pi \cdot 2\widetilde{M} \|v_1 - v_2\|_{C(\overline{Q}_T)}\right) \widetilde{M}^2 t \\
= & \left(8\pi^2 \widetilde{M}^3 + 12\pi^2 \widetilde{M}^2\right) \|v_1 - v_2\|_{C(\overline{Q}_T)} t,
\end{aligned}
$$

one obtains

$$
\begin{aligned}
\|u_1 - u_2\|_{C(\overline{Q}_T)} & \leq \left(1 + \frac{8\pi^2 \widetilde{M}^3 + 12\pi^2 \widetilde{M}^2}{4\pi^2 \widetilde{m}^4}\right) \|v_1 - v_2\|_{C(\overline{Q}_T)} t \\
& = \frac{\widetilde{m}^4 + 2\widetilde{M}^3 + 3\widetilde{M}^2}{\widetilde{m}^4} \|v_1 - v_2\|_{C(\overline{Q}_T)} t \\
& \leq \frac{1}{2} \|v_1 - v_2\|_{C(\overline{Q}_T)}.
\end{aligned}
$$

Hence, $\mathbb{T}$ is a contraction operator from V to V. By Banach's contraction mapping principle, there exists a unique fixed point of $\mathbb{T}$. So one gets $\kappa \in C(\overline{Q}_T)$ such that $\widetilde{m} \leq \kappa \leq \widetilde{M}$ and $\mathbb{T}(\kappa) = \kappa$, i.e., the Cauchy problem (1.2) has a unique solution on Q_T.

It is a straightforward to show that κ satisfies the closing condition (1.3). The higher order regularity of $\kappa(\theta, t)$ is an immediate consequence of the smoothness of $\kappa_0(\theta)$. Q. E. D.

Combining this theorem and Theorem 2.1, one has the local existence of the flow (1.1). Since $\widetilde{m} \leq \kappa \leq \widetilde{M}$, the evolving curve $X(\cdot, t)$ is convex and it has no singularities for each t in $[0, T)$.

Corollary 2.3 *The curve flow (1.1) has a unique smooth solution on Q_T, where T is defined in Theorem 2.2.*

By estimating of the non-local quantity $E_3(t) = \int_0^L \kappa^3 ds = \int_0^{2\pi} \kappa d\theta$ and the curvature κ, we can prove the long time existence of the flow (1.1).

Theorem 2.4 *The evolution equation (1.1) has a long time solution on $[0, 2\pi] \times [0, +\infty)$.*

Proof. Let us set $\kappa_{\min}(t) \triangleq \min\{\kappa(\theta, t) | \theta \in S^1\}, \kappa_{\max}(t) \triangleq \max\{\kappa(\theta, t) | \theta \in S^1\}$. First, if we assume that $\kappa_{\min}(t) > 0$ and $\kappa_{\max}(t) < \infty$ for $t \in [0, T)$, then

$$
\begin{aligned}
\frac{d}{dt} E_2(t) & = \int_0^{2\pi} \left(\kappa - \frac{E_2(t)}{E_3(t)} \kappa^2\right) d\theta \\
& = E_2(t) - \frac{E_2(t)}{E_3(t)} E_3(t) = 0.
\end{aligned}
$$

So one obtains $E_2(t) \equiv E_2(0)$, i.e., the flow (1.1) preserves the elastic energy of the evolving curve. By using Cauchy-Schwartz inequality $E_3(t) = \int_0^{2\pi} \kappa^2 d\theta \geq \frac{1}{2\pi}\left(\int_0^{2\pi} \kappa d\theta\right)^2 = \frac{1}{2\pi}E_2(0)^2$. One gets

$$\begin{aligned} \frac{\partial \kappa}{\partial t} &= \kappa - \frac{E_2(t)}{E_3(t)}\kappa^2 = \kappa - \frac{E_2(0)}{\int_0^{2\pi}\kappa^2 d\theta}\kappa^2 \\ &\geq -\frac{E_2(0)}{\frac{(E_2(0))^2}{2\pi}}\kappa^2 = -\frac{2\pi}{E_2(0)}\kappa^2, \end{aligned}$$

namely,

$$\begin{aligned} \frac{\partial}{\partial t}\frac{1}{\kappa(\theta,t)} &\leq \frac{2\pi}{E_2(0)}, \\ \frac{1}{\kappa(\theta,t)} - \frac{1}{\kappa_0(\theta)} &\leq \frac{2\pi}{E_2(0)}t, \end{aligned}$$

$$\kappa(\theta,t) \geq \frac{1}{\frac{1}{\kappa_0(\theta)} + \frac{2\pi}{E_2(0)}t} > 0. \tag{2.6}$$

If there is a positive time $T_0 > 0$ such that $\kappa_{\min}(T_0) = 0$ and $\kappa_{\min}(t) > 0$ for $t \in [0, T_0)$ then (2.6) tells us $\kappa(\theta, T_0) > 0$, a contradiction. So the flow (1.1) preserves the convexity of the evolving curve if no singularity occurs in the same time interval. Second, by the evolution equation of κ, one gets $\frac{\partial \kappa}{\partial t} \leq \kappa$, which implies that

$$\kappa_{\max}(t) \leq \kappa_{\max}(0)e^t,$$

here $t \in [0, \infty)$. The singularity will never happen as time goes. Combining the above two steps gives us

$$0 < \frac{E_2(0)\kappa_0(\theta)}{E_2(0) + 2\pi\kappa_0(\theta)t} \leq \kappa(\theta,t) \leq \kappa_{\max}(0)e^t, \tag{2.7}$$

for any positive t. So we obtain the long time existence of the flow (1.1). Q. E. D.

3 Convergence

In this section, the geometric properties of the flow (1.1) will be investigated and we will complete the proof of the Main Theorem. Now, let us introduce several useful inequalities.

Lemma 3.1 *Let $X(\theta)$ be a closed and convex curve with length denoted by L, here θ is the tangential angle of this curve. Set A the area bounded by X. Let r_{in} and r_{out} be the radius of the maximum incircle and the minimum circumcircle, respectively. And $\kappa(\theta)$ represents the curvature of the curve X at θ. We have the following inequalities for X.*
Bonnesen's inequality (c.f. [2] and [21]):

$$\pi^2(r_{out} - r_{in})^2 \leq L^2 - 4\pi A. \tag{3.1}$$

Gage's inequality (c.f. [4]):

$$\int_0^{2\pi} \kappa d\theta \quad \geq \quad \frac{\pi L}{A}. \tag{3.2}$$

Lin-Tai's inequality (c.f. [15]):

$$\int_0^{2\pi} \kappa^2 d\theta \quad \geq \quad \frac{L}{2A}\int_0^{2\pi} \kappa d\theta. \tag{3.3}$$

Lemma 3.2 *(Hausdorff Convergence) Under the flow (1.1), the evolving curve converges in the Hausdorff metric to a finite circle. The center of this limiting circle is the original point of the plane and the radius of this circle is equal to* $\frac{2\pi}{E_2(0)}$.

Proof. Gage's variation formulae of length $L(t)$ and area $A(t)$ in [5] give us

$$\begin{aligned}
\frac{dL}{dt} &= -\int_0^{2\pi}\left(p - \frac{E_2(0)}{E_3(t)}\right) d\theta = -L + 2\pi\frac{E_2(0)}{E_3(t)}, \\
\frac{dA}{dt} &= -\int_0^{2\pi} \rho\left(p - \frac{E_2(0)}{E_3(t)}\right)\, d\theta = -2A + \frac{LE_2(0)}{E_3(t)}.
\end{aligned}$$

The Cauchy-Schwartz inequality gives us

$$E_2(0) = \int_0^{2\pi} \kappa d\theta = \int_0^{L(t)} \kappa^2 ds \geq \frac{4\pi^2}{L(t)},$$

$$\frac{dL}{dt} \leq -L + 2\pi\frac{E_2(0)}{\frac{(E_2(0))^2}{2\pi}} = -L + \frac{4\pi^2}{E_2(0)} \leq 0.$$

So one obtains that

$$\frac{4\pi^2}{E_2(0)} \leq L(t) \leq L_0. \tag{3.4}$$

Since $L(t)$ is monotone decreasing and it has a lower bound, one gets the limit of $L(t)$ as $t \to \infty$. Denote by L_∞ the limit of $L(t)$ as $t \to \infty$. By Lin-Tai's inequality, one gets $\frac{dA}{dt} \leq 0$, i.e., the area of the domain enclosed by the evolving curve is monotone decreasing. Gage's inequality (3.2) and Inequality (3.4) imply that

$$A(t) \geq \frac{\pi L}{E_2(0)} \geq \frac{\pi\frac{4\pi^2}{E_2(0)}}{E_2(0)} = \frac{4\pi^3}{(E_2(0))^2}.$$

Hence,

$$\frac{4\pi^3}{(E_2(0))^2} \leq A(t) \leq A(0). \tag{3.5}$$

By the evolution equation of $L(t)$ and $A(t)$,

$$\begin{aligned}
\frac{d}{dt}(L^2 - 4\pi A) &= 2L\left(-L + 2\pi\frac{E_2(0)}{E_3(t)}\right) - 4\pi\left(-2A + \frac{LE_2(0)}{E_3(t)}\right) \\
&= -2(L^2 - 4\pi A),
\end{aligned}$$

namely,

$$L^2 - 4\pi A = (L(0)^2 - 4\pi A(0))e^{-2t}. \tag{3.6}$$

Combining with Bonnesen's inequality (3.2) and the above equation, one has the Hausdorff convergence of the flow (1.1).

Since the elastic energy of the limit circle equals to $E_2(0)$, the radius of the limit circle is $\frac{2\pi}{E_2(0)}$. In order to determine the center of the limit circle, one needs to find the limit of the Steiner center of the evolving curve. According to [10], the Steiner center S of a convex curve is defined by

$$S = \left(\int_0^{2\pi} p\cos\theta d\theta, \int_0^{2\pi} p\sin\theta d\theta\right).$$

Noticing that $p(\theta, t) = -\langle X(\theta, t), N(\theta, t)\rangle$, one has the evolution equation of the support function as follows

$$\frac{\partial p}{\partial t}(\theta, t) = -\left\langle \frac{\partial X}{\partial t}(\theta, t), N(\theta, t)\right\rangle = -p(\theta, t) + \frac{E_2(0)}{E_3(t)}.$$

So

$$\frac{d}{dt}\int_0^{2\pi} p\cos\theta d\theta = \int_0^{2\pi}\left(\frac{E_2(0)}{E_3(t)} - p\right)\cos\theta d\theta = -\int_0^{2\pi} p\cos\theta d\theta.$$

Similarly, we have $\frac{d}{dt}\int_0^{2\pi} p\sin\theta d\theta = -\int_0^{2\pi} p\sin\theta d\theta$. Therefore, $\frac{d}{dt}S(t) = -S(t)$, that is

$$S(t) = e^{-t}S(0). \tag{3.7}$$

Hence the Steiner center of the evolving curve runs to the original point of the plane as time goes to infinity. Since the Steiner center of a circle is its center, the center of the limiting circle is the original point of the plane. Q.E.D.

Although it is shown that the flow (1.1) has long time solution on $S^1 \times [0, \infty)$ by (2.6), one can obtain better estimate of the curvature given in the following lemma.

Lemma 3.3 *(Uniform Bound of the Curvature) There exist two positive constants C_1, C_2 such that*

$$C_1 \le \kappa(\theta, t) \le C_2,$$

for all $(\theta, t) \in S^1 \times [0, \infty)$.

Proof. Define $\kappa_{\min}(t) = \min\{\kappa(\theta, t)|\theta \in [0, 2\pi]\}$ and $\kappa_{\max}(t) = \max\{\kappa(\theta, t)|\theta \in [0, 2\pi]\}$. By the evolution equation of κ and the Cauchy-Schwartz inequality, one gets that

$$\frac{\partial \log\kappa}{\partial t} = 1 - \frac{E_2}{E_3(t)}\kappa \ge 1 - \frac{2\pi}{E_2}\kappa.$$

Since $\kappa_{\min}(t) \le \frac{E_2}{2\pi}$, $\log\kappa_{\min}(t)$ is an increasing function. Thus one has a lower bound of $\kappa_{\min}(t)$:

$$\kappa_{\min}(t) \ge \kappa_{\min}(0) > 0.$$

Compute that

$$\frac{d}{dt}\left[\left(\frac{\partial\kappa}{\partial\theta}\right)^2\right] = 2\left(\frac{\partial\kappa}{\partial\theta}\right)^2 - 4\kappa\left(\frac{\partial\kappa}{\partial\theta}\right)^2\frac{E_2}{E_3}.$$

If $\left(\frac{\partial\kappa}{\partial\theta}\right)^2 > 0$ holds in some interval then $\frac{\partial}{\partial t}\log\left(\frac{\partial\kappa}{\partial\theta}\right)^2 = 2 - 4\kappa\frac{E_2}{E_3(t)}$ and furthermore,

$$\frac{\partial}{\partial t}\left(\log\left(\frac{\partial\kappa}{\partial\theta}\right)^2 - 2\log\kappa\right) = -2\kappa\frac{E_2}{E_3(t)} < 0.$$

Hence,

$$\left(\frac{\partial\kappa}{\partial\theta}\right)^2 \leq \left(\frac{\partial\kappa_0}{\partial\theta}\right)^2\frac{\kappa^2}{\kappa_0^2}.$$

Now the Hanarck estimate of κ can be deduced in the following trick,

$$\begin{aligned}
\log\kappa_{\max}(t) - \log\kappa_{\min}(t) &= \int_{\theta_1}^{\theta_2}\frac{\partial\kappa}{\partial\theta}(\theta,t)/\kappa(\theta,t)d\theta \\
&\leq \int_0^{2\pi}\left|\frac{\partial\kappa}{\partial\theta}(\theta,t)\right|\frac{1}{\kappa(\theta,t)}d\theta \\
&\leq \sqrt{2\pi}\sqrt{\int_0^{2\pi}\left|\frac{\partial\kappa}{\partial\theta}(\theta,t)\right|^2\frac{1}{\kappa(\theta,t)^2}d\theta} \\
&\leq \sqrt{2\pi}\sqrt{\int_0^{2\pi}\left|\frac{\partial\kappa_0}{\partial\theta}\right|^2\frac{1}{\kappa_0^2}d\theta} \triangleq C.
\end{aligned}$$

So one obtains the upper bound of κ: $\kappa_{\max}(t) \leq \kappa_{\min}(t)e^C \leq \frac{E_2}{2\pi}e^C$. Q.E.D.

Since both κ and $\frac{\partial\kappa}{\partial\theta}$ are uniformly bounded on $[0,2\pi]\times[0,\infty)$, the Arzela-Ascoli Theorem implies that there exists a convergent sequence, denoted by $\kappa(\theta,t_i)$, as t_i tends to infinity. So one can study the C^2 and C^∞ convergence of the flow (1.1) by showing that κ converges to a positive constant and all the derivatives $\frac{\partial^i\kappa}{\partial\theta^i}$ tend to 0 as $t\to\infty$, $i=1,2,\cdots$. However, there is another choice to give the proof of the convergence.

Lemma 3.4 *(C^∞ Convergence) Under the flow (1.1), the radius of the curvature $\rho(\theta,t)$ converges to a constant $\frac{L_\infty}{2\pi}$ and all the derivatives of the radius of curvature $\frac{\partial^i\rho}{\partial\theta^i}(\theta,t)$ converges to 0 as time tends to infinity.*

Proof. By the evolution equation of the curvature, the radius of curvature of the evolving curve satisfies

$$\frac{\partial\rho}{\partial t}(\theta,t) = \frac{E_2(0)}{E_3(t)} - \rho(\theta,t). \tag{3.8}$$

So

$$\frac{d}{dt}\left(\rho(\theta,t) - \frac{L(t)}{2\pi}\right) = -\left(\rho(\theta,t) - \frac{L(t)}{2\pi}\right),$$

$$\rho(\theta,t)-\frac{L(t)}{2\pi}=\left(\rho(\theta,0)-\frac{L(0)}{2\pi}\right)e^{-t}. \tag{3.9}$$

This equation tells us

$$\lim_{t\to\infty}\rho(\theta,t)=\frac{L_\infty}{2\pi},$$

i.e., the evolving curve converges to the limiting circle in the C^2 metric. Thus one gets the C^2 convergence of the flow (1.1). By Equation (3.8), one gets

$$\frac{\partial^k\rho}{\partial\theta^k}(\theta,t)=\frac{\partial^k\rho}{\partial\theta^k}(\theta,0)e^{-t},$$

which gives us the C^∞ convergence of the flow (1.1) by using the uniform bound of the curvature in Lemma 3.3. Q.E.D.

Now, combining Theorem 2.1, Corollary 2.3, Theorem 2.4, Lemma 3.2 and Lemma 3.4 can give us the proof the Main Theorem. As an application of the flow (1.1), we prove an inequality for convex plane curves.

Theorem 3.5 *Let $X_0(\theta)$ be a smooth and strictly convex curve in the plane, where θ is the tangential angle of the curve. Let us choose a point in the domain enclosed by X_0 to be the original point of the plane. Denote by p_0 the support function of this curve, i.e., $p_0(\theta) = -\langle X_0(\theta), N(\theta)\rangle$. The following inequality holds for the curve X_0:*

$$\int_0^{2\pi}p_0^2d\theta\geq\frac{8\pi^3}{E_2^2}+\frac{1}{2\pi}(L_0^2-4\pi A_0), \tag{3.10}$$

where E_2 is the elastic energy of X_0, L_0 and A_0 are the perimeter and the area enclosed by X_0, respectively. The equality holds in (3.10) if and only if X_0 is a circle and the original point of the plane is the center of this circle.

Proof. Let $X_0(\theta)$ evolve according to the flow (2.2). Then one gets a family of convex curves denoted by $X(\theta,t)$. Under this flow the support function of the evolving curve satisfies

$$\frac{\partial p}{\partial t}(\theta,t)=\frac{E_2}{E_3(t)}-p(\theta,t),$$

where $E_3(t)=\int_0^{2\pi}\kappa^2(\theta,t)d\theta$ and $\kappa(\theta,t)$ is the curvature of the evolving curve X at (θ,t). Since the flow (2.2) preserves the elastic energy of $X(\cdot,t)$, $E_2(t)\equiv E_2(0)=\int_0^{2\pi}\kappa_0(\theta)d\theta$. By the definition of p, one has the Cauchy formula $\int_0^{2\pi}pd\theta=L$. Compute that

$$\frac{d}{dt}\int_0^{2\pi}p^2d\theta \quad=\quad 2\int_0^{2\pi}p\left(\frac{E_2}{E_3(t)}-p\right)d\theta=2\frac{E_2}{E_3(t)}L-2\int_0^{2\pi}p^2d\theta.$$

By Lin-Tai's inequality (3.3) and the Cauchy-Schwartz inequality, one obtains that

$$\frac{d}{dt}\int_0^{2\pi}p^2d\theta \quad\leq\quad 2\frac{2A(t)}{L(t)}L(t)-\frac{L(t)^2}{\pi}=\frac{1}{\pi}(4\pi A(t)-L(t)^2)=\frac{1}{2\pi}\frac{d}{dt}(L(t)^2-4\pi A(t)).$$

Integrating the both sides of the above inequality with respect to t can give us:

$$\int_0^{2\pi} p^2 d\theta - \int_0^{2\pi} p_0^2 d\theta \leq \frac{1}{2\pi}(L(t)^2 - 4\pi A(t)) - \frac{1}{2\pi}(L_0^2 - 4\pi A_0).$$

Noticing that $\lim_{t\to\infty} p(\theta,t) = \frac{L_\infty}{2\pi} = \frac{2\pi}{E_2}$ (imitate the proof of Equation (3.9)), one obtains

$$\frac{8\pi^3}{E_2^2} - \int_0^{2\pi} p_0^2 d\theta \leq -\frac{1}{2\pi}(L_0^2 - 4\pi A_0).$$

If X_0 is a circle and the original point of the plane is the center of this circle then p is equal to the radius and the equality of (3.10) holds. Now, suppose the equality holds in (3.4). The Cauchy-Schwartz inequality and Gage's inequality implies that

$$\frac{L_0^2}{2\pi} \leq \int_0^{2\pi} p_0^2 d\theta = \frac{8\pi^3}{E_2^2} + \frac{1}{2\pi}(L_0^2 - 4\pi A_0) \leq \frac{8\pi A_0^2}{L_0^2} + \frac{1}{2\pi}(L_0^2 - 4\pi A_0).$$

Comparing the both sides can give us $0 \leq \frac{4\pi A_0}{L_0^2} - 1$. So $L_0^2 - 4\pi A_0 = 0$, i.e., X_0 is a circle. So the equality (3.10) is $\int_0^{2\pi} p_0^2 d\theta = \frac{8\pi^3}{E_2^2}$. Since $\int_0^{2\pi} p_0^2 d\theta \geq \frac{L_0^2}{2\pi}$ and $\frac{L_0^2}{2\pi} \geq \frac{8\pi^3}{E_2^2}$, the Cauchy-Schwartz inequality tells us p is a constant, i.e., the original point of the plane is the center of the circle X_0. Q.E.D.

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