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Quantum information inequalities via tracial positive linear maps

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ABSTRACT

We present some generalizations of quantum information inequalities involving tracial positive linear maps between C^*-algebras. Among several results, we establish a noncommutative Heisenberg uncertainty relation. More precisely, we show that if $\Phi : \mathcal{A} \to \mathcal{B}$ is a tracial positive linear map between C^*-algebras, $\rho \in \mathcal{A}$ is a Φ-density element and A, B are self-adjoint operators of $\mathcal{A}$ such that $\mathrm{sp}(-\mathrm{i}\rho^{\frac{1}{2}}[A,B]\rho^{\frac{1}{2}}) \subseteq [m,M]$ for some scalers $0 < m < M$, then under some conditions

$$V_{\rho,\Phi}(A)\sharp V_{\rho,\Phi}(B) \geq \frac{1}{2\sqrt{K_{m,M}(\rho[A,B])}}\,|\Phi(\rho[A,B])|\,, \tag{0.1}$$

where $K_{m,M}(\rho[A,B])$ is the Kantorovich constant of the operator $-\mathrm{i}\rho^{\frac{1}{2}}[A,B]\rho^{\frac{1}{2}}$ and $V_{\rho,\Phi}(X)$ is the generalized variance of X. In addition, we use some arguments differing from the scalar theory to present some inequalities related to the generalized correlation and the generalized Wigner–Yanase–Dyson skew information.

1. Introduction and preliminaries

In quantum measurement theory, the classical expectation value of an observable (self-adjoint operator) A in a quantum state (density operator) ρ is expressed by $\mathrm{Tr}(\rho A)$. Also, the classical variance for a quantum state ρ and an observable operator A is defined by $V_\rho(A) := \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}(\rho A))^2$. The Heisenberg uncertainty relation asserts that

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2 \tag{1.1}$$

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for a quantum state ρ, where $[A,B] := AB - BA$ is the commutator of two observables A and B; see [6]. It gives a fundamental limit for the measurements of incompatible observables. A further strong result was given by Schrödinger [15] as

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}(\mathrm{Cov}_\rho(A,B))|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2, \tag{1.2}$$

where the classical covariance is defined by $\mathrm{Cov}_\rho(A) := \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)$.

Yanagi et al. [18] defined the one-parameter correlation and the one-parameter Wigner–Yanase skew information (is known as the Wigner–Yanase–Dyson skew information; cf. [10]) for operators A, B, respectively, as follows

$$\mathrm{Corr}_\rho^\alpha(A,B) := \mathrm{Tr}(\rho A^*B) - \mathrm{Tr}(\rho^{1-\alpha}A^*\rho^\alpha B) \quad \text{and} \quad I_\rho^\alpha(A) := \mathrm{Corr}_\rho^\alpha(A,A),$$

where $\alpha \in [0,1]$. They showed a trace inequality representing the relation between these two quantities as

$$\left|\mathrm{Re}(\mathrm{Corr}_\rho^\alpha(A,B))\right|^2 \leq I_\rho^\alpha(A)I_\rho^\alpha(B). \tag{1.3}$$

In the case that $\alpha = \frac{1}{2}$, we get the classical notions of the correlation $\mathrm{Corr}_\rho(A,B)$ and the Wigner–Yanase skew information $I_\rho(A)$. The classical Wigner–Yanase skew information represents a for non-commutativity between a quantum state ρ and an observable A.

Luo [11] introduced the quantity $U_\rho(A)$ as a measure of uncertainty by

$$U_\rho(A) = \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_\rho(A))^2}.$$

He then showed a Heisenberg-type uncertainty relation on $U_\rho(A)$ as

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A,B])|^2. \tag{1.4}$$

These inequalities was studied and extended by a number of mathematicians. For further information we refer interested readers to [3,5,8,17].

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with the unit I. If $\mathcal{H} = \mathbb{C}^n$, we identify $\mathbb{B}(\mathbb{C}^n)$ with the matrix algebra of $n \times n$ complex matrices $M_n(\mathbb{C})$. We consider the usual Löwner order $\leq$ on the real space of self-adjoint operators. Throughout the paper, a capital letter means an operator in $\mathbb{B}(\mathcal{H})$. An operator A is said to be strictly positive (denoted by $A > 0$) if it is a positive invertible operator. According to the Gelfand–Naimark–Segal theorem, every C^*-algebra can be regarded as a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. So we may consider elements of $\mathcal{A}$ as Hilbert space operators. We use $\mathcal{A}, \mathcal{B}, \cdots$ to denote C^*-algebras. We denote by $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ the real and imaginary parts of A, respectively. The geometric mean is defined by $A\sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$ for operators $A > 0$ and $B \geq 0$. A W^*-algebra is a $*$-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology and contains the identity operator. The C^*-algebra of complex valued continuous functions on the compact Hausdorff space Ω is denoted by $C(\Omega)$.

A linear map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ between C^*-algebras is said to be $*$-linear if $\Phi(A^*) = \Phi(A)^*$. It is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is called strictly positive if $A > 0$, then $\Phi(A) > 0$. We say that Φ is unital if $\mathcal{A}, \mathcal{B}$ are unital and Φ preserves the unit. A linear map Φ is called n-positive if the map $\Phi_n : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$ defined by $\Phi_n([a_{ij}]) = [\Phi(a_{ij})]$ is positive, where $M_n(\mathcal{A})$ stands for the C^*-algebra of $n \times n$ matrices with entries in $\mathcal{A}$. A map Φ is said to be completely positive if it is n-positive for every $n \in \mathbb{N}$. According to [16, Theorem 1.2.4] if the range of the positive linear map Φ is commutative, then Φ is completely positive. It is known (see, e.g., [4]) that if Φ is a unital positive linear map, then

$$\Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B). \tag{1.5}$$

A map Φ is called tracial if $\Phi(AB) = \Phi(BA)$. The usual trace on the trace class operators acting on a Hilbert space is a tracial positive linear functional. For a given closed two sided ideal $\mathcal{I}$ of a C^*-algebra $\mathcal{A}$, the existence of a tracial positive linear map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfying $\Phi(\Phi(A)) = \Phi(A)$ and $\Phi(A) - A \in \mathcal{I}$ is equivalent to the commutativity of the quotient $\mathcal{A}/\mathcal{I}$; see [2] for more examples and implications of the definition. For a tracial positive linear map Φ, a positive operator $\rho \in \mathcal{A}$ is said to be Φ-density if $\Phi(\rho) = I$. A unital C^*-algebra $\mathcal{B}$ is said to be injective whenever for every unital C^*-algebra $\mathcal{A}$ and for every self-adjoint subspace S of $\mathcal{A}$, each unital completely positive linear map from S into $\mathcal{B}$, can be extended to a completely positive linear map from $\mathcal{A}$ into $\mathcal{B}$. Our investigation is based on the following definition.

Definition 1.1. Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive linear map and ρ be a Φ-density operator. Then

$$\mathrm{Cov}_{\rho,\Phi}(A, B) := \Phi(\rho A^* B) - \Phi(\rho A^*)\Phi(\rho B) \text{ and } V_{\rho,\Phi}(A) := \mathrm{Cov}_{\rho,\Phi}(A, A),$$

are called the generalized covariance and the generalized variance A, B, respectively. Further, the generalized correlation and the generalized Wigner–Yanase–Dyson skew information of two operators A, B are defined by

$$\mathrm{Corr}^{\alpha}_{\rho,\Phi}(A, B) := \Phi(\rho A^* B) - \Phi(\rho^{1-\alpha} A^* \rho^{\alpha} B) \quad \text{and} \quad I^{\alpha}_{\rho,\Phi}(A) := \mathrm{Corr}^{\alpha}_{\rho,\Phi}(A, A)\,,$$

respectively.

It is known that for every tracial positive linear map, the matrix

$$\begin{bmatrix} V_{\rho,\Phi}(A) & \mathrm{Cov}_{\rho,\Phi}(B, A) \\ \mathrm{Cov}_{\rho,\Phi}(A, B) & V_{\rho,\Phi}(B) \end{bmatrix}$$

is positive, which is equivalent to

$$V_{\rho,\Phi}(A) \geq \mathrm{Cov}_{\rho,\Phi}(B, A)(V_{\rho,\Phi}(B))^{-1}\mathrm{Cov}_{\rho,\Phi}(A, B), \tag{1.6}$$

which is called the variance–covariance inequality; see [14,13] for technical discussions.

If $\mathcal{A}$ is a C^*-algebra and $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, then a conditional expectation $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{B}$ is a positive contractive linear map such that $\mathcal{E}(BAC) = B\mathcal{E}(A)C$ for every $A \in \mathcal{A}$ and all $B, C \in \mathcal{B}$.

If $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ is a semi-inner product module over a C^*-algebra $\mathcal{A}$, then the Cauchy–Schwarz inequality for $x, y \in \mathcal{X}$ asserts that (see [9,1])

$$\langle x, y\rangle\langle y, x\rangle \leq \|\langle y, y\rangle\|\langle x, x\rangle.$$

If $\langle y, y\rangle \in \mathcal{Z}(\mathcal{A})$, where $\mathcal{Z}(\mathcal{A})$ is the center of the C^*-algebra $\mathcal{A}$, then the latter inequality turns into (see [7])

$$\langle x, y\rangle\langle y, x\rangle \leq \langle y, y\rangle\langle x, x\rangle. \tag{1.7}$$

In Section 1, we use some techniques in the non-commutative setting to give some generalizations of inequalities (1.2) and (1.1) for tracial positive linear maps between C^*-algebras. More precisely, for a tracial positive linear map Φ between C^*-algebras under certain conditions we show that

$$V_{\rho,\Phi}(A)\sharp V_{\rho,\Phi}(B) \geq \frac{1}{2\sqrt{K_{m,M}(\rho[A, B])}}\,|\Phi(\rho[A, B])|$$

for every self adjoint operators A, B. Section 2 deals with generalizations of inequalities (1.3) and (1.4) for conditional expectation maps. Among other things, we prove that

$$0 \leq I^{\alpha}_{\rho,\Phi}(A) \leq I_{\rho,\Phi}(A) \leq V_{\rho,\Phi}(A)$$

for every self-adjoint operator A. In addition, we generalize some significant inequalities for trace in the quantum mechanical systems to inequalities for tracial positive linear maps between C^*-algebras. We indeed use some arguments differing from the classical theory to present some inequalities related to the generalized correlation and the generalized Wigner–Yanase–Dyson skew information.

2. Inequalities for generalized covariance and variance

We start this section by giving a generalization of inequality (1.2). In fact we prove inequality (1.2) for a tracial positive linear map between C^*-algebras under some mild conditions. We need the following notions slightly differing from the notions defined in Definition 1.1.

Definition 2.1. For a tracial positive linear map Φ from a C^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ and positive operator $\rho \in \mathcal{A}$ and for operators $A, B \in \mathcal{A}$ we set

$$\mathrm{Cov}'_{\rho,\Phi}(A,B) = \Phi(\rho A^* B) - \Phi(\rho A^*)\Phi(\rho)^{-1}\Phi(\rho B) \text{ and } V'_{\rho,\Phi}(A) = \mathrm{Cov}'_{\rho,\Phi}(A,A).$$

To achieve our result we need the following lemma.

Lemma 2.2. *[2, Lemma 2.1] Let $A > 0, B \geq 0$ be two operators in $\mathcal{A}$. Then the block matrix $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive if and only if $B \geq X^* A^{-1} X$.*

We are ready to prove our first result.

Theorem 2.3. *Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive linear map between C^*-algebras and $\rho \in \mathcal{A}$ be a positive operator such that $\Phi(\rho) > 0$. If $\Phi(\mathcal{A})$ is a commutative subspace of $\mathcal{B}$, then*

$$V'_{\rho,\Phi}(A)V'_{\rho,\Phi}(B) - |\mathrm{Re}(\mathrm{Cov}'_{\rho,\Phi}(A,B))|^2 \geq \frac{1}{4}|\Phi(\rho[A,B])|^2$$

for all self-adjoint operators A, B. In particular,

$$V'_{\rho,\Phi}(A)V'_{\rho,\Phi}(B) \geq \frac{1}{4}|\Phi(\rho[A,B])|^2.$$

Proof. A simple calculation shows that

$$\begin{aligned} \mathrm{Cov}'_{\rho,\Phi}(A,B) - \mathrm{Cov}'_{\rho,\Phi}(B,A) &= \Phi(\rho AB) - \Phi(\rho A)\Phi(\rho)^{-1}\Phi(\rho B) \\ &\quad - \Phi(\rho BA) + \Phi(\rho B)\Phi(\rho)^{-1}\Phi(\rho A) \\ &= \Phi(\rho[A,B]) \quad (\text{since, } \Phi(\mathcal{A}) \text{ is commutative}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Cov}'_{\rho,\Psi}(A,B) + \mathrm{Cov}'_{\rho,\Phi}(B,A) &= \mathrm{Cov}'_{\rho,\Psi}(A,B) + \left(\mathrm{Cov}'_{\rho,\Phi}(A,B)\right)^* \\ &= 2\mathrm{Re}(\mathrm{Cov}'_{\rho,\Phi}(A,B)). \end{aligned}$$

Summing both sides of the above inequalities, we get

$$2\mathrm{Cov}'_{\rho,\Phi}(A,B) = \Phi(\rho[A,B]) + 2\mathrm{Re}(\mathrm{Cov}'_{\rho,\Phi}(A,B)).$$

Since $\Phi(\rho[A,B])^* = -\Phi(\rho[A,B])$ and $2\mathrm{Re}(\mathrm{Cov}'_{\rho\Phi}(A,B))$ is self-adjoint, it follows from the commutativity of $\Phi(\mathcal{A})$ that

$$|\mathrm{Cov}'_{\rho,\Phi}(A,B)|^2 = |\mathrm{Re}(\mathrm{Cov}'_{\rho,\Phi}(A,B))|^2 + \frac{1}{4}|\Phi(\rho[A,B])|^2. \tag{2.1}$$

Moreover,

$$\begin{bmatrix} \rho^{\frac{1}{2}}A^*A\rho^{\frac{1}{2}} & \rho^{\frac{1}{2}}A^*B\rho^{\frac{1}{2}} & \rho^{\frac{1}{2}}A^*\rho^{\frac{1}{2}} \\ \rho^{\frac{1}{2}}B^*A\rho^{\frac{1}{2}} & \rho^{\frac{1}{2}}B^*B\rho^{\frac{1}{2}} & \rho^{\frac{1}{2}}B^*\rho^{\frac{1}{2}} \\ \rho^{\frac{1}{2}}A\rho^{\frac{1}{2}} & \rho^{\frac{1}{2}}B\rho^{\frac{1}{2}} & \rho \end{bmatrix} = \begin{bmatrix} \rho^{\frac{1}{2}}A^* & 0 & 0 \\ \rho^{\frac{1}{2}}B^* & 0 & 0 \\ \rho^{\frac{1}{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} A\rho^{\frac{1}{2}} & B\rho^{\frac{1}{2}} & \rho^{\frac{1}{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0.$$

It follows from the complete positivity (and indeed the 3-positivity) and the tracial property of Φ that

$$\begin{bmatrix} \Phi(\rho A^*A) & \Phi(\rho A^*B) & \Phi(\rho A^*) \\ \Phi(\rho B^*A) & \Phi(\rho B^*B) & \Phi(\rho B^*) \\ \Phi(\rho A) & \Phi(\rho B) & \Phi(\rho) \end{bmatrix} \geq 0.$$

Hence, by applying Lemma 2.2, we have

$$\begin{bmatrix} \Phi(\rho A^*A) & \Phi(\rho A^*B) \\ \Phi(\rho B^*A) & \Phi(\rho B^*B) \end{bmatrix} \geq \begin{bmatrix} \Phi(\rho A)^* & 0 \\ \Phi(\rho B)^* & 0 \end{bmatrix} \begin{bmatrix} \Phi(\rho)^{-1} & 0 \\ 0 & \Phi(\rho)^{-1} \end{bmatrix} \begin{bmatrix} \Phi(\rho A) & \Phi(\rho B) \\ 0 & 0 \end{bmatrix},$$

whence

$$\begin{bmatrix} \Phi(\rho A^*A) & \Phi(\rho A^*B) \\ \Phi(\rho B^*A) & \Phi(\rho B^*B) \end{bmatrix} \geq \begin{bmatrix} \Phi(\rho A)^*\Phi(\rho)^{-1}\Phi(\rho A) & \Phi(\rho A)^*\Phi(\rho)^{-1}\Phi(\rho B) \\ \Phi(\rho B)^*\Phi(\rho)^{-1}\Phi(\rho A) & \Phi(\rho B)^*\Phi(\rho)^{-1}\Phi(\rho B) \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} V'_{\rho,\Phi}(A) & \mathrm{Cov}'_{\rho,\Phi}(B,A) \\ \mathrm{Cov}'_{\rho,\Phi}(A,B) & V'_{\rho,\Phi}(B) \end{bmatrix} \geq 0.$$

It follows from Lemma 2.2 that

$$V'_{\rho,\Phi}(A) \geq \mathrm{Cov}'_{\rho,\Phi}(A,B)^* \left(V'_{\rho,\Phi}(B)\right)^{-1} \mathrm{Cov}'_{\rho,\Phi}(A,B).$$

Applying the commutativity $\Phi(\mathcal{A})$, we get

$$V'_{\rho,\Phi}(A)V'_{\rho,\Phi}(B) \geq |\mathrm{Cov}'_{\rho,\Phi}(A,B)|^2.$$

Consequently, if A, B are self-adjoint operators, then

$$\begin{aligned} \left(\Phi(\rho A^2) - \Phi(\rho A)^2\Phi(\rho)^{-1}\right)\left(\Phi(\rho B^2) - \Phi(\rho B)^2\Phi(\rho)^{-1}\right) &= V'_{\rho,\Phi}(A)V'_{\rho,\Phi}(B) \\ &\geq |\mathrm{Re}(\mathrm{Cov}'_{\rho,\Psi}(A,B))|^2 + \frac{1}{4}|\Phi(\rho[A,B])|^2 \\ &\qquad \text{(by equality (2.1))}. \quad \square \end{aligned}$$

Corollary 2.4. *Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive linear map between C^*-algebras and $\rho \in \mathcal{A}$ be a Φ-density operator. If $\Phi(\mathcal{A})$ is a commutative subspace of $\mathcal{B}$, then*

$$V_{\rho,\Phi}(A)V_{\rho,\Phi}(B) - |\mathrm{Re}(\mathrm{Cov}_{\rho,\Phi}(A,B))|^2 \geq \frac{1}{4}|\Phi(\rho[A,B])|^2$$

for all self-adjoint operators A, B.

Proof. Obviously, if ρ is a Φ-density operator, then $\Phi(\rho)^{-1} = I$. Now Theorem 2.3 yields the required inequality. □

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. If $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{B}$ is a tracial conditional expectation, then

$$B\mathcal{E}(A) = \mathcal{E}(BA) = \mathcal{E}(AB) = \mathcal{E}(A)B \tag{2.2}$$

for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Using this fact we give the following corollary.

Corollary 2.5. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. If $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{B}$ is a tracial conditional expectation, then*

$$V_{\rho,\mathcal{E}}(A)V_{\rho,\mathcal{E}}(B) - |\mathrm{Re}(\mathrm{Cov}_{\rho,\mathcal{E}}(A,B))|^2 \geq \frac{1}{4}|\mathcal{E}(\rho[A,B])|^2$$

for all self-adjoint operators $A, B \in \mathcal{A}$ and each $\mathcal{E}$-density operator $\rho \in \mathcal{A}$.

Now we give a version of Heisenberg's uncertainty relation, in the case that $\mathcal{B}$ is not a commutative C^*-algebra. To get this result we need some lemmas.

Lemma 2.6 *(Choi–Tsui). [2, pp. 59–60] Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras such that either one of them is W^*-algebra or $\mathcal{B}$ is an injective C^*-algebra. Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive linear map. Then there exist a commutative C^*-algebra $C(X)$ and tracial positive linear maps $\phi_1 : A \longrightarrow C(X)$ and $\phi_2 : C(X) \longrightarrow \mathcal{B}$ such that $\Phi = \phi_2 \circ \phi_1$. Moreover, in the case that Φ is unital, then ϕ_1 and ϕ_2 can be chosen to be unital. In particular, Φ is completely positive.*

Lemma 2.7 *(Kadison's inequality). [4, Chapter 1] If $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a unital 2-positive linear map between unital C^*-algebras, then*

$$\Phi(|A|^2) \geq |\Phi(A)|^2$$

for every $A \in \mathcal{A}$.

In the case that A is a positive operator of $\mathcal{A}$ satisfying $0 < mI \leq A \leq MI$ for some scalers $m < M$, by [4, Theorem 1.32], the reverse inequality

$$\Phi(A^2) \leq \frac{(M+m)^2}{4Mm}\Phi(A)^2 \tag{2.3}$$

holds.

Lemma 2.8. *Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital 2-positive linear map between unital C^*-algebras. If A is an operator of $\mathcal{A}$ satisfying $\mathrm{sp}(A) \subseteq [m, M] \cup [-M, -m]$ for some scalers $0 < m < M$, then*

$$|\Phi(A)| \leq \sqrt{\frac{(M+m)^2}{4Mm}}\Phi(|A|). \tag{2.4}$$

Proof. Using Lemma 2.7 and inequality (2.3), we get

$$|\Phi(A)|^2 \leq \Phi(|A|^2) \leq \frac{(M+m)^2}{4Mm}(\Phi(|A|))^2.$$

Taking square roots of both sides of the latter inequality we obtain inequality (2.4). □

In this paper, we denote $\dfrac{(M+m)^2}{4Mm}$ for an operator $m \leq A \leq M$ by $K_{M,m}(A)$, which is called the Kantorovich constant of A.

The next theorem gives a Heisenberg's type uncertainty relation for tracial positive linear maps between C^*-algebras.

Theorem 2.9. *Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras and Ω be a compact Hausdorff space. Let $\phi_1 : \mathcal{A} \longrightarrow C(\Omega)$ be a unital tracial positive linear map and $\phi_2 : C(\Omega) \longrightarrow \mathcal{B}$ be a unital positive linear map and $\Phi := \phi_2 \circ \phi_1$. If $\rho \in \mathcal{A}$ is a Φ-density operator and A, B are self-adjoint operators of $\mathcal{A}$ such that $\mathrm{sp}(\text{-}i\rho^{\frac{1}{2}}[A, B]\rho^{\frac{1}{2}}) \subseteq [m, M]$ for some scalers $0 < m < M$, then*

$$V_{\rho,\Phi}(A)\sharp V_{\rho,\Phi}(B) \geq \frac{1}{2\sqrt{K_{m,M}(\rho[A,B])}}\,|\Phi(\rho[A,B])|\,, \tag{2.5}$$

where $K_{m,M}(\rho[A, B])$ is the Kantorovich constant of the operator $\text{-}i\rho^{\frac{1}{2}}[A, B]\rho^{\frac{1}{2}}$.

Proof. By a continuity argument we can assume that $\rho > 0$. Due to $0 < mI \leq \text{-i}\rho^{\frac{1}{2}}[A, B]\rho^{\frac{1}{2}} \leq MI$ and ϕ_1 is a unital and tracial positive linear map, we infer that $mI \leq \text{-i}\phi_1(\rho[A, B]) \leq MI$. It follows that

$$mI \leq |\phi_1(\rho[A, B])| \leq MI. \tag{2.6}$$

Using the fact that ϕ_1 is a unital positive linear map (and so strictly positive) and applying Theorem 2.3 for ϕ_1 we get

$$\left(\phi_1(\rho A^2) - \phi_1(\rho A)^2\phi_1(\rho)^{-1}\right)\left(\phi_1(\rho B^2) - \phi_1(\rho B)^2\phi_1(\rho)^{-1}\right) \geq \frac{1}{4}|\phi_1(\rho[A, B])|^2.$$

Applying the commutativity of range of ϕ_1, we get

$$\left(\phi_1(\rho A^2) - \phi_1(\rho A)^2\phi_1(\rho)^{-1}\right)\sharp\left(\phi_1(\rho B^2) - \phi_1(\rho B)^2\phi_1(\rho)^{-1}\right) \geq \frac{1}{2}|\phi_1(\rho[A, B])|. \tag{2.7}$$

Using the fact that $(\phi_2 \circ \phi_1)(\rho) = I$, we can write

$$\begin{aligned}
&\left(\Phi(\rho A^2) - \Phi(\rho A)^2\right)\sharp\left(\Phi(\rho B^2) - \Phi(\rho B)^2\right)\\
&\quad = \left((\phi_2 \circ \phi_1)(\rho A^2) - (\phi_2 \circ \phi_1)(\rho A)^2\right)\\
&\qquad \sharp\left((\phi_2 \circ \phi_1)(\rho B^2) - (\phi_2 \circ \phi_1)(\rho B)^2\right)\\
&\quad = \left((\phi_2 \circ \phi_1)(\rho A^2) - (\phi_2 \circ \phi_1)(\rho A)(\phi_2 \circ \phi_1)(\rho)(\phi_2 \circ \phi_1)(\rho A)\right)\\
&\qquad \sharp\left((\phi_2 \circ \phi_1)(\rho B^2) - (\phi_2 \circ \phi_1)(\rho B)(\phi_2 \circ \phi_1)(\rho)(\phi_2 \circ \phi_1)(\rho B)\right).
\end{aligned}$$

We claim that

$$\begin{aligned}&\big((\phi_2\circ\phi_1)(\rho A^2)-(\phi_2\circ\phi_1)(\rho A)(\phi_2\circ\phi_1)(\rho)(\phi_2\circ\phi_1)(\rho A)\big)\\&\qquad\sharp\big((\phi_2\circ\phi_1)(\rho B^2)-(\phi_2\circ\phi_1)(\rho B)(\phi_2\circ\phi_1)(\rho)(\phi_2\circ\phi_1)(\rho B)\big)\\&\qquad\geq\phi_2\big(\phi_1(\rho A^2)-\phi_1(\rho A)^2\phi_1(\rho)^{-1}\big)\sharp\phi_2\big(\phi_1(\rho B^2)-\phi_1(\rho B)^2\phi_1(\rho)^{-1}\big).\end{aligned}$$

Since the range of ϕ_1 is commutative, to prove our claim, it is enough to show that

$$\phi_2\left(\phi_1(\rho X)\phi_1(\rho)^{-1}\phi_1(\rho X)\right)\geq(\phi_2\circ\phi_1)(\rho X)(\phi_2\circ\phi_1)(\rho)(\phi_2\circ\phi_1)(\rho X)\tag{2.8}$$

for every self-adjoint operator $X\in\mathcal{A}$ and $(\phi_2\circ\phi_1)$-density operator $\rho\in\mathcal{A}$. Clearly, matrix $\begin{pmatrix}\phi_1(\rho X)\phi_1(\rho)^{-1}\phi_1(\rho X) & \phi_1(\rho X)\\ \phi_1(\rho X) & \phi_1(\rho)\end{pmatrix}$ is positive. Since Φ_2 is completely positive (and so 2-positive), we get

$$\begin{pmatrix}\phi_2(\phi_1(\rho X)\phi_1(\rho)^{-1}\phi_1(\rho X)) & (\phi_2\circ\phi_1)(\rho X)\\ (\phi_2\circ\phi_1)(\rho X) & (\phi_2\circ\phi_1)(\rho)\end{pmatrix}\geq 0,$$

which ensures the validity of inequality (2.8). Therefore,

$$\begin{aligned}V_{\rho,\Phi}(A)\sharp V_{\rho,\Phi}(B)&=\big(\Phi(\rho A^2)-\Phi(\rho A)^2\big)\,\sharp\,\big(\Phi(\rho B^2)-\Phi(\rho B)^2\big)\\&=\big((\phi_2\circ\phi_1)(\rho A^2)-(\phi_2\circ\phi_1)(\rho A)^2\big)\\&\quad\sharp\big((\phi_2\circ\phi_1)(\rho B^2)-(\phi_2\circ\phi_1)(\rho B)^2\big)\\&=\big((\phi_2\circ\phi_1)(\rho A^2)-(\phi_2\circ\phi_1)(\rho A)(\phi_2\circ\phi_1)(\rho)(\phi_2\circ\phi_1)(\rho A)\big)\\&\quad\sharp\big((\phi_2\circ\phi_1)(\rho B^2)-(\phi_2\circ\phi_1)(\rho B)(\phi_2\circ\phi_1)(\rho)(\phi_2\circ\phi_1)(\rho B)\big)\\&\geq\phi_2\big(\phi_1(\rho A^2)-\phi_1(\rho A)^2\phi_1(\rho)^{-1}\big)\\&\quad\sharp\phi_2\big(\phi_1(\rho B^2)-\phi_1(\rho B)^2\phi_1(\rho)^{-1}\big)\\&\qquad\qquad\text{(by inequality (2.8))}\\&\geq\phi_2\Big(\big(\phi_1(\rho A^2)-\phi_1(\rho A)^2\phi_1(\rho)^{-1}\big)\\&\quad\sharp\big(\phi_1(\rho B^2)-\phi_1(\rho B)^2\phi_1(\rho)^{-1}\big)\Big)\\&\qquad\qquad\text{(by inequality (1.5))}\\&\geq\frac{1}{2}\phi_2(|\phi_1(\rho[A,B])|)\qquad\qquad(2.9)\\&\qquad\qquad\text{(by inequality (2.7))}\\&\geq\frac{1}{2\sqrt{K_{m,M}(\rho[A,B])}}\,|\phi_2\circ\phi_1(\rho[A,B])|\\&\qquad\qquad\text{(by inequality (2.4))}\\&=\frac{1}{2\sqrt{K_{m,M}(\rho[A,B])}}|\Phi(\rho[A,B])|.\quad\square\end{aligned}$$

Applying Lemma 2.6, we immediately get the following corollary.

Corollary 2.10. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras such that either one of them is W^*-algebra or $\mathcal{B}$ is an injective C^*-algebra. If $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a tracial positive linear map and $\rho \in \mathcal{A}$ is a Φ-density operator and A, B are self-adjoint operators in $\mathcal{A}$ such that $\mathrm{sp}(-i\rho^{\frac{1}{2}}[A,B]\rho^{\frac{1}{2}}) \subseteq [m, M]$ for some scalars $0 < m < M$, then*

$$V_{\rho,\Phi}(A) \sharp V_{\rho,\Phi}(B) \geq \frac{1}{2\sqrt{K_{m,M}(\rho[A,B])}} \left|\Phi(\rho[A,B])\right|,$$

where $K_{[m,M]}(\rho[A,B])$ is the Kantorovich constant of the operator $-i\rho^{\frac{1}{2}}[A,B]\rho^{\frac{1}{2}}$.

If $\phi_1 := \mathrm{Tr}$ is the usual trace and ϕ_2 is the identity map on $\mathbb{C}$, then it immediately follows from inequality (2.9) that

$$V_\rho(A)^{\frac{1}{2}} V_\rho(B)^{\frac{1}{2}} \geq \frac{1}{2} \left|\mathrm{Tr}(\rho[A,B])\right|.$$

So we get the following result.

Corollary 2.11. *For every self-adjoint operators A, B and each density operator ρ it holds that*

$$V_\rho(A) V_\rho(B) \geq \frac{1}{4} \left|\mathrm{Tr}(\rho[A,B])\right|^2.$$

Remark 2.12. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. By readout the proof of Theorem 2.3, Theorem 2.9 and Corollary 2.4, if we put $\rho := I$, then we can replace the condition "tracial positive linear map" with "unital positive linear map".

3. Inequalities for generalized correlation and Wigner–Yanase skew information

We aim to give generalizations of inequality (1.3) and inequality (1.4). In addition we generalize some inequalities related to classical Wigner–Yanase–Dyson skew information to tracial positive linear map. As mentioned in the introduction, in the case that $\alpha = \frac{1}{2}$ we denote $I^{\alpha}_{\rho,\Phi}$ by $I_{\rho,\Phi}$. In some cases we prove our result by assuming that ρ is only a positive operator. To get these generalizations we need some lemmas. The technique of the first lemma is classic.

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a tracial positive linear map, then*

$$I^{\alpha}_{\rho,\Phi}(A) = \Phi(\rho A^2) - \Phi(\rho^{1-\alpha} A \rho^{\alpha} A) \geq 0 \qquad (\alpha \in [0,1]) \tag{3.1}$$

for every self-adjoint operator $A \in \mathcal{A}$ and each positive operator $\rho \in \mathcal{A}$.

Proof. Put $\Delta = \{\alpha \in [0,1] : \Phi(\rho A^2) \geq \Phi(\rho^{\alpha} A \rho^{1-\alpha} A)\}$. Clearly $\{0,1\} \subseteq \Delta$ and the set Δ is closed, since the map $\alpha \longrightarrow \Phi(\rho^{\alpha} A \rho^{1-\alpha} A)$ is norm continuous. Therefore, to prove $[0,1] \subseteq \Delta$ it is enough to show that $\alpha, \beta \in \Delta$ implies $\frac{\alpha+\beta}{2} \in \Delta$. It follows from the tracial positivity of Φ that

$$\begin{aligned}
0 &\leq \Phi\left((\rho^{\frac{1-\alpha}{2}} A \rho^{\frac{\alpha}{2}} - \rho^{\frac{1-\beta}{2}} A \rho^{\frac{\beta}{2}})^*(\rho^{\frac{1-\alpha}{2}} A \rho^{\frac{\alpha}{2}} - \rho^{\frac{1-\beta}{2}} A \rho^{\frac{\beta}{2}})\right) \\
&= \Phi\left((\rho^{\frac{\alpha}{2}} A \rho^{\frac{1-\alpha}{2}} - \rho^{\frac{\beta}{2}} A \rho^{\frac{1-\beta}{2}})(\rho^{\frac{1-\alpha}{2}} A \rho^{\frac{\alpha}{2}} - \rho^{\frac{1-\beta}{2}} A \rho^{\frac{\beta}{2}})\right) \\
&= \Phi\left(\rho^{\frac{\alpha}{2}} A \rho^{1-\alpha} A \rho^{\frac{\alpha}{2}} - \rho^{\frac{\alpha}{2}} A \rho^{1-\frac{\alpha+\beta}{2}} A \rho^{\frac{\beta}{2}} - \rho^{\frac{\beta}{2}} A \rho^{1-\frac{\alpha+\beta}{2}} A \rho^{\frac{\alpha}{2}} + \rho^{\frac{\beta}{2}} A \rho^{1-\beta} A \rho^{\frac{\beta}{2}}\right) \\
&= \Phi(\rho^{\alpha} A \rho^{1-\alpha} A) - \Phi(\rho^{\frac{\alpha+\beta}{2}} A \rho^{1-\frac{\alpha+\beta}{2}} A) - \Phi(\rho^{\frac{\alpha+\beta}{2}} A \rho^{1-\frac{\alpha+\beta}{2}} A) + \Phi(\rho^{\beta} A \rho^{1-\beta} A).
\end{aligned}$$

Hence,

$$\Phi(\rho^{\alpha}A\rho^{1-\alpha}A)+\Phi(\rho^{\beta}A\rho^{1-\beta}A)\geq 2\Phi(\rho^{\frac{\alpha+\beta}{2}}A\rho^{1-\frac{\alpha+\beta}{2}}A), \tag{3.2}$$

since $\alpha,\beta\in\Delta$, we infer

$$\Phi(\rho A^2)\geq\Phi(\rho^{\frac{\alpha+\beta}{2}}A\rho^{1-\frac{\alpha+\beta}{2}}A),$$

which implies that $\frac{\alpha+\beta}{2}\in\Delta$. □

Remark 3.2. Inequality (3.2) shows that the map $\alpha\to\Phi(\rho^{\alpha}A\rho^{1-\alpha}A)$ is convex. In addition, we have

$$I^{\alpha}_{\rho,\Phi}(A)\leq I_{\rho,\Phi}(A)\quad(\alpha\in\mathbb{R}) \tag{3.3}$$

for every self-adjoint operator A and each positive operator ρ. Indeed, using the positivity and the tracial property of Φ, we get

$$\begin{aligned}0&\leq\Phi\left((\rho^{\frac{\alpha}{2}}A\rho^{\frac{1-\alpha}{2}}-\rho^{\frac{1-\alpha}{2}}A\rho^{\frac{\alpha}{2}})(\rho^{\frac{1-\alpha}{2}}A\rho^{\frac{\alpha}{2}}-\rho^{\frac{\alpha}{2}}A\rho^{\frac{\alpha}{2}})\right)\\&=2\Phi(\rho^{\alpha}A\rho^{1-\alpha}A)-2\Phi(\rho^{\frac12}A\rho^{\frac12}A).\end{aligned}$$

Therefore $\Phi(\rho^{\alpha}A\rho^{1-\alpha}A)\geq\Phi(\rho^{\frac12}A\rho^{\frac12}A)$ and so

$$I^{\alpha}_{\rho,\Phi}(A)=\Phi(\rho A^2)-\Phi(\rho^{\alpha}A\rho^{1-\alpha}A)\leq\Phi(\rho A^2)-\Phi(\rho^{\frac12}A\rho^{\frac12}A)=I_{\rho,\Phi}(A).$$

Let Φ be a tracial positive linear map between C^* algebras. It follows from Lemma 3.1 that $I^{\alpha}_{\rho,\Phi}(A)\geq 0$ for every self-adjoint operator A, but it is not true in general when A is an arbitrary operator. Hence even if Φ is a tracial positive functional, then $\mathrm{Corr}^{\alpha}_{\rho,\Phi}(\cdot,\cdot)$ cannot induce a complex valued semi-inner product and we cannot use the Cauchy–Schwarz inequality; see [18, Remark IV.2]. The next lemma helps us to give a positive definite version of the generalized correlation.

Lemma 3.3. *Let $\Phi:\mathcal{A}\longrightarrow\mathcal{B}$ be a tracial positive linear map and $\rho\in\mathcal{A}$ be a positive operator. Then*

$$I^{\alpha}_{\rho,\Phi}(A)+I^{\alpha}_{\rho,\Phi}(A^*)\geq 0$$

for every $A\in\mathcal{A}$.

Proof. We define $\tilde{\Phi}:M_2(\mathcal{A})\longrightarrow B$ by $\tilde{\Phi}\left(\begin{bmatrix}A&B\\C&D\end{bmatrix}\right)=\frac12\Phi(A+D)$. It is obvious that $\tilde{\Phi}$ is a tracial positive linear map. Take $\tilde{A}=\begin{bmatrix}0&A^*\\A&0\end{bmatrix}$ and $\tilde{\rho}=\begin{bmatrix}\rho&0\\0&\rho\end{bmatrix}$. Clearly $\tilde{A}$ is a self-adjoint operator in $M_2(\mathcal{A})$ and $\tilde{\rho}$ is a $\tilde{\Phi}$-density operator. Using Lemma 3.1 for $\tilde{\Phi}$, we get

$$\begin{aligned}\frac12\left(\Phi(\rho A^*A)+\Phi(\rho AA^*)\right)&=\tilde{\Phi}\left(\begin{bmatrix}\rho&0\\0&\rho\end{bmatrix}\begin{bmatrix}0&A^*\\A&0\end{bmatrix}\begin{bmatrix}0&A^*\\A&0\end{bmatrix}\right)\\&=\tilde{\Phi}(\tilde{\rho}\tilde{A}^2)\\&\geq\tilde{\Phi}(\tilde{\rho}^{1-\alpha}\tilde{A}\tilde{\rho}^{\alpha}\tilde{A})\\&\qquad\text{(by Lemma 3.1)}\end{aligned}$$

$$= \tilde{\Phi}\left(\begin{bmatrix} \rho^{1-\alpha} & 0 \\ 0 & \rho^{1-\alpha} \end{bmatrix}\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\begin{bmatrix} \rho^{\alpha} & 0 \\ 0 & \rho^{\alpha} \end{bmatrix}\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\right)$$
$$= \frac{1}{2}(\Phi(\rho^{1-\alpha}A^*\rho^{\alpha}A) + \Phi(\rho^{1-\alpha}A\rho^{\alpha}A^*)).$$

Hence,

$$I^{\alpha}_{\rho,\Phi}(A) + I^{\alpha}_{\rho,\Phi}(A^*) = \Phi(\rho A^*A) + \Phi(\rho AA^*) - \Phi(\rho^{1-\alpha}A^*\rho^{\alpha}A) - \Phi(\rho^{1-\alpha}A\rho^{\alpha}A^*) \geq 0. \quad \square$$

Definition 3.4. Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive linear map and $\rho \in \mathcal{A}$ be a Φ-density operator. Then for every operators $A, B \in \mathcal{A}$, we set

$$\mathrm{Corr}'^{\alpha}_{\rho,\Phi}(A,B) := \frac{1}{2}\left(\mathrm{Corr}^{\alpha}_{\rho,\Phi}(A,B) + \mathrm{Corr}^{\alpha}_{\rho,\Phi}(B^*,A^*)\right) \text{ and } I'^{\alpha}_{\rho,\Phi}(A) := \mathrm{Corr}'^{\alpha}_{\rho,\Phi}(A,A).$$

It is easy to check that $\mathrm{Corr}'^{\alpha}_{\rho,\Phi}(A,B)$ has the following properties:

(i) $\mathrm{Corr}'^{\alpha}_{\rho,\Phi}(A,A) \geq 0$, for every $A \in \mathcal{A}$ (by Lemma 3.3),
(ii) $\mathrm{Corr}'^{\alpha}_{\rho,\Phi}(A,B+\lambda C) = \mathrm{Corr}'^{\alpha}_{\rho,\Phi}(A,B) + \lambda\mathrm{Corr}'^{\alpha}_{\rho,\Phi}(A,C)$, for all $A, B \in \mathcal{A}$ and every $\lambda \in \mathbb{C}$,
(iii) $\mathrm{Corr}'^{\alpha}_{\rho,\Phi}(A,B)^* = \mathrm{Corr}'^{\alpha}_{\rho,\Phi}(B,A)$.

Next we give a generalization of inequality (1.3).

Theorem 3.5. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be C^*-subalgebra of $\mathcal{A}$. If $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{B}$ is a tracial conditional expectation, then*

$$\left|\mathrm{Re}(\mathrm{Corr}^{\alpha}_{\rho,\mathcal{E}}(A,B))\right|^2 \leq I^{\alpha}_{\rho,\mathcal{E}}(A)I^{\alpha}_{\rho,\mathcal{E}}(B)$$

for all self-adjoint operators $A, B \in \mathcal{A}$ and each $\mathcal{E}$-density operator $\rho \in \mathcal{A}$.

Proof. Define the map $\langle\cdot,\cdot\rangle : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{B}$ by $\langle A,B\rangle = \mathrm{Corr}'^{\alpha}_{\rho,\mathcal{E}}(A,B)$. If $A, B \in \mathcal{A}$ and $C \in \mathcal{B}$, then

$$\begin{aligned}
\langle A, BC\rangle &= \mathrm{Corr}'^{\alpha}_{\rho,\mathcal{E}}(A,BC) \\
&= \frac{1}{2}\left(\mathrm{Corr}^{\alpha}_{\rho,\mathcal{E}}(A,BC) + \mathrm{Corr}^{\alpha}_{\rho,\mathcal{E}}(C^*B^*,A^*)\right) \\
&= \frac{1}{2}\left(\mathcal{E}(\rho A^*BC) - \mathcal{E}(\rho^{1-\alpha}A^*\rho^{\alpha}BC) + \mathcal{E}(\rho BCA^*) - \mathcal{E}(\rho^{1-\alpha}BC\rho^{\alpha}A^*)\right) \\
&= \frac{1}{2}\left(\mathcal{E}(\rho A^*BC) - \mathcal{E}(\rho^{1-\alpha}A^*\rho^{\alpha}BC) + \mathcal{E}(CA^*\rho B) - \mathcal{E}(C\rho^{\alpha}A^*\rho^{1-\alpha}B)\right) \\
&\qquad\qquad \text{(since } \mathcal{E} \text{ is tracial)} \\
&= \frac{1}{2}\left(\mathcal{E}(\rho A^*B)C - \mathcal{E}(\rho^{1-\alpha}A^*\rho^{\alpha}B)C + \mathcal{E}(\rho BA^*)C - \mathcal{E}(\rho^{1-\alpha}B\rho^{\alpha}A^*)C\right) \\
&\qquad \text{(since } \mathcal{E} \text{ is a conditional expectation and by equality (2.2))} \\
&= \mathrm{Corr}'^{\alpha}_{\rho,\mathcal{E}}(A,B)C \\
&= \langle A,B\rangle C.
\end{aligned}$$

Using this fact and Definition 3.4 we see that $(\mathcal{A},\langle\cdot,\cdot\rangle)$ is a semi-inner product $\mathcal{B}$-module. Moreover, equality (2.2) shows that $\mathrm{ran}(Z) \subseteq \mathcal{Z}(\mathcal{B})$. If A and B are self-adjoint operators in $\mathcal{A}$, then we get

$$\begin{aligned}\left|\mathrm{Re}(\mathrm{Corr}^{\alpha}_{\rho,\mathcal{E}}(A,B))\right|^2 &= \left|\frac{1}{2}\left(\mathrm{Corr}^{\alpha}_{\rho,\mathcal{E}}(A,B)+\mathrm{Corr}^{\alpha}_{\rho,\mathcal{E}}(B,A)\right)\right|^2\\ &= \left|\mathrm{Corr}'^{\alpha}_{\rho,\mathcal{E}}(A,B)\right|^2\\ &= |\langle A,B\rangle|^2\\ &\le \langle A,A\rangle\langle B,B\rangle \qquad \text{(by inequality (1.7))}\\ &= I^{\alpha}_{\rho,\mathcal{E}}(A)I^{\alpha}_{\rho,\mathcal{E}}(B) \qquad \text{(since } A,B \text{ are self-adjoint).}\ \square\end{aligned}$$

Let A be a self-adjoint operator and ρ be a density operator. According to [12, Section III] we have $I_\rho(A) \le V_\rho(A)$. We give a generalization of this inequality for a tracial 2-positive linear map Φ and a Φ-density operator ρ. It follows from Lemma 2.2 that the matrix

$$\begin{bmatrix} \rho^{\frac{1}{4}}A\rho^{\frac{1}{4}}\rho^{\frac{1}{4}}A\rho^{\frac{1}{4}} & \rho^{\frac{1}{4}}A\rho^{\frac{1}{4}}\rho^{\frac{1}{2}} \\ \rho^{\frac{1}{2}}\rho^{\frac{1}{4}}A\rho^{\frac{1}{4}} & \rho \end{bmatrix}$$

is positive. Since Φ is 2-positive, we have

$$\begin{bmatrix} \Phi(\rho^{\frac{1}{4}}A\rho^{\frac{1}{2}}\rho^{\frac{1}{2}}A\rho^{\frac{1}{4}}) & \Phi(\rho^{\frac{1}{4}}A\rho^{\frac{1}{4}}\rho^{\frac{1}{2}}) \\ \Phi(\rho^{\frac{1}{2}}\rho^{\frac{1}{4}}A\rho^{\frac{1}{4}}) & \Phi(\rho) \end{bmatrix} \ge 0.$$

Therefore, by using Lemma 2.2 and applying the tracial property of Φ we get

$$\Phi(\rho^{\frac{1}{2}}A\rho^{\frac{1}{2}}A) \ge \Phi(\rho A)\Phi(\rho)^{-1}\Phi(\rho A) = \Phi(\rho A)^2, \tag{3.4}$$

which implies that $I_{\rho,\Phi}(A) \le V_{\rho,\Phi}(A)$. Consequently, by using inequality (3.3), we reach $I^{\alpha}_{\rho,\Phi}(A) \le V_{\rho,\Phi}(A)$.

For a tracial positive linear map Φ and a self-adjoint operator A, we set

$$J_{\rho,\Phi}(A) := 2V_{\rho,\Phi}(A) - I_{\rho,\Phi}(A) \text{ and } U_{\rho,\Phi}(A) := I_{\rho,\Phi}(A)\sharp J_{\rho,\Phi}(A).$$

Since $U_{\rho,\Phi}(A) \le V_{\rho,\Phi}(A)$ (by the arithmetic–geometric mean inequality), the next theorem is a refinement of Theorem 2.9 in the case that Φ is a conditional expectation. To establish it, we model the classical techniques (see [11]) to the non-commutative framework.

Theorem 3.6. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. If $\mathcal{E}:\mathcal{A}\longrightarrow\mathcal{B}$ is a tracial conditional expectation, then*

$$U_{\rho,\mathcal{E}}(A)U_{\rho,\mathcal{E}}(B) \ge \frac{1}{4}|\mathcal{E}(\rho[A,B])|^2 \tag{3.5}$$

for all self-adjoint operators $A,B\in\mathcal{A}$ and each $\mathcal{E}$-density operator $\rho\in\mathcal{A}$.

Proof. Consider self-adjoint operators $A_0 = A - \mathcal{E}(\rho A)$ and $B_0 = B - \mathcal{E}(\rho B)$. A simple calculation shows that

$$I_{\rho,\mathcal{E}}(A) = \frac{1}{2}\mathcal{E}((\mathrm{i}[\rho^{\frac{1}{2}},A_0])^2) \text{ and } J_{\rho,\mathcal{E}}(B) = \frac{1}{2}\mathcal{E}(\{\rho^{\frac{1}{2}},B_0\}^2), \tag{3.6}$$

where $\{\rho^{\frac{1}{2}},B\} = \rho^{\frac{1}{2}}B_0 + B_0\rho^{\frac{1}{2}}$. Indeed,

$$\begin{aligned}
\mathcal{E}((\mathrm{i}[\rho^{\frac{1}{2}}, A_0])^2) &= -\mathcal{E}(\rho^{\frac{1}{2}} A_0 \rho^{\frac{1}{2}} A_0 - \rho^{\frac{1}{2}} A_0^2 \rho^{\frac{1}{2}} - A_0 \rho^{\frac{1}{2}} \rho^{\frac{1}{2}} A_0 + A_0 \rho^{\frac{1}{2}} A_0 \rho^{\frac{1}{2}}) \\
&= 2\mathcal{E}(\rho A_0^2) - 2\mathcal{E}(\rho^{\frac{1}{2}} A_0 \rho^{\frac{1}{2}} A_0) \\
&\qquad \text{(by the tracial property of } \mathcal{E}) \\
&= 2\mathcal{E}\Big(\rho(A - \mathcal{E}(\rho A))^2\Big) - 2\mathcal{E}\Big(\rho^{\frac{1}{2}}(A - \mathcal{E}(\rho A))\rho^{\frac{1}{2}}(A - \mathcal{E}(\rho A))\Big) \\
&= 2\mathcal{E}(\rho A^2) - 2(\mathcal{E}(\rho A))^2 - 2\mathcal{E}(\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}} A) + 2(\mathcal{E}(\rho A))^2 \\
&\qquad \text{(since } \mathcal{E} \text{ is a conditional expectation)} \\
&= 2\mathcal{E}(\rho A^2) - 2\mathcal{E}(\rho^{\frac{1}{2}} A \rho^{\frac{1}{2}} A).
\end{aligned}$$

Similarly, we can establish the other inequality in (3.6).

Let $Z \in \mathcal{B}$. Then

$$\begin{aligned}
&\mathcal{E}\Big(\mathrm{i}Z[\rho^{\frac{1}{2}}, A_0]\{\rho^{\frac{1}{2}}, B_0\} + \mathrm{i}\{\rho^{\frac{1}{2}}, B_0\}[\rho^{\frac{1}{2}}, A_0]Z\Big) \\
&\qquad = Z\mathcal{E}\Big(\mathrm{i}\big(\rho^{\frac{1}{2}} A_0 \rho^{\frac{1}{2}} B_0 + \rho^{\frac{1}{2}} A_0 B_0 \rho^{\frac{1}{2}} - A_0 \rho^{\frac{1}{2}} \rho^{\frac{1}{2}} B_0 \\
&\qquad\quad - A_0 \rho^{\frac{1}{2}} B_0 \rho^{\frac{1}{2}} + \rho^{\frac{1}{2}} B_0 \rho^{\frac{1}{2}} A_0 \\
&\qquad\quad - \rho^{\frac{1}{2}} B_0 A_0 \rho^{\frac{1}{2}} + B_0 \rho^{\frac{1}{2}} \rho^{\frac{1}{2}} A_0 - B_0 \rho^{\frac{1}{2}} A_0 \rho^{\frac{1}{2}}\big)\Big) \\
&\qquad\qquad \text{(by equality (2.2))} \\
&\qquad = 2\mathrm{i}Z\mathcal{E}(\rho[A_0, B_0]) \\
&\qquad = 2\mathrm{i}Z\mathcal{E}\big(\rho\,[A - \mathcal{E}(\rho A), B - \mathcal{E}(\rho B)]\,\big) \\
&\qquad = 2\mathrm{i}Z\mathcal{E}\Big(\rho\big((A - \mathcal{E}(\rho A))(B - \mathcal{E}(\rho B)) \\
&\qquad\quad - (B - \mathcal{E}(\rho B))(A - \mathcal{E}(\rho A))\big)\Big) \\
&\qquad = 2\mathrm{i}Z\mathcal{E}\Big(\rho\big(AB - A\mathcal{E}(\rho B) - \mathcal{E}(\rho A)B + \mathcal{E}(\rho A)\mathcal{E}(\rho B) \\
&\qquad\quad - BA + B\mathcal{E}(\rho A) + \mathcal{E}(\rho B)A - \mathcal{E}(\rho B)\mathcal{E}(\rho A)\big)\Big) \\
&\qquad = 2\mathrm{i}Z\big(\mathcal{E}(\rho AB) - \mathcal{E}(\rho A)\mathcal{E}(\rho B) - \mathcal{E}(\rho A)\mathcal{E}(\rho B) \\
&\qquad\quad + \mathcal{E}(\rho)\mathcal{E}(\rho A)\mathcal{E}(\rho B) - \mathcal{E}(\rho BA) + \mathcal{E}(\rho B)\mathcal{E}(\rho A) \\
&\qquad\quad + \mathcal{E}(\rho B)\mathcal{E}(\rho A) - \mathcal{E}(\rho)\mathcal{E}(\rho B)\mathcal{E}(\rho A)\big) \\
&\qquad = 2\mathrm{i}Z\mathcal{E}(\rho[A, B]) \\
&\qquad\qquad \text{(by equality (2.2)).}
\end{aligned}$$

Hence,

$$\mathcal{E}\Big(\mathrm{i}Z[\rho^{\frac{1}{2}}, A_0]\{\rho^{\frac{1}{2}}, B_0\} + \mathrm{i}\{\rho^{\frac{1}{2}}, B_0\}[\rho^{\frac{1}{2}}, A_0]Z\Big) = 2\mathrm{i}Z\mathcal{E}(\rho[A, B]). \tag{3.7}$$

Let $Z \in \mathcal{B}$ be a self-adjoint operator and $X = \mathrm{i}[\rho^{\frac{1}{2}}, A_0]Z + \{\rho^{\frac{1}{2}}, B_0\}$. Then

$$0 \le \mathcal{E}(X^*X) = \mathcal{E}\Big(\big(\mathrm{i}Z[\rho^{\frac{1}{2}}, A_0] + \{\rho^{\frac{1}{2}}, B_0\}\big)\big(\mathrm{i}[\rho^{\frac{1}{2}}, A_0]Z + \{\rho^{\frac{1}{2}}, B_0\}\big)\Big)$$

(because $\mathrm{i}[\rho^{\frac{1}{2}}, A_0]$ is self-adjoint)

$$
\begin{aligned}
&= \mathcal{E}\Big(- Z[\rho^{\frac{1}{2}}, A_0][\rho^{\frac{1}{2}}, A_0]Z + iZ[\rho^{\frac{1}{2}}, A_0]\{\rho^{\frac{1}{2}}, B_0\} \\
&\quad + \mathrm{i}\{\rho^{\frac{1}{2}}, B_0\}[\rho^{\frac{1}{2}}, A_0]Z + \{\rho^{\frac{1}{2}}, B_0\}^2\Big) \\
&= 2I_{\rho,\mathcal{E}}(A)Z^2 + 2\mathrm{i}\mathcal{E}(\rho[A, B])Z + 2J_{\rho,\mathcal{E}}(B) \\
&\qquad\qquad \text{(by equality (3.6) and equality (3.7)).}
\end{aligned}
$$

Without loss of the generality we can assume that $I_{\rho,\mathcal{E}}(A) > 0$. If we put $Z := -\frac{\mathrm{i}}{2} I_{\rho,\mathcal{E}}(A)^{-1}\mathcal{E}(\rho[A, B])$, then we get

$$
-\frac{1}{4} I_{\rho,\mathcal{E}}(A)^{-1}\mathcal{E}(\rho[A, B])^2 + \frac{1}{2} I_{\rho,\mathcal{E}}(A)^{-1}\mathcal{E}(\rho[A, B])^2 + J_{\rho,\mathcal{E}}(B) \geq 0,
$$

or equivalently,

$$
I_{\rho,\mathcal{E}}(A)J_{\rho,\mathcal{E}}(B) \geq -\frac{1}{4}\mathcal{E}(\rho[A, B])^2 = \frac{1}{4}|\mathcal{E}(\rho[A, B])|^2, \tag{3.8}
$$

since $\mathcal{E}(\rho[A, B])^* = -\mathcal{E}(\rho[A, B])$. It follows from the fact that for every $X \in \mathcal{A}$, $\mathcal{E}(X) \subseteq \mathcal{Z}(\mathcal{B})$ (equality (2.2)), we have

$$
\begin{aligned}
U_{\rho,\mathcal{E}}(A)U_{\rho,\mathcal{E}}(B) &= \big(I_{\rho,\mathcal{E}}(A)\sharp J_{\rho,\mathcal{E}}(A)\big)\big(I_{\rho,\mathcal{E}}(B)\sharp J_{\rho,\mathcal{E}}(B)\big) \\
&= \big(I_{\rho,\mathcal{E}}(A)J_{\rho,\mathcal{E}}(B)\big)^{\frac{1}{2}}\big(I_{\rho,\mathcal{E}}(B)J_{\rho,\mathcal{E}}(A)\big)^{\frac{1}{2}} \\
&\qquad \text{(by the commutativity property in equality (2.2))} \\
&\geq \frac{1}{4}|\mathcal{E}(\rho[A, B])|^2 \\
&\qquad \text{(by inequality (3.8)).} \quad \square
\end{aligned}
$$

As a consequence we get the following result of Luo [11].

Corollary 3.7. *[11, p. 2] If A, B are two self-adjoint operators, then*

$$
U_\rho(A)U_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2.
$$

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