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Inverse spectral analysis for a class of infinite band symmetric matrices ☆

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A R T I C L E I N F O

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A B S T R A C T

This work deals with the direct and inverse spectral analysis for a class of infinite band symmetric matrices. This class corresponds to operators arising from difference equations with usual and *inner* boundary conditions. We give a characterization of the spectral functions for the operators and provide necessary and sufficient conditions for a matrix-valued function to be a spectral function of the operators. Additionally, we give an algorithm for recovering the matrix from the spectral function. The approach to the inverse problem is based on the rational interpolation theory.

1. Introduction

In this paper, the direct and inverse spectral analysis of a class of infinite real symmetric band matrices, denoted $\mathcal{M}(n,\infty)$, is carried out with emphasis in the inverse problems of characterization and reconstruction. The matrices under consideration, defined in the paragraphs below, arise from difference equations with initial and left endpoint boundary conditions together with the so called *inner* boundary conditions. Inner boundary conditions are given by degenerations of the diagonals (see the paragraphs above Definition 1 and equation (2.4)). Each matrix in $\mathcal{M}(n,\infty)$ generates uniquely a closed symmetric operator for which we give a spectral characterization. More specifically, we provide necessary and sufficient conditions for a matrix-valued function to be a spectral function of the operators stemming from our class of matrices (see Definition 5 and Theorems 5.1 and 5.2). As a byproduct of the spectral analysis of the operators

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corresponding to matrices in $\mathcal{M}(n,\infty)$, we find an if-and-only-if criterion for degeneration in terms of the properties of polynomials in an L_2 space (see Theorem 3.1).

Although the inverse spectral problems for Jacobi matrices have been studied extensively (see for instance [7–9,14,19,22–24,34–36] for the finite case and [10,11,13,14,20,21,37,38] for the infinite case), works dealing with band matrices, not necessarily tridiagonal, are not so abundant (see [5,17,18,27–29,32,41,42] for the finite case and [3,16] for the infinite case).

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and fix an orthonormal basis $\{\delta_k\}_{k=1}^{\infty}$ in it. We study the symmetric operator A whose matrix representation with respect to $\{\delta_k\}_{k=1}^{\infty}$ is a real symmetric band matrix which is denoted by $\mathcal{A}$ (see [2, Sec. 47] for the definition of the matrix representation of an unbounded symmetric operator).

We assume that the matrix $\mathcal{A}$ has $2n+1$ band diagonals ($n \in \mathbb{N}$), that is, $2n+1$ diagonals not necessarily zero. The band diagonals satisfy the following conditions. The band diagonal farthest from the main one, which is given by the diagonal matrix $\operatorname{diag}\{d_k^{(n)}\}_{k=1}^{\infty}$, denoted by $\mathcal{D}_n$, is such that, for some $m_1 \in \mathbb{N}$, all the numbers $d_1^{(n)}, \dots, d_{m_1-1}^{(n)}$ are positive and $d_k^{(n)} = 0$ for all $k \geq m_1$ with

$$m_1 > 1\,. \tag{1.1}$$

It may happen that all the elements of the sequence $\{d_k^{(n)}\}_{k\in\mathbb{N}}$ are positive which we convene to mean that $m_1 = \infty$.

Now, if $m_1 < \infty$, then the elements $\{d_{m_1+k}^{(n-1)}\}_{k=1}^{\infty}$ of the diagonal next to the farthest, $\mathcal{D}_{n-1}$, behave in the same way as the elements of $\mathcal{D}_n$, that is, there is m_2, satisfying

$$m_1 < m_2\,, \tag{1.2}$$

such that $d_{m_1+1}^{(n-1)}, \dots, d_{m_2-1}^{(n-1)} > 0$ and $d_k^{(n-1)} = 0$ for all $k \geq m_2$. Here, it is also possible that $m_2 = \infty$ in which case $d_k^{(n-1)} > 0$ for all $k > m_1$.

We continue applying the same rule as long as $m_1, \dots, m_j$ are finite. Thus, if $m_j < \infty$, there is m_{j+1}, satisfying

$$m_j < m_{j+1}\,, \tag{1.3}$$

such that $d_{m_j+1}^{(n-j)}, \dots, d_{m_{j+1}-1}^{(n-j)} > 0$ (here we assume that $m_j + 1 < m_{j+1}$) and $d_k^{(n-j)} = 0$ for all $k \geq m_{j+1}$. If $m_j = \infty$, then $d_k^{(n-j)} > 0$ for all $k > m_j$. Eventually, there is $j_0 \leq n-1$ such that $m_{j_0+1} = \infty$. We allow j_0 to be zero, which accordingly means that $m_1 = \infty$.

If $j_0 \geq 1$, as long as $j < j_0$, we say that the diagonal corresponding to $\mathcal{D}_{n-j}$ undergoes degeneration at m_{j+1}. Note that the diagonal corresponding to $\mathcal{D}_{n-j_0}$ does not degenerate. Also, j_0 defines the number of degenerations that the matrix $\mathcal{A}$ has.

Definition 1. For a natural number n, the set of matrices satisfying the above properties is denoted by $\mathcal{M}(n,\infty)$. The set of numbers $\{m_j\}_{j=1}^{j_0}$ characterizes the degenerations of the diagonals. For a matrix without degenerations, this set is empty.

A matrix in $\mathcal{M}(n,\infty)$ has the particular structure illustrated in Fig. 1. Due to transformations similar to the one given in [39, Lem. 1.6], this class of matrices is wider than it seems. We shall see in Section 2 that the rows where there are degenerations and the ones where there are not (cf. (2.12) and (2.10)) give rise to difference equations playing different roles in the spectral analysis of the operator corresponding to the matrix. This is so even when all entries denoted by gray squares are zero.

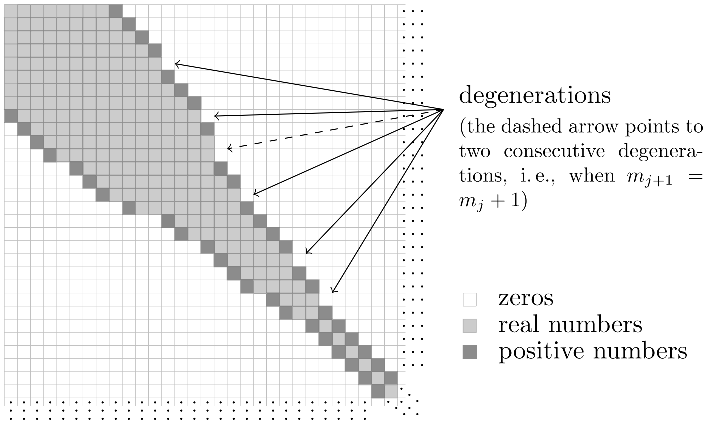


Fig. 1. The structure of a matrix in $\mathcal{M}(n, \infty)$.

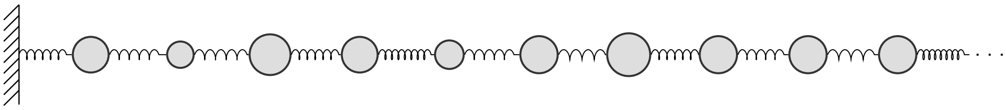


Fig. 2. Mass-spring system corresponding to a Jacobi matrix.

Remark 1. Define the number $n_0 := n - j_0$. Note that the "tail" of the matrix, that is, the semi-infinite submatrix obtained by removing the first m_{j_0} columns and rows, has $2n_0 + 1$ diagonals and the diagonal $\mathcal{D}_{n_0}$ has only positive numbers.

An example of a matrix in $\mathcal{M}(3, \infty)$, when $m_1 = 3$ and $m_2 = 5$, is the following.

$$\mathcal{A} = \begin{pmatrix} d_1^{(0)} & d_1^{(1)} & d_1^{(2)} & d_1^{(3)} & 0 & 0 & 0 & \dots \\ d_1^{(1)} & d_2^{(0)} & d_2^{(1)} & d_2^{(2)} & d_2^{(3)} & 0 & 0 & \\ d_1^{(2)} & d_2^{(1)} & d_3^{(0)} & d_3^{(1)} & d_3^{(2)} & 0 & 0 & \ddots \\ d_1^{(3)} & d_2^{(2)} & d_3^{(1)} & d_4^{(0)} & d_4^{(1)} & d_4^{(2)} & 0 & \ddots \\ 0 & d_2^{(3)} & d_3^{(2)} & d_4^{(1)} & d_5^{(0)} & d_5^{(1)} & 0 & \ddots \\ 0 & 0 & 0 & d_4^{(2)} & d_5^{(1)} & d_6^{(0)} & d_6^{(1)} & \ddots \\ 0 & 0 & 0 & 0 & 0 & d_6^{(1)} & d_7^{(0)} & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.4}$$

For this realization of $\mathcal{A}$, we say that it underwent a degeneration of the diagonal $\mathcal{D}_3$ in $m_1 = 3$ and a degeneration of $\mathcal{D}_2$ in $m_2 = 5$. Note that, in this case, $j_0 = 2$.

It is known that the dynamics of an infinite linear mass-spring system (see Fig. 2) is characterized by the spectral properties of a semi-infinite Jacobi matrix [10,11] when the system is within the regime of validity of the Hooke law (see [15,31] for an explanation of how to obtain the matrix from the mass-spring system in the finite case). The entries of the Jacobi matrix are determined by the masses and spring constants of the system [9–11,15,31]. The movement of the mechanical system of Fig. 2 is a superposition of harmonic oscillations whose frequencies are the square roots of absolute values of the Jacobi operator's eigenvalues. Analogously, one can deduce that a self-adjoint extension of the minimal closed operator generated by a matrix in $\mathcal{M}(n, \infty)$ models a linear mass-spring system where the interaction extends to all the n neighbors

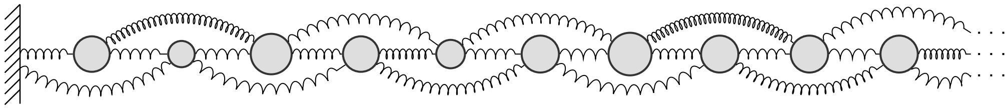


Fig. 3. Mass-spring system of a matrix in $\mathcal{M}(2, \infty)$: nondegenerated case.

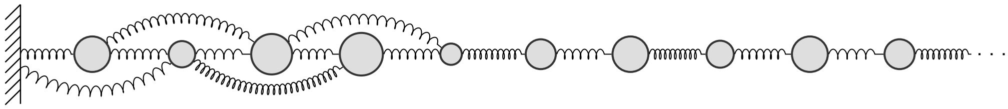


Fig. 4. Mass-spring system of a matrix in $\mathcal{M}(2, \infty)$: degenerated case.

of each mass (cf. [29, Appendix]). For instance, if the matrix is in $\mathcal{M}(2, \infty)$ and no degeneration of the diagonals occurs, viz. $m_1 = \infty$, the corresponding mass-spring system is given in Fig. 3. If for another matrix in $\mathcal{M}(2, \infty)$, one has degeneration of the diagonals, for instance $m_1 = 4$, the corresponding mass-spring system is given in Fig. 4.

In this work, the approach to the inverse spectral analysis of the operators whose matrix representation belongs to $\mathcal{M}(n, \infty)$ is based on the one used in [27–29] which deals with the finite dimensional case. As in those papers, an important ingredient of the inverse spectral analysis is the linear interpolation of vector polynomials. For this paper, as well as in [29], we use the theory of linear interpolation of n-dimensional vector polynomials, recently developed in [30].

This paper is organized as follows. In Section 2, we present the results obtained in [29] on the spectral measures of the operators corresponding to finite dimensional matrices being an upper-left corner of a matrix in $\mathcal{M}(n, \infty)$. These finite dimensional operators play an auxiliary role in the spectral analysis of operator A. Later, in Section 3, we construct a matrix-valued function for each element of $\mathcal{M}(n, \infty)$ having the properties of a spectral function. Section 4 deals with various criteria for the operator A to be self-adjoint and gives the spectral function of A, touching upon some of their properties. Finally, in Section 5, we deal with the problem of reconstruction and characterization.

2. Spectral analysis of submatrices

Fix $N > n$. The spectral analysis of the operator A is carried out by means of the auxiliary operator $P_{\mathcal{H}_N} A \restriction_{\mathcal{H}_N}$, where $\mathcal{H}_N = \operatorname{span}\{\delta_i\}_{i=1}^N$, and $P_{\mathcal{H}_N}$ is the orthogonal projection onto the subspace $\mathcal{H}_N$. Note that $P_{\mathcal{H}_N} A \restriction_{\mathcal{H}_N}$ can be identified with the operator whose matrix representation is the finite dimensional submatrix corresponding to the $N \times N$ upper-left corner of a matrix in $\mathcal{M}(n, \infty)$ (cf. (1.4)). We denote the class of these $N \times N$ matrices by $\mathcal{M}(n, N)$ and the corresponding operator in $\mathcal{H}_N$ is denoted by $\widetilde{A}_N$.

According to [29, Sec. 2], the spectral analysis of the operator $\widetilde{A}_N$ can be carried out by studying a system of N equations, where each equation, given by a fixed $k \in \{1, \dots, N\}$, is of the form (cf. [29, Eq. (2.2)])

$$\sum_{i=0}^{n-1} d_{k-n+i}^{(n-i)} \varphi_{k-n+i} + d_k^{(0)} \varphi_k + \sum_{i=1}^{n} d_k^{(i)} \varphi_{k+i} = z\varphi_k \,, \tag{2.1}$$

where it has been assumed that

$$\varphi_k = 0 \,, \quad \text{for } k < 1 \,, \tag{2.2a}$$

$$\varphi_k = 0 \,, \quad \text{for } k > N \,. \tag{2.2b}$$

One can consider (2.2) as boundary conditions where (2.2a) is the condition at the left endpoint and (2.2b) is the condition at the right endpoint.

The system (2.1) with (2.2), restricted to $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^{j_0}$ when there are degenerations, can be solved recursively whenever the first n entries of the vector φ are given. Let $\varphi^{(j)}(z)$ ($j \in \{1, \dots, n\}$) be a solution of (2.1) for all $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^{j_0}$ such that

$$\left\langle \delta_i, \varphi^{(j)}(z) \right\rangle = t_{ji}, \text{ for } i = 1, \dots, n\,, \tag{2.3}$$

where $\mathcal{T} = \{t_{ji}\}_{j,i=1}^{n}$ satisfies

I) $\mathcal{T}$ is $n \times n$ upper triangular with real entries.
II) $\prod_{i=1}^{n} t_{ii} \neq 0$.

Each matrix $\mathcal{T}$ yields a system of solutions $\{\varphi^{(j)}(z)\}_{j=1}^{n}$ that constitutes a basis in the space of solutions of (2.1) and (2.2a).

The condition given by (2.3) can be seen as the initial conditions for the system (2.1) and (2.2a). We emphasize that, given the boundary condition at the left endpoint (2.2a) and the initial condition (2.3), the system (2.1), restricted to $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^{j_0}$ in the presence of degenerations, has a unique solution for any fixed $j \in \{1, \dots, n\}$ and $z \in \mathbb{C}$.

The degenerations, which the diagonals of matrices in $\mathcal{M}(n, N)$ undergo, are related to another kind of "boundary conditions". Indeed, the equations of the system (2.1), when $k \in \{m_i\}_{i=1}^{j_0}$, give rise to the inner boundary conditions (of the right endpoint type) (cf. [29, Eq. (2.8)]).

Let $\{x_l\}_{l=1}^{N}$ be the spectrum of the operator $\widetilde{A}_N$, denoted $\operatorname{spec} \widetilde{A}_N$. The eigenvector $\alpha(x_l)$ corresponding to the eigenvalue x_l, normalized in such a way that $\|\alpha(x_l)\| = 1$, can be decomposed as follows

$$\alpha(x_l) = \sum_{j=1}^{n} \alpha_j(x_l)\varphi^{(j)}(x_l)\,, \tag{2.4}$$

where $\alpha_j(x_l) \in \mathbb{C}$. It follows from (2.1), (2.2), and (2.3), that

$$\sum_{j=1}^{n} |\alpha_j(x_k)| > 0 \quad \text{for all } \; k \in \{1, \dots, N\}$$

and

$$\sum_{k=1}^{N} |\alpha_j(x_k)| > 0 \quad \text{for all } \; j \in \{1, \dots, n\}\,. \tag{2.5}$$

The operator $\widetilde{A}_N$ has a matrix-valued spectral function

$$\sigma_N^{\mathcal{T}}(t) = \sum_{x_l < t} \begin{pmatrix} |\alpha_1(x_l)|^2 & \overline{\alpha_1(x_l)}\alpha_2(x_l) & \dots & \overline{\alpha_1(x_l)}\alpha_n(x_l) \\ \overline{\alpha_2(x_l)}\alpha_1(x_l) & |\alpha_2(x_l)|^2 & \dots & \overline{\alpha_2(x_l)}\alpha_n(x_l) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\alpha_n(x_l)}\alpha_1(x_l) & \overline{\alpha_n(x_l)}\alpha_2(x_l) & \dots & |\alpha_n(x_l)|^2 \end{pmatrix} \tag{2.6}$$

with the following properties:

a) It is a nondecreasing monotone step function which is continuous from the left.
b) Each jump is a matrix of rank not greater than n.
c) The sum of the ranks of all jumps equals N.

Note that the matrices in the sum on the right-hand side of (2.6) are the tensor product of the vector $\begin{pmatrix} \overline{\alpha_1(x_l)} \\ \vdots \\ \overline{\alpha_n(x_l)} \end{pmatrix}$ with the complex conjugate of itself.

The relationship between the spectral functions $\sigma_N^{\mathcal{T}}$ for an arbitrary $\mathcal{T}$ and the case $\mathcal{T} = I$ is given by the following equation which is proven in [29, Pro. 2.1].

$$\mathcal{T}^* \int_{\mathbb{R}} d\,\sigma_N^{\mathcal{T}} \mathcal{T} = \int_{\mathbb{R}} d\,\sigma_N^{I} = I\,. \tag{2.7}$$

Consider the Hilbert space $L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})$ with the usual inner product which we assume to be antilinear in the first argument (for the definition of $L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})$, see [2, Sec. 72]). Clearly, property c) above implies that $L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})$ is an N-dimensional space. By polynomial interpolation, one verifies that in each equivalence class there is an n-dimensional vector polynomial.

Define the vectors

$$\boldsymbol{p}_k := \mathcal{T}\boldsymbol{e}_k \qquad \text{for } k = 1, \ldots, n\,, \tag{2.8}$$

where $\{\boldsymbol{e}_k\}_{k=1}^n$ is the canonical basis in $\mathbb{C}^n$, i.e.,

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \boldsymbol{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{2.9}$$

Taking $\{\boldsymbol{p}_k\}_{k=1}^n$ as initial conditions of the recurrence equation

$$\sum_{i=0}^{n-1} d_{k-n+i}^{(n-i)} \boldsymbol{p}_{k-n+i}(z) + d_k^{(0)} \boldsymbol{p}_k(z) + \sum_{i=1}^{n} d_k^{(i)} \boldsymbol{p}_{k+i}(z) = z\boldsymbol{p}_k(z)\,, \quad k \in \mathbb{N} \setminus \{m_j\}_{j=1}^{j_0}\,, \tag{2.10}$$

where it is assumed that

$$\boldsymbol{p}_k = 0\,, \quad \text{for} \quad k < 1\,, \tag{2.11}$$

one obtains a sequence $\{\boldsymbol{p}_k(z)\}_{k=1}^{\infty}$ of vector polynomials. The next assertion is proven in [29, Lem. 2.2].

Proposition 2.1. *For any natural number $N > n$, the vector polynomials $\{\boldsymbol{p}_k(z)\}_{k=1}^N$, defined by (2.10), satisfy*

$$\left\langle \boldsymbol{p}_j, \boldsymbol{p}_k \right\rangle_{L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})} = \delta_{jk}$$

for $j, k \in \{1, \ldots, N\}$.

Let $U : \mathcal{H}_N \to L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})$ be the isometry given by $U\delta_k \mapsto \boldsymbol{p}_k$, for all $k \in \{1, \ldots, N\}$. Under this isometry the operator $\widetilde{A}_N$ becomes the operator of multiplication by the independent variable in $L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})$ (see [29, Sec. 2]).

Define

$$\boldsymbol{q}_j(z) := (z - d_{m_j}^{(0)})\boldsymbol{p}_{m_j}(z) - \sum_{k=0}^{n-1} d_{m_j-n+k}^{(n-k)} \boldsymbol{p}_{m_j-n+k}(z) - \sum_{k=1}^{n-j} d_{m_j}^{(k)} \boldsymbol{p}_{m_j+k}(z) \tag{2.12}$$

for $j \in \{1, \ldots, j_0\}$.

Using the same reasoning as in [29, Thm. 3.1], one proves that, for any natural number $N \geq n_0 + m_{j_0}$ (see Remark 1), the vector polynomials $\{\boldsymbol{q}_j(z)\}_{k=1}^{j_0}$ satisfy

$$\left\langle \boldsymbol{q}_j, \boldsymbol{q}_j \right\rangle_{L_2(\mathbb{R},\sigma_N^{\mathcal{T}})} = 0\,. \tag{2.13}$$

The existence of polynomials of zero norm in $L_2(\mathbb{R}, \sigma_N^{\mathcal{T}})$ is related to a linear interpolation problem consisting in the following: Given collections of numbers $\{z_k\}_{k=1}^N$ and $\{\alpha_j(k)\}_{j=1}^n$ ($k = 1, \dots, N$), find the scalar polynomials $R_j(z)$, ($j = 1, \dots, n$), which satisfy the equation

$$\sum_{j=1}^{n} \alpha_j(k) R_j(z_k) = 0\,, \quad \forall k \in \{1, \dots, N\}\,.$$

This is equivalent (see [30, Sec. 4]) to finding n-dimensional vector polynomials satisfying

$$\langle \boldsymbol{r}(z), \boldsymbol{r}(z) \rangle_{L_2(\mathbb{R},\sigma_N^{\mathcal{T}})} = 0\,, \qquad \boldsymbol{r}(z) = (R_1(z), R_2(z), \dots, R_n(z))^t\,. \tag{2.14}$$

In [30], it was found that the solutions of the linear interpolation problem given by (2.14) are determined by a set of n vector polynomials called generators [30, Thm. 5.3]. Let us introduce some concepts related to the generators of a linear interpolation problem (for a detailed account of this, see [30, Secs. 2 and 4]).

Definition 2. Let $\boldsymbol{r}(z) = (R_1(z), R_2(z), \dots, R_n(z))^t$ be an n-dimensional vector polynomial. The height of $\boldsymbol{r}(z)$ is the number

$$h(\boldsymbol{r}) := \max_{j \in \{1,\dots,n\}} \{n \deg(R_j) + j - 1\}\,,$$

where it is assumed that $\deg 0 := -\infty$ and $h(\boldsymbol{0}) := -\infty$.

Note that we have defined the vector polynomials $\{\boldsymbol{e}_k\}_{k=1}^n$ so that

$$h(\boldsymbol{e}_k) = k - 1\,. \tag{2.15}$$

The following assertion is proven in [30, Thm. 2.1] (see also [29, Prop. 3.1]). We reproduce it here for the reader's convenience.

Proposition 2.2. *If a collection of n-dimensional vector polynomials $\{\boldsymbol{g}_i(z)\}_{i=1}^j$ ($j \geq 1$) satisfy $h(\boldsymbol{g}_i) = i - 1$ for all $i \in \{1, \dots, j\}$, then any vector polynomial $\boldsymbol{r}(z)$ with height $j - 1$ can be written as follows*

$$\boldsymbol{r}(z) = \sum_{i=1}^{j} c_i \boldsymbol{g}_i(z)\,,$$

where $c_i \in \mathbb{C}$ for all $i \in \{1, \dots, n\}$ and $c_j \neq 0$.

Definition 3. The first generator of the linear interpolation problem given by (2.14) is the n-dimensional vector polynomial satisfying (2.14) and having the least height. For $k > 1$, the k-th generator $\boldsymbol{q}_k$ of the linear interpolation problem is a solution of (2.14) with least height such that it cannot be written as a linear combination of

$$\{R_j(z)\boldsymbol{q}_j(z)\}_{j=1}^{k-1}\,,$$

where $\boldsymbol{q}_j(z)$ is the j-th generator of the linear interpolation problem and $R_j(z)$ is a scalar polynomial.

Having the concepts of height of a vector polynomial and generator of the interpolation problem (2.14) at hand, we invoke results from [29] and [30]. First we agree on the following.

Convention 1. From now on, we consider the natural number N to be no less than $n_0 + m_{j_0}$ (see Remark 1).

Proposition 2.3. *([29, Thm. 3.1]) The vector polynomials $\{\boldsymbol{q}_j(z)\}_{j=1}^{j_0}$ are the first j_0 generators of the linear interpolation problem given by (2.14) (see [29, Sec. 3]). Moreover, for $j = 1, \dots, j_0$, the numbers $h(\boldsymbol{q}_j)$ are different elements of the factor space $\mathbb{Z}/n\mathbb{Z}$ [30, Lem. 4.3].*

The heights of the vector polynomials $\{\boldsymbol{p}_k\}_{k=n+1}^{\infty}$ are determined recursively by means of the system (2.10). Indeed, for any $m_j < k < m_{j+1}$, with $j = 0, \dots, j_0$, one has the equation

$$\cdots + d_k^{(0)} \boldsymbol{p}_k + d_k^{(1)} \boldsymbol{p}_{k+1} + \cdots + d_k^{(n-j)} \boldsymbol{p}_{k+n-j} = z\boldsymbol{p}_k \,,$$

where we have assumed that $m_0 = 0$. Since $d_k^{(n-j)}$ never vanishes, the height of $\boldsymbol{p}_{k+n-j}$ coincides with the one of $z\boldsymbol{p}_k$. Thus

$$h(\boldsymbol{p}_{k+n-j}) = n + h(\boldsymbol{p}_k) \,. \tag{2.16}$$

If there are no degenerations of the diagonals, then (2.16) implies that

$$h(\boldsymbol{p}_k) = k - 1 \,, \quad \text{for all } k \in \mathbb{N} \,. \tag{2.17}$$

On the other hand, in the presence of degenerations, one verifies from (2.12) and (2.16) that

$$h(\boldsymbol{p}_k) \neq h(\boldsymbol{p}_{m_j}) + n = h(\boldsymbol{q}_j) \,, \tag{2.18}$$

for any $k \in \mathbb{N}$ and $j = 1, \dots, j_0$.

Lemma 2.1. *For any nonnegative integer s, there exist $k \in \mathbb{N}$ or a pair $j \in \{1, \dots, j_0\}$ and $l \in \mathbb{N} \cup \{0\}$ such that either $s = h(\boldsymbol{p}_k)$ or $s = h(\boldsymbol{q}_j) + nl$.*

Proof. This proof repeats the one of [29, Lem. 3.3]. We have reproduced it here for the reader's convenience. Due to (2.16), it follows from (2.8) and (2.15) that

$$h(\boldsymbol{p}_k) = k - 1 \quad \text{for } k = 1, \dots, h(\boldsymbol{q}_1) \tag{2.19}$$

(cf. (2.17)).

Suppose that there is $s \in \mathbb{N}$ ($s > n$) such that $s \neq h(\boldsymbol{p}_k)$ for all $k \in \mathbb{N}$ and $s \neq h(\boldsymbol{q}_j) + nl$ for all $j \in \{1, \dots, j_0\}$ and $l \in \mathbb{N} \cup \{0\}$. Let $\hat{l}$ be an integer such that $s - n\hat{l} \in \{h(\boldsymbol{p}_k)\}_{k=1}^{\infty} \cup \{h(\boldsymbol{q}_j) + nl\}$ ($j \in \{1, \dots, j_0\}$ and $l \in \mathbb{N} \cup \{0\}$). There is always such an integer due to (2.19) and $h(\boldsymbol{q}_1) > n$ (see (2.18)). We take $\hat{l}_0$ to be the minimum of all $\hat{l}$'s. Thus, there is $k' \in \mathbb{N}$ or $j' \in \{1, \dots, j_0\}$, respectively, such that either

a) $s - n\hat{l}_0 = h(\boldsymbol{p}_{k'})$ or
b) $s - n\hat{l}_0 = h(\boldsymbol{q}_{j'}) + nl$, with $l \in \mathbb{N} \cup \{0\}$.

In the case a), we prove that $\hat{l}_0$ is not the minimum integer, this implies the assertion of the lemma. Indeed, if there is $j \in \{1, \dots, j_0\}$ such that $k' = m_j$, then $s - n\hat{l}_0 + n = h(\boldsymbol{p}_{m_j}) + n = h(\boldsymbol{q}_j)$ due to (2.18). If there is not such j, then $m_j < k' < m_{j+1}$ and (2.16) implies $s - n\hat{l}_0 + n = h(\boldsymbol{p}_{k'}) + n = h(\boldsymbol{p}_{k'+n-j})$.

For the case b), if $s - n\hat{l}_0 = h(\boldsymbol{q}_{j'}) + nl$, then $s = h(\boldsymbol{q}_{j'}) + n(l + \hat{l}_0)$ which is a contradiction. □

As a consequence of Proposition 2.2, the above lemma yields the following result.

Corollary 2.1. *Any vector polynomial $\boldsymbol{r}(z)$ is a finite linear combination of*

$$\{\boldsymbol{p}_k(z) : k \in \mathbb{N}\} \cup \{z^l \boldsymbol{q}_j(z) : l \in \mathbb{N} \cup \{0\},\ j \in \{1, \dots, j_0\}\}\,, \tag{2.20}$$

where if $j_0 = 0$, the second set in (2.20) is empty.

To conclude this section, we use the canonical basis of $\mathbb{C}^n$ (see (2.9)) to define a family of vector polynomials for $k \in \mathbb{N}$ and $i = 1, \dots, n$.

$$\boldsymbol{e}_{nk+i}(z) := z^k \boldsymbol{e}_i\,. \tag{2.21}$$

Observe that

$$\langle \boldsymbol{e}_{nk+i}(t), \boldsymbol{e}_{nl+j}(t)\rangle_{L_2(\mathbb{R},\sigma_N^{\mathcal{T}})} = \int\limits_{\mathbb{R}} t^{k+l} d\,\sigma_N^{\mathcal{T}}(i,j)\,, \tag{2.22}$$

where $\sigma_N^{\mathcal{T}}(i,j)$ is given by $\left\langle \boldsymbol{e}_i, \sigma_N^{\mathcal{T}}(t)\boldsymbol{e}_j\right\rangle$ (the entry at i, j of the matrix-valued function (2.6)).

For $k = 0, 1, \dots, \left\lceil \frac{2h(\boldsymbol{p}_N)}{n} \right\rceil$, where $\lceil\cdot\rceil$ is the ceiling function, denote the matrix moments of $\sigma_N^{\mathcal{T}}$ by

$$S_k(\mathcal{T}) := \int\limits_{\mathbb{R}} t^k\, d\sigma_N^{\mathcal{T}}\,. \tag{2.23}$$

On the basis of Corollary 2.1, one verifies that the matrix moments of $\sigma_N^{\mathcal{T}}$ coincide with the ones of $\sigma_{\widetilde{N}}^{\mathcal{T}}$ for any $\widetilde{N} \geq N$. This explains why we have dropped the N in the notation of $S_k(\mathcal{T})$.

Remark 2. Note that, for any natural number k, there exists $N \in \mathbb{N}$ such that $S_{2k}(\mathcal{T})$ is given by (2.23) and it is a positive definite matrix.

3. Spectral analysis of infinite symmetric band matrices

In this section, we construct a matrix-valued function for each element of $\mathcal{M}(n,\infty)$ having the properties of a spectral function. To this end, we give defining criteria for a measure to be a spectral function of a *matrix* in the class $\mathcal{M}(n,\infty)$. By our definition, any spectral function σ of $\mathcal{A}$ in $\mathcal{M}(n,\infty)$ is the spectral function of some self-adjoint extension of the minimal closed operator A generated by $\mathcal{A}$ (see [2, Sec. 47]) so that this self-adjoint operator is transformed by a unitary isometric map, which can be regarded as a Fourier transform, into the operator of multiplication by the independent variable defined on its maximal domain in some space $L_2(\mathbb{R},\sigma)$ (for the definition of this space, see [2, Sec. 72]). It is worth remarking that not all the spectral functions of a matrix in $\mathcal{M}(n,\infty)$ correspond to a self-adjoint extension $\widetilde{A}$ of the minimal closed operator generated by $\mathcal{A}$ such that $\widetilde{A} \subset A^*$ (see Remark 4).

The results of this section and the next one provide a complete description of all possible spectral functions that can be associated with some element of $\mathcal{M}(n,\infty)$ by our criteria.

Definition 4. A nondecreasing $n \times n$ matrix-valued function σ with finite moments, such that $\int_{\mathbb{R}} d\sigma$ is invertible, is called a spectral function of a matrix $\mathcal{A}$ in $\mathcal{M}(n,\infty)$ if and only if there exists $\mathcal{T}$ satisfying I) and II) (see (2.3)) such that $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ is an orthonormal sequence in $L_2(\mathbb{R},\sigma)$ and, for each $j \in \{1, \dots, j_0\}$, $\boldsymbol{q}_j$ is in the equivalence class of zero in $L_2(\mathbb{R},\sigma)$.

Remark 3. The spectral function σ of a matrix in $\mathcal{M}(n,\infty)$ has an infinite number of growth points. Indeed, if σ had a finite number of growth points, then $L_2(\mathbb{R},\sigma)$ would be a finite dimensional space and correspondingly the sequence of vector polynomials $\{\boldsymbol{p}_k\}_k$ would be finite.

As a consequence of Corollary 2.1, all the vector polynomials are in $L_2(\mathbb{R},\sigma)$ when σ is a spectral function of a matrix in $\mathcal{M}(n,\infty)$. Moreover the polynomials are dense in $L_2(\mathbb{R},\sigma)$ when the orthonormal system $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ turns out to be complete.

On the basis of Definition 4, one can construct an isometric map between the original space $\mathcal{H}$ and the subspace being the closure of the polynomials in $L_2(\mathbb{R},\sigma)$. This isometric map, which will be denoted by U, is realized by associating the orthonormal basis $\{\delta_k\}_{k=1}^{\infty}$ with the orthonormal system $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$, i.e., $U\delta_k = \boldsymbol{p}_k$ for all $k \in \mathbb{N}$. Furthermore, under this map, the operator A is transformed into some restriction of the operator of multiplication by the independent variable. Indeed, if $\varphi = \sum_{k=1}^{\infty}\varphi_k\delta_k$ is an element of the domain of A, then $f = \sum_{k=1}^{\infty}\varphi_k\boldsymbol{p}_k$ is in the domain of the operator of multiplication by the independent variable and

$$UAU^{-1}f(t) = tf(t)\,.$$

Lemma 3.1. *Let $\mathcal{A}$ be an element of $\mathcal{M}(n,\infty)$ and $\sigma_N^{\mathfrak{T}}$ be the matrix-valued spectral function of the corresponding operator A_N for a fixed matrix $\mathfrak{T}$ satisfying I) and II). Then, there exists a subsequence $\{\sigma_{N_i}^{\mathfrak{T}}\}_{i=1}^{\infty}$ converging pointwise to a matrix-valued function $\sigma^{\mathfrak{T}}$.*

Proof. In view of (2.7), the hypothesis of Helly's first theorem for bounded operators [6, Thm. 4.3] is satisfied in any bounded interval (cf. [33, Sec. 8.4] for the scalar case), therefore the statement follows. The generalization of Helly's first theorem given in [6, Thm. 4.3] is based on applying the scalar theorem to the bilinear form of the sequence of operators (for fixed elements in the Hilbert space) in a diagonal process fashion using the boundedness of the operators and the separability of the space. This yields the assertion in the sense of weak convergence. Using the fact that uniform and weak convergence are equivalent in finite dimensional spaces, one obtains the assertion. □

The following proposition is obtained by applying [6, Thm. 4.4] to the result above and taking into account that the matrix $\sigma_N^{\mathfrak{T}}$ is finite dimensional for any $N > n$.

Proposition 3.1 *(Helly's generalized second theorem). Suppose that the function $f(t)$ is continuous in the real interval $[a,b]$, where a and b are points of continuity of $\sigma^{\mathfrak{T}}(t)$ (see Lemma 3.1). Then there exists a subsequence $\{\sigma_{N_i}^{\mathfrak{T}}\}_{i=1}^{\infty}$ such that*

$$\int_a^b f(t)\,d\sigma_{N_i}^{\mathfrak{T}}(t) \xrightarrow[i\to\infty]{} \int_a^b f(t)\,d\sigma^{\mathfrak{T}}(t)\,.$$

With these results at hand, we prove the following assertions.

Lemma 3.2. *There exists a subsequence $\{\sigma_{N_i}^{\mathfrak{T}}\}_{i=1}^{\infty}$ such that*

$$\int_{\mathbb{R}} t^k\,d\sigma_{N_i}^{\mathfrak{T}} = \int_{\mathbb{R}} t^k\,d\sigma^{\mathfrak{T}}$$

for any nonnegative integer $k \le \left\lceil \frac{2h(\boldsymbol{p}_{N_i})}{n} \right\rceil$ (see (2.23)).

Proof. If one assumes that $-a < 0$ and $b > 0$ are two points of continuity of $\sigma^{\mathfrak{T}}(t)$, then, it follows from Proposition 3.1 that

$$\int_{-a}^{b} t^k \, d\sigma^{\mathfrak{T}} = \lim_{i\to\infty} \int_{-a}^{b} t^k \, d\sigma^{\mathfrak{T}}_{N_i} .$$

On the other hand, given a number r such that $r > k$, then for $r < \left\lceil \frac{2h(\boldsymbol{p}_{N_i})}{n} \right\rceil$

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} - \int_{-a}^{b} t^k \, d\sigma^{\mathfrak{T}}_{N_i} \right\| &= \left\| \int_{-\infty}^{-a} + \int_{b}^{\infty} t^k \, d\sigma^{\mathfrak{T}}_{N_i} \right\| = \left\| \int_{-\infty}^{-a} + \int_{b}^{\infty} \frac{t^r}{t^{r-k}} \, d\sigma^{\mathfrak{T}}_{N_i} \right\| \\ &\le \frac{1}{c^{r-k}} \left\| \int_{-\infty}^{-a} + \int_{b}^{\infty} t^r \, d\sigma^{\mathfrak{T}}_{N_i} \right\| \le \frac{\|S_r(\mathfrak{T})\|}{c^{r-k}} , \end{aligned}$$

where $c = \min\{a, b\}$ and $S_r(\mathfrak{T}) = \int_{\mathbb{R}} t^r \, d\sigma^{\mathfrak{T}}_{N_i}$ (the integral is convergent due to Proposition 2.1). Thus,

$$\left\| S_k(\mathfrak{T}) - \int_{-a}^{b} t^k \, d\sigma^{\mathfrak{T}} \right\| \le \frac{\|S_r(\mathfrak{T})\|}{c^{r-k}} .$$

This yields the assertion when one makes a and b tend to ∞ in such a way that $-a$ and b are all the time points of continuity of $\sigma^{\mathfrak{T}}(t)$. $\square$

From the previous lemma, one directly obtains the following result.

Corollary 3.1. *The spectral function $\sigma^{\mathfrak{T}}$ to which a subsequence of $\{\sigma^{\mathfrak{T}}_N\}_{N=2}^{\infty}$ converges according to Lemma 3.1 is a solution of a certain matrix moment problem given by $\{S_k(\mathfrak{T})\}_{k=0}^{\infty}$.*

Lemma 3.3. *Any $\mathcal{A} \in \mathcal{M}(n, \infty)$ has at least one spectral function (in the sense of Definition 4).*

Proof. It follows directly from Proposition 2.1 and Lemma 3.2 that the vector polynomials $\{\boldsymbol{p}_k(z)\}_{k=1}^{\infty}$, defined by (2.9) and (2.10), satisfy

$$\langle \boldsymbol{p}_j, \boldsymbol{p}_k \rangle_{L_2(\mathbb{R},\sigma^{\mathfrak{T}})} = \delta_{jk}$$

for $j, k \in \mathbb{N}$, where $\sigma^{\mathfrak{T}}$ is the function given by Lemma 3.1. Now, fix $j \in \{1, \dots, j_0\}$ and consider N according to Convention 1. Thus,

$$0 = \|\boldsymbol{q}_j\|^2_{L_2(\mathbb{R},\sigma^{\mathfrak{T}}_N)} = \int_{\mathbb{R}} \langle \boldsymbol{q}_j, d\sigma^{\mathfrak{T}}_N \boldsymbol{q}_j \rangle .$$

By Lemma 3.2 there is a subsequence $\{\sigma^{\mathfrak{T}}_{N_i}\}_{i=1}^{\infty}$ such that, beginning from some $i \in \mathbb{N}$,

$$0 = \int_{\mathbb{R}} \langle \boldsymbol{q}_j, d\sigma^{\mathfrak{T}}_{N_i} \boldsymbol{q}_j \rangle = \int_{\mathbb{R}} \langle \boldsymbol{q}_j, d\sigma^{\mathfrak{T}} \boldsymbol{q}_j \rangle = \|\boldsymbol{q}_j\|^2_{L_2(\mathbb{R},\sigma^{\mathfrak{T}})} . \quad \square$$

Remark 4. Let $\mathcal{A}$ be in $\mathcal{M}(n,\infty)$ and σ be the spectral function of $\mathcal{A}$ according to Definition 4. If the moment problem associated with σ turns out to be *determinate*, then there is just one solution of the moment problem and this solution, i.e. σ, corresponds to a spectral function of the operator A which turns out to be self-adjoint [12, Sec. 2]. In this case, the function $\sigma^{\mathfrak{T}}$ given by Lemma 3.1 will be the unique solution to *its* corresponding moment problem. Note that one could associate another spectral function to A by considering a different matrix $\mathfrak{T}$ (see (2.3)), but the moment problem for it would be different (and also determinate). If the moment problem is *indeterminate*, then there are various solutions of the moment problem and each solution $\widehat{\sigma}$ is a spectral function of $\mathcal{A}$ since the sequence of polynomials $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ is orthonormal in $L_2(\mathbb{R},\widehat{\sigma})$ for any solution $\widehat{\sigma}$. In this case, $\widehat{\sigma}$ not necessarily corresponds to the spectral function of a canonical self-adjoint extension of the operator A (by a canonical self-adjoint extension of the symmetric operator A we mean a self-adjoint restriction of A^*). Indeed, the solution $\widehat{\sigma}$ is the spectral function of a canonical self-adjoint extension if and only if the polynomials are dense in $L_2(\mathbb{R},\widehat{\sigma})$. We expect that the spectral function $\sigma^{\mathfrak{T}}$, to which a subsequence of $\{\sigma_N^{\mathfrak{T}}\}_{N=2}^{\infty}$ converges according to Lemma 3.1, be such that the polynomials are dense in $L_2(\mathbb{R},\sigma^{\mathfrak{T}})$. This matter, together with other questions on characterization of the functions $\sigma^{\mathfrak{T}}$ will be dealt with in a forthcoming paper.

Definition 5. The set of all $n\times n$-matrix-valued functions with an infinite number of growing points such that all the moments $\{S_k\}_{k=1}^{\infty}$ exist and S_0 is invertible is denoted by $\mathfrak{M}(n,\infty)$. Besides, $\mathfrak{M}_d(n,\infty)$ denotes the subset of $\mathfrak{M}(n,\infty)$ for which the sequence of matrix moments generates a determinate matrix moment problem.

Theorem 3.1. *Let $\mathcal{A}$ be in $\mathcal{M}(n,\infty)$ and j_0 be the number of degenerations of $\mathcal{A}$ (see the paragraph above Definition 1). For any spectral function σ of $\mathcal{A}$, it holds true that:*

i) (Nondegenerate case) If $j_0=0$, i.e., the matrix $\mathcal{A}$ does not undergo degenerations, then there are no vector polynomials in the equivalence class of the zero of the space $L_2(\mathbb{R},\sigma)$, i.e.,

$$\langle \boldsymbol{r}(z),\boldsymbol{r}(z)\rangle_{L_2(\mathbb{R},\sigma)}=0 \iff \boldsymbol{r}\equiv 0\,.$$

ii) (Degenerate case) If $j_0>0$, then all the polynomials $\boldsymbol{q}_1,\dots,\boldsymbol{q}_{j_0}$ are in the equivalence class of zero and any polynomial $\boldsymbol{r}(z)$ in this equivalence class can be written as

$$\boldsymbol{r}(z)=\sum_{j=1}^{j_0} R_j(z)\boldsymbol{q}_j(z)\,, \tag{3.1}$$

where $R_j(z)$ is a scalar polynomial.

Proof. First one proves ii). The first part of the assertion follows immediately from Definition 4. Suppose that there is a nontrivial vector polynomial $\boldsymbol{r}(z)$ in the equivalence class of zero with height r. Therefore, by Corollary 2.1

$$\boldsymbol{r}(z)=\sum_{k=1}^{l} c_k\boldsymbol{p}_k(z)+\sum_{j=1}^{j_0} R_j(z)\boldsymbol{q}_j(z)\,, \tag{3.2}$$

where $\max\{h(\boldsymbol{p}_l),\max_{j=1,\dots,j_0}\{h(R_j\boldsymbol{q}_j)\}\}=r$. Furthermore,

$$c_k=\langle \boldsymbol{r}(z),\boldsymbol{p}_k(z)\rangle_{L_2(\mathbb{R},\sigma)} \qquad \text{for all } k\in\{1,\dots,l\}\,. \tag{3.3}$$

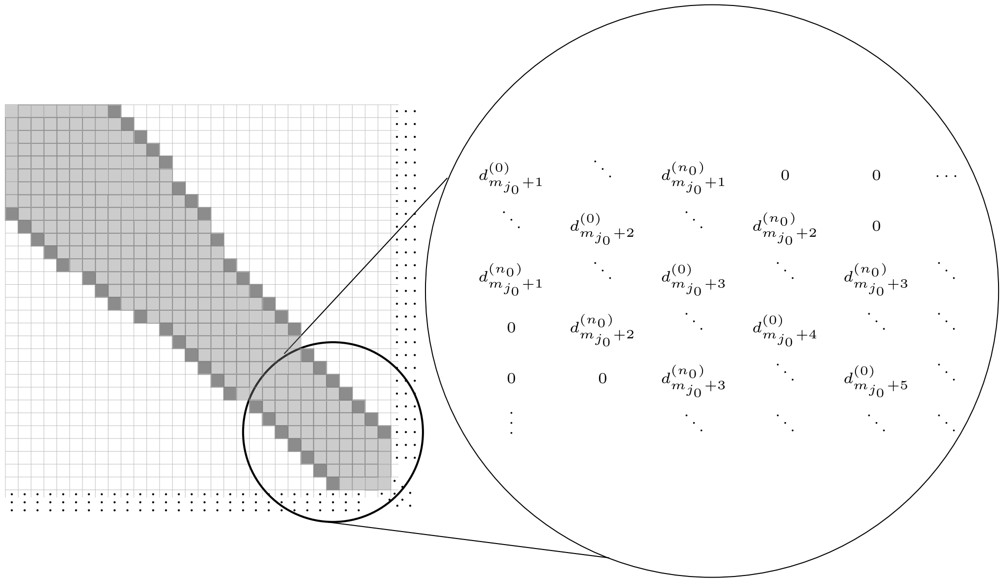


Fig. 5. Tail of matrix $\mathcal{A}$.

And, since $\boldsymbol{r}(z)$ is in the zero class, the right-hand side of the equality in (3.3) is always zero. Hence, (3.1) holds true.

To prove i), one uses again (3.2) taking into account Corollary 2.1. Then (3.3) shows that the only vector polynomial in the zero class is the zero polynomial. □

Remark 5. The assertion ii) of Theorem 3.1 can be interpreted as follows. If the spectral function of $\mathcal{A}$ has a countable set of growth points not accumulating anywhere, then the spectrum of the operator of multiplication consists only of eigenvalues which, due to the fact that σ is an $n \times n$ matrix, have multiplicity not greater than n. Let $\{x_l\}_{l=1}^{\infty}$ be the eigenvalues of the multiplication operator by the independent variable in $L_2(\mathbb{R}, \sigma)$ enumerated taking into account their multiplicity. Hence the vector polynomials $\{\boldsymbol{q}_j\}_{j=1}^{j_0}$ are generators of the interpolation problem

$$\langle \boldsymbol{r}(x_l), \sigma_l \boldsymbol{r}(x_l) \rangle_{\mathbb{C}^n} = 0\,, \quad l \in \mathbb{N}, \tag{3.4}$$

where σ_l is a matrix of the same form as right-hand side of (2.6) and has the same properties. Note that (3.4) yields a linear interpolation problem with an infinite set of nodes of interpolation.

4. Spectral functions in the self-adjoint case

The operator A is symmetric and, by definition, closed. In this section, we are interested in the case when $A = A^*$. So let us touch upon some criteria for self-adjointness of A.

Our first criterion is based on the fact that any semi-infinite band matrix can be considered as a block semi-infinite Jacobi matrix. Indeed, any semi-infinite band matrix with $2n + 1$ diagonals is equivalent to a semi-infinite Jacobi matrix where each entry is a $p \times p$ matrix with $p \geq n$. Since the operator A^* is the operator defined by the matrix $\mathcal{A}$ in the maximal domain [2, Sec. 47], the fact that the operator A is self-adjoint depends exclusively on the asymptotic behavior of the diagonal sequences $\{d_k^{(n)}\}_{k=1}^{\infty}$ of its matrix representation $\mathcal{A}$.

For any matrix in $\mathcal{M}(n, \infty)$, consider the semi-infinite submatrix after the last degeneration, which we called the "tail of the matrix" (see Remark 1). This "tail" can be seen as a semi-infinite block Jacobi matrix (see Fig. 5).

Let us denote

$$\begin{pmatrix} Q_1 & B_1^* & 0 & 0 & \cdots \\ B_1 & Q_2 & B_2^* & 0 & \\ 0 & B_2 & Q_3 & B_3^* & \ddots \\ 0 & 0 & B_3 & Q_4 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} := \begin{pmatrix} d^{(0)}_{m_{j_0}+1} & \ddots & d^{(n_0)}_{m_{j_0}+1} & 0 & 0 & \cdots \\ \ddots & d^{(0)}_{m_{j_0}+2} & \ddots & d^{(n_0)}_{m_{j_0}+2} & 0 & \\ d^{(n_0)}_{m_{j_0}+1} & \ddots & d^{(0)}_{m_{j_0}+3} & \ddots & d^{(n_0)}_{m_{j_0}+3} & \ddots \\ 0 & d^{(n_0)}_{m_{j_0}+2} & \ddots & d^{(0)}_{m_{j_0}+4} & \ddots & \ddots \\ 0 & 0 & d^{(n_0)}_{m_{j_0}+3} & \ddots & d^{(0)}_{m_{j_0}+5} & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where each entry is an $n_0 \times n_0$ matrix ($n_0 := n - j_0$). Clearly, the elements of the block diagonal adjacent to the main diagonal, i.e., the matrices $\{B_k\}_{k\in\mathbb{N}}$ and $\{B_k^*\}_{k\in\mathbb{N}}$, are upper and, respectively, lower triangular matrices such that the main diagonal entries are positive numbers.

The following proposition is the analogue of the Carleman criterion [1, Chap. 1, Addenda and Problems] for block Jacobi matrices.

Proposition 4.1. *([4, Ch. 7, Thm. 2.9]) If $\sum_{j=1}^{\infty} 1/\|B_j\|$ diverges, then A is self-adjoint.*

In [26, Cor. 2.5], the following necessary conditions for self-adjointness are given. These conditions generalize well known criteria for a Jacobi operator to be self-adjoint.

Proposition 4.2. *Suppose that, starting from some k_0, all the matrices Q_k are invertible. If*

$$\lim_{k\to+\infty} \|Q_k^{-1}\| = 0\,, \ \textit{and} \ \limsup_{k\to+\infty}\{\|Q_k^{-1}B_k\| + \|Q_k^{-1}B_k^*\|\} < 1\,,$$

then the operator A is self-adjoint.

Another criterion is given by perturbation theory and is related to the if-and-only-if criterion given above. Indeed, consider the operators D_j ($j = 0, 1, \dots, n$), whose matrix representation with respect to $\{\delta_k\}_{k\in\mathbb{N}}$ is a diagonal matrix, i.e., $D_j\delta_k = d_k^{(j)}\delta_k$ for all $k \in \mathbb{N}$, where $d_k^{(j)}$ is a real number (see [2, Sec. 47]). Note that $\mathcal{D}_j$, given in the Introduction, is the matrix representation of the operator D_j with respect to $\{\delta_k\}_{k\in\mathbb{N}}$. Define the shift operator S as follows

$$S\delta_k = \delta_{k+1}\,, \quad \text{for all } k \in \mathbb{N}\,,$$

where by linearity, it is defined on $\operatorname{span}\{\delta_k\}_{k=1}^{\infty}$ and then extended to $\mathcal{H}$ by continuity. Consider the symmetric operator

$$A' := D_0 + \sum_{j=1}^{n} S^j D_j + \sum_{j=1}^{n} D_j (S^*)^j\,. \tag{4.1}$$

Now, if the operator $\sum_{j=1}^{n} S^j D_j + \sum_{j=1}^{n} D_j(S^*)^j$ is D_0-bounded with D_0-bound smaller than 1 (see [40, Sec. 5.1]), one can resort to the Rellich–Kato theorem [25, Thm. 4.3] to show that A' is self-adjoint. When this happens, it can be shown that $A = A'$.

Let us assume from this point to the end of this section that the operator A is self-adjoint. Our approach to constructing the spectral functions of A is based on techniques of perturbation theory related to the strong resolvent convergence (see [40, Sec. 9.3]).

We begin by recalling the following definition.

Definition 6. A subset D of the domain of a closeable operator B is called a core of B when $\overline{B\upharpoonright_D} = B$.

Also, we recur to the following known results (cf. [25, Chap. 8, Cor. 1.6 and Thm. 1.15]):

Proposition 4.3. *[40, Thm. 9.16] Let $\{B_N\}_{N\in\mathbb{N}}$ and B be self-adjoint operators on $\mathcal{H}$. If there is a core D of B such that for every $\varphi \in D$ there is an $N_0 \in \mathbb{N}$ which satisfies $\varphi \in \operatorname{dom}(B_N)$ for $N \geq N_0$ and $B_N\varphi \xrightarrow[N\to\infty]{} B\varphi$, then the sequence $\{(B_N - zI)^{-1}\}_{N\in\mathbb{N}}$ converges strongly to $(B - zI)^{-1}$ (denoted $(B_N - zI)^{-1} \xrightarrow[N\to\infty]{s} (B - zI)^{-1}$) for all $z \in \mathbb{C} \setminus \mathbb{R}$.*

Proposition 4.4. *[40, Thm. 9.19] Let $\{B_N\}_{N\in\mathbb{N}}$ and B be self-adjoint operators on $\mathcal{H}$, such that the sequence $\{(B_N - iI)^{-1}\}_{N\in\mathbb{N}}$ converges strongly to $(B - iI)^{-1}$. Then*

$$\begin{array}{r} E_{B_N}(t) \xrightarrow[N\to\infty]{s} E_B(t) \\ E_{B_N}(t+0) \xrightarrow[N\to\infty]{s} E_B(t) \end{array}, \quad \text{for all } t \in \mathbb{R} \text{ such that } E_B(t) = E_B(t+0)\,.$$

Here, $E_{B_N}(t)$ and $E_B(t)$ are the spectral resolutions of the identity of B_N and B, respectively.

Recall the finite dimensional operator $\widetilde{A}_N$ studied in Section 2 and define

$$A_N := \widetilde{A}_N \oplus \mathbb{O}\,,$$

where $\mathbb{O}$ is the zero-operator in the infinite dimensional space $\mathcal{H} \ominus \mathcal{H}_N$. For any $N > n$, the operator A_N is self-adjoint, so we take advantage of the spectral theorem. Let us introduce the following notation for the matrix-valued spectral functions

$$\sigma_N(t) := \{\langle \delta_i, E_{A_N}(t)\delta_j\rangle\}_{i,j=1}^{\infty} \qquad \text{for any } N > n \tag{4.2}$$

$$\sigma(t) := \{\langle \delta_i, E_A(t)\delta_j\rangle\}_{i,j=1}^{\infty}\,. \tag{4.3}$$

Lemma 4.1. *The matrix-valued functions $\sigma_N(t)$ given in (4.2) converge to the matrix-valued function $\sigma(t)$, defined by (4.3), at all points of continuity of $\sigma(t)$, i.e.,*

$$\sigma_N(t) \xrightarrow[N\to\infty]{} \sigma(t)\,, \quad t \text{ being a point of continuity of } \sigma(t)\,. \tag{4.4}$$

Proof. Let $l_{\text{fin}}(\mathbb{N})$ be the linear space of sequences with a finite number of nonzero elements. This space is a core of the operator A. Given an element $\varphi = \sum_{k=1}^{s} \varphi_k\delta_k \in l_{\text{fin}}(\mathbb{N})$, one verifies that, for all $N \geq N_0 = s + n$, $A_N\varphi = A\varphi$. Therefore, the conditions of Proposition 4.3 are satisfied. So, by Proposition 4.4, one obtains the result. □

Corollary 4.1. *For any $k \in \mathbb{N} \cup \{0\}$, the integral*

$$\int_{\mathbb{R}} t^k d\sigma$$

converges. Moreover, $\int_{\mathbb{R}} d\sigma$ is the identity matrix.

Proof. The first part of the assertion is a consequence of Lemmas 3.2 and 4.1. The second part follows from the fact that σ is the spectral function of the self-adjoint operator A. $\square$

On the basis of the previous result, let us denote

$$S_k := \int_{\mathbb{R}} t^k d\sigma$$

for any $k \in \mathbb{N} \cup \{0\}$.

Definition 7. Given the spectral function σ of the self-adjoint operator A, denote

$$\sigma_{\mathcal{T}} := \mathcal{T}\sigma\mathcal{T}^*\,,$$

where $\mathcal{T}$ is a matrix satisfying I) and II) (see (2.3)).

Using (2.7), one obtains

$$\sigma_N^{\mathcal{T}}(t) \xrightarrow[N\to\infty]{} \sigma_{\mathcal{T}}(t)\,, \quad \text{for } t \text{ being a point of continuity of } \sigma(t)\,, \tag{4.5}$$

where $\sigma_N^{\mathcal{T}}$ is the function given in (2.6). To verify this, observe that σ_N, given by (4.2), corresponds to σ_N^I (cf. the paragraph below [29, Remark 2.2]). It also holds that

$$\mathcal{T}S_k\mathcal{T}^* = \int_{\mathbb{R}} t^k\, d\sigma_{\mathcal{T}}. \tag{4.6}$$

Lemma 4.2. *For any matrix $\mathcal{T}$ satisfying I) and II), the function $\sigma_{\mathcal{T}}$, given in Definition 7, is in $\mathfrak{M}_d(n,\infty)$ (see Definition 5).*

Proof. It follows from [12, Sec. 2] that the sequence $\{S_k\}_{k=0}^{\infty}$ defines a determinate moment problem. In view of (4.6), the sequence $\{\mathcal{T}S_k\mathcal{T}^*\}_{k=0}^{\infty}$ also has only one solution for any $\mathcal{T}$. $\square$

5. Reconstruction of the matrix

In this section, the starting point is a matrix-valued function $\widetilde{\sigma} \in \mathfrak{M}(n,\infty)$ (see Definition 5) and we construct a matrix $\mathcal{A}$ in the class $\mathcal{M}(n,\infty)$ from this function. Furthermore, we verify that, for some matrix $\mathcal{T}$ which gives the initial conditions (see (2.3)), $\widetilde{\sigma}$ is a spectral function of the reconstructed matrix $\mathcal{A}$ according to Definition 4. Hence, any matrix in $\mathcal{M}(n,\infty)$ can be reconstructed from its function in $\mathfrak{M}(n,\infty)$.

Consider the Hilbert space $L_2(\mathbb{R},\widetilde{\sigma})$ with $\widetilde{\sigma} \in \mathfrak{M}(n,\infty)$. All n-dimensional vector polynomials are in $L_2(\mathbb{R},\widetilde{\sigma})$ and either there are polynomials of zero norm in this space or there are not. Let us apply the Gram–Schmidt procedure of orthonormalization to the sequence of vector polynomials given by (2.21). If there exist nonzero polynomials whose norm is zero, then the Gram–Schmidt algorithm yields vector polynomials of zero norm. Indeed, let $\boldsymbol{r} \not\equiv 0$ be a vector polynomial of zero norm of minimal height h_1 (that is, any nonzero polynomial of zero norm has height no less than h_1), and let $\{\widetilde{\boldsymbol{p}}_k\}_{k=1}^{h_1}$ be the orthonormalized vector polynomials obtained by the first h_1 iterations of the Gram–Schmidt procedure. Hence, if one defines

$$\boldsymbol{s} = \boldsymbol{e}_{h_1+1} - \sum_{i=1}^{h_1} \langle \widetilde{\boldsymbol{p}}_i, \boldsymbol{e}_{h_1+1}\rangle\, \widetilde{\boldsymbol{p}}_i\,,$$

then, in view of the fact that $h(\widetilde{\boldsymbol{p}}_k) = k-1$ for $k = 1,\dots,h_1$, and taking into account Proposition 2.2, one has

$$\boldsymbol{e}_{h_1+1} = a\boldsymbol{r} + \sum_{i=1}^{h_1} a_i \widetilde{\boldsymbol{p}}_i$$

which in turn leads to

$$\boldsymbol{s} = a\boldsymbol{r} + \sum_{k=1}^{h_1} \widetilde{a}_k \widetilde{\boldsymbol{p}}_k \,. \tag{5.1}$$

This implies that $\|\boldsymbol{s}\|_{L_2(\mathbb{R},\widetilde{\sigma})} = 0$ since $\boldsymbol{r}$ has zero norm and $\boldsymbol{s} \perp \widetilde{\boldsymbol{p}}_k$ for $k = 1, \dots, h_1$ by construction. Thus, the Gram–Schmidt procedure yields vector polynomials of zero norm.

Having found a vector polynomial of zero norm, one continues with the procedure taking the next vector of the sequence (2.21). Observe that if the Gram–Schmidt technique has produced a vector polynomial of zero norm $\boldsymbol{q}$ of height h, then for any integer number l, the vector polynomial $\boldsymbol{t}$ that is obtained at the $h+1+nl$-th iteration of the Gram–Schmidt process, viz.,

$$\boldsymbol{t} = \boldsymbol{e}_{h+1+nl} - \sum_{h(\widetilde{\boldsymbol{p}}_i)<h+nl} \langle \widetilde{\boldsymbol{p}}_i, \boldsymbol{e}_{h+1} \rangle \, \widetilde{\boldsymbol{p}}_i \,,$$

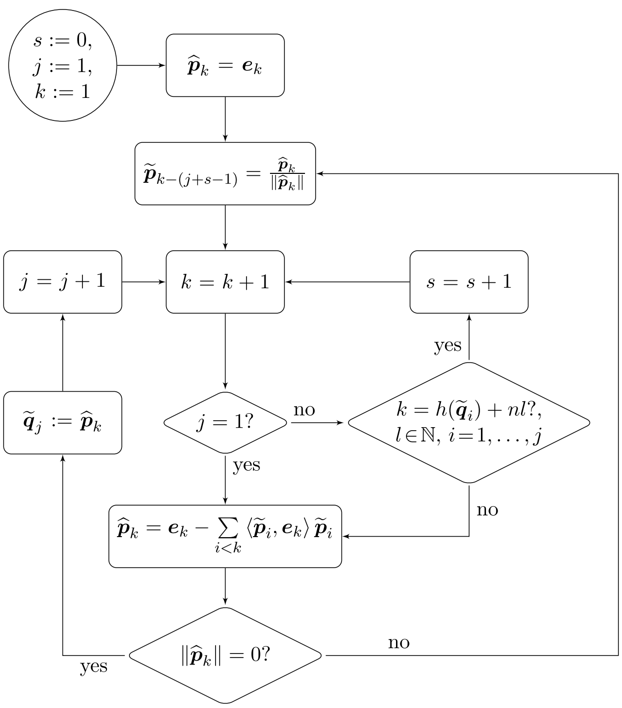
satisfies that $\|\boldsymbol{t}\|_{L_2(\mathbb{R},\widetilde{\sigma})} = 0$ (for all $l \in \mathbb{N}$). Indeed, due to Proposition 2.2, one has

$$\boldsymbol{e}_{h+1+nl} = z^l \boldsymbol{q} + \sum_{h(\widetilde{\boldsymbol{p}}_i)<h+nl} c_i \widetilde{\boldsymbol{p}}_i + \sum_{h(\boldsymbol{r})<h+nl} \boldsymbol{r} \,,$$

where each $\boldsymbol{r}$ is a vector polynomial of zero norm obtained from the Gram–Schmidt procedure.

Remark 6. Since $\widetilde{\sigma}$ has an infinite number of growth points, the Hilbert space $L_2(\mathbb{R},\widetilde{\sigma})$ is infinite dimensional [2, Sec. 72]. Thus, the Gram–Schmidt procedure renders an infinite sequence of orthonormal vectors.

The following flow chart shows that the Gram–Schmidt procedure applied to the sequence (2.21) gives not only the orthonormalized sequence, but also a sequence of null vector polynomials such that at any step of the algorithm these two sequences together are a basis of the space of vector polynomials (see Proposition 2.2 and compare with (2.20)).



Remark 7. Another consequence of the fact that the measure has an infinite number of growth points is that one cannot obtain more than $n-1$ null vectors from the Gram–Schmidt procedure applied to the sequence of vector polynomials given by (2.21). Indeed, if one finds the n-th vector polynomial $\widetilde{\boldsymbol{q}}_n$, by repeating the argument described above and taking into account

$$\{h(\widetilde{\boldsymbol{q}}_1), \dots, h(\widetilde{\boldsymbol{q}}_n)\} = \mathbb{Z}/n\mathbb{Z}\,,$$

one obtains that all the vectors provided by this procedure have zero norm beginning from some vector. This would correspond to an infinite loop in the left side of the flow chart and to a measure with finite support since $L_2(\mathbb{R}, \widetilde{\sigma})$ would be finite dimensional.

Lemma 5.1. *Any vector polynomial $\boldsymbol{r}(z)$ is a finite linear combination of*

$$\{\widetilde{\boldsymbol{p}}_k(z) : k \in \mathbb{N}\} \cup \{z^l \widetilde{\boldsymbol{q}}_j(z) : l \in \mathbb{N} \cup \{0\},\, j \in \{1, \dots, j_0\}\}\,.$$

Proof. Note that the vector polynomials defined in (2.21) satisfy that $h(\boldsymbol{e}_i) = i - 1$. Due to the fact that

$$h\left(\boldsymbol{e}_k - \sum_{h(\widetilde{\boldsymbol{p}}_i) < k-1} \langle \widetilde{\boldsymbol{p}}_i, \boldsymbol{e}_k\rangle \widetilde{\boldsymbol{p}}_i\right) = h(\boldsymbol{e}_k)\,, \tag{5.2}$$

one concludes that the heights of the set $\{\widetilde{\boldsymbol{p}}_k(z)\}_{k=1}^{\infty} \cup \{z^l \widetilde{\boldsymbol{q}}_i(z)\}_{i=1}^{n}$ $(l \in \mathbb{N} \cup \{0\})$ are in one-to-one correspondence with the set $\mathbb{N} \cup \{0\}$. To complete the proof, it only remains to use Proposition 2.2. □

By the argumentation given above and the same reasoning used in the proof of Theorem 3.1 ii), one arrives at the following assertion.

Proposition 5.1. *Let $\widetilde{\sigma}$ be in $\mathfrak{M}(n, \infty)$. There exist at most $n-1$ vector polynomials $\{\widetilde{\boldsymbol{q}}_i\}_{i=1}^{j_0}$ $(j_0 \le n-1)$ such that any vector polynomial $\boldsymbol{r}$ of zero norm can be written as*

$$\boldsymbol{r} = \sum_{i=1}^{j_0} R_i \widetilde{\boldsymbol{q}}_i\,,$$

where R_i is a scalar polynomial for any $i \in \{1, \dots, j_0\}$.

Let $\widetilde{\sigma}(t)$ be a matrix valued function in $\mathfrak{M}(n, \infty)$ and consider the sequences $\{\widetilde{\boldsymbol{p}}_k\}_{k\in\mathbb{N}}$ and $\{\widetilde{\boldsymbol{q}}_{\boldsymbol{i}}\}_{i=1}^{j_0}$ obtained by applying the Gram–Schmidt process to the sequence (2.21). Since for any $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $h(z\widetilde{\boldsymbol{p}}_k) \le h(\widetilde{\boldsymbol{p}}_l)$, one has by Lemma 5.1 that

$$z\widetilde{\boldsymbol{p}}_k(z) = \sum_{i=1}^{l} c_{ik} \widetilde{\boldsymbol{p}}_i(z) + \sum_{j=1}^{j_0} R_{kj}(z) \widetilde{\boldsymbol{q}}_j(z)\,, \tag{5.3}$$

where $c_{ik} \in \mathbb{C}$ and $R_{kj}(z)$ is a scalar polynomial.

Remark 8. By comparing the heights of the left and right hand sides of (5.3), one obtains the following relations given in items i) and ii) below. To verify item iii), one has to take into account that the leading coefficient of $\boldsymbol{e}_k$ is positive for $k \in \mathbb{N}$ and therefore the Gram–Schmidt procedure yields the sequence $\{\widetilde{\boldsymbol{p}}_k\}_{k=1}^{\infty}$ with its elements having positive leading coefficients (cf. [29, Rem. 4]).

i) $c_{lk} = 0$ if $h(z\widetilde{\boldsymbol{p}}_k) < h(\widetilde{\boldsymbol{p}}_l)$,

ii) $R_{kj}(z) = 0$ if $h(z\widetilde{\boldsymbol{p}}_k) < h(R_{kj}(z)\widetilde{\boldsymbol{q}}_j)$,
iii) $c_{lk} > 0$ if there is $l \in \mathbb{N}$ such that $h(z\widetilde{\boldsymbol{p}}_k) = h(\widetilde{\boldsymbol{p}}_l)$.

Clearly (recall that our inner product is antilinear in its first argument),

$$c_{lk} = \langle \widetilde{\boldsymbol{p}}_l, z\widetilde{\boldsymbol{p}}_k \rangle_{L_2(\mathbb{R},\widetilde{\sigma})} = \langle z\widetilde{\boldsymbol{p}}_l, \widetilde{\boldsymbol{p}}_k \rangle_{L_2(\mathbb{R},\widetilde{\sigma})} = c_{kl}\,. \tag{5.4}$$

In [29, Sec. 3], a reconstruction algorithm is provided for recovering the finite band matrix associated to the operator A_N from its spectral function. The proof of [29, Lem. 4.1] proves the following assertion

Proposition 5.2. *If $|l - k| > n$. Then, the complex numbers c_{ki} in (5.3) obey*

$$c_{kl} = c_{lk} = 0\,.$$

Proposition 5.2 shows that $\{c_{lk}\}_{l,k=1}^{\infty}$ is a band matrix. Let us turn to the question of characterizing the diagonals of the matrix $\{c_{lk}\}_{l,k=1}^{\infty}$. It will be shown that they undergo the kind of degeneration given in the Introduction.

For a fixed number $i \in \{0, \dots, n\}$, we define the numbers

$$d_k^{(i)} := c_{k+i,k} = c_{k,k+i} \tag{5.5}$$

for $k \in \mathbb{N}$. The proof of the following assertion repeats the one of [29, Lem. 4.2].

Proposition 5.3. *Fix $j \in \{0, \dots, j_0 - 1\}$.*

i) If k is such that $h(\widetilde{\boldsymbol{q}}_j) < h(z\widetilde{\boldsymbol{p}}_k) < h(\widetilde{\boldsymbol{q}}_{j+1})$, then $d_k^{(n-j)} > 0$. Here one assumes that $h(\boldsymbol{q}_0) := n - 1$.
ii) If k is such that $h(z\widetilde{\boldsymbol{p}}_k) \geq h(\widetilde{\boldsymbol{q}}_{j+1})$, then $d_k^{(n-j)} = 0$.

Corollary 5.1. *If c_{ik} are the coefficients given in (5.3), then the matrix $\{c_{kl}\}_{k,l=1}^{\infty}$ is in $\mathcal{M}(n,\infty)$ and it is the matrix representation with respect to $\{\widetilde{\boldsymbol{p}}_k\}_{k=1}^{\infty}$ of a symmetric restriction of the operator of multiplication by the independent variable in $\overline{\operatorname{span}\{\widetilde{\boldsymbol{p}}_k\}_{k=1}^{\infty}} \subset L_2(\mathbb{R},\widetilde{\sigma})$. (The restriction of the operator could be improper, i.e., the case when the restriction coincides with the multiplication operator is not excluded.)*

Proof. Taking into account (5.5), it follows from Propositions 5.2 and 5.3 that the matrix $\{c_{kl}\}_{k,l=1}^{\infty}$ is in the class $\mathcal{M}(n,\infty)$. Now, in view of (5.4), the operator of multiplication by the independent variable is an extension of the minimal closed symmetric operator B in $\overline{\operatorname{span}\{\widetilde{\boldsymbol{p}}_k\}_{k=1}^{\infty}} \subset L_2(\mathbb{R},\widetilde{\sigma})$ satisfying

$$c_{kl} = \langle \widetilde{\boldsymbol{p}}_k, B\widetilde{\boldsymbol{p}}_l \rangle\,. \quad \square$$

Remark 9. It follows from (2.5) that $\boldsymbol{e}_i(z)$ is not in the equivalence class of zero in $L(\mathbb{R},\widetilde{\sigma})$ for $i \in \{1, \dots, n\}$. Therefore, if one defines

$$t_{ij} := \left\langle \delta_i, \widetilde{\boldsymbol{p}}_j \right\rangle_{\mathbb{C}^n}\,, \quad \forall i,j \in \{1,\dots,n\}\,, \tag{5.6}$$

the matrix $\mathcal{T} = \{t_{ij}\}_{i,j=1}^{n}$ satisfies I) and II) (Section 2). Now, for this matrix $\mathcal{T}$ and the matrix $\{c_{kl}\}_{k,l=1}^{\infty}$ construct the polynomials $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ according to (2.8)–(2.11). By means of the constructed sequence $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$, one defines $\{\boldsymbol{q}_k\}_{k=1}^{j_0}$ using (2.12).

Theorem 5.1. *Let $\widetilde{\sigma}$ be an element of $\mathfrak{M}(n,\infty)$ and c_{ik} be the coefficients given in (5.3). Then $\widetilde{\sigma}$ is a spectral function of the matrix $\{c_{kl}\}_{k,l=1}^{\infty}$ according to Definition 4.*

Proof. Since the recurrence equation for the orthonormal sequence $\{\widetilde{\boldsymbol{p}}_k\}_{k=1}^{\infty}$ and the sequence of polynomials $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ given by Remark 9 are related in the same way as in the finite dimensional case (see [29, Eqs. (2.17) and (4.15)]), one can use the argumentation of the proofs of [29, Lem. 4.3] to obtain that the vector polynomials $\{\boldsymbol{p}_k(z)\}_{k=1}^{\infty}$ and $\{\widetilde{\boldsymbol{p}}_k(z)\}_{k=1}^{\infty}$ satisfy

$$\boldsymbol{p}_k(z) = \widetilde{\boldsymbol{p}}_k(z) + \boldsymbol{r}_k(z)\,, \tag{5.7}$$

where $\|\boldsymbol{r}_k\|_{L_2(\mathbb{R},\widetilde{\sigma})} = 0$. Analogously, when $j_0 \neq 0$ it can also be proven that the vector polynomials $\{\widetilde{\boldsymbol{q}}_j(z)\}_{j=1}^{j_0}$ and the vector polynomials $\{\boldsymbol{q}_j(z)\}_{j=1}^{j_0}$ given in Remark 9 satisfy

$$\boldsymbol{q}_j(z) = \sum_{i\leq j} R_i(z)\widetilde{\boldsymbol{q}}_i(z)\,, \quad R_j \neq 0\,, \tag{5.8}$$

where $R_i(z)$ are scalar polynomials (see [29, Lem. 4.4]). Due to (5.7) and (5.8) $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ is an orthonormal sequence in $L_2(\mathbb{R},\widetilde{\sigma})$ and $\boldsymbol{q}_j$ is in the equivalence class of zero in this space for any $j \in \{1,\dots,j_0\}$. □

Theorem 5.2. *Consider the space $L_2(\mathbb{R},\widetilde{\sigma})$ and let $\widetilde{A}$ be the operator of multiplication by the independent variable in it, defined on its maximal domain. If $\widetilde{\sigma}$ is in $\mathfrak{M}_d(n,\infty)$, then $\{c_{ik}\}_{i,k=1}^{\infty}$ (where c_{ik} are the coefficients given in (5.3)) is the matrix representation of $\widetilde{A}$ with respect to $\{\widetilde{\boldsymbol{p}}_k\}_{k=1}^{\infty}$. Moreover, there is $\mathfrak{T}$ satisfying I) and II) such that $\sigma_{\mathfrak{T}}$, given in Definition 7, coincides with $\widetilde{\sigma}$.*

Proof. According to Remark 9 and Theorem 5.1, there is $\mathfrak{T}$ such that the vector polynomials $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$, generated by $\{c_{kl}\}_{k,l=1}^{\infty}$ and $\mathfrak{T}$, are orthonormal in $L_2(\mathbb{R},\widetilde{\sigma})$. Since $\widetilde{\sigma}$ is the unique solution of the moment problem

$$\left\{\int_{\mathbb{R}} t^k d\widetilde{\sigma}\right\}_{k=0}^{\infty}\,,$$

it follows from Remark 4 (see [12, Sec. 2]) that the orthonormal system $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ is a basis and $\{c_{kl}\}_{k,l=1}^{\infty}$ is the corresponding matrix representation of the operator of multiplication by the independent variable with respect to $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ or, equivalently, $\{\widetilde{\boldsymbol{p}}_k\}_{k=1}^{\infty}$. For proving the second part of the assertion, first observe that, due to (4.5), one can apply Lemma 3.2 to the sequence $\{\sigma_N^{\mathfrak{T}}\}_{N>n}$ and the function $\sigma_{\mathfrak{T}}$. This yields the existence of a sequence $\{N_i\}_{i=1}^{\infty}$ such that

$$\int_{\mathbb{R}} t^k\, d\sigma_{N_i}^{\mathfrak{T}} = \int_{\mathbb{R}} t^k\, d\sigma_{\mathfrak{T}}\,.$$

This equality and the fact that, for any $k, l \in \mathbb{N}$ and N sufficiently large,

$$\langle \boldsymbol{p}_k, \boldsymbol{p}_l\rangle_{L_2(\mathbb{R},\sigma_N^{\mathfrak{T}})} = \delta_{kl}$$

imply that $\{\boldsymbol{p}_k\}_{k=1}^{\infty}$ is orthonormal in $L_2(\mathbb{R},\sigma_{\mathfrak{T}})$. Thus, by Corollary 2.1 and Lemma 5.1, $\widetilde{\sigma}$ and $\sigma_{\mathfrak{T}}$ have the same moments. □

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