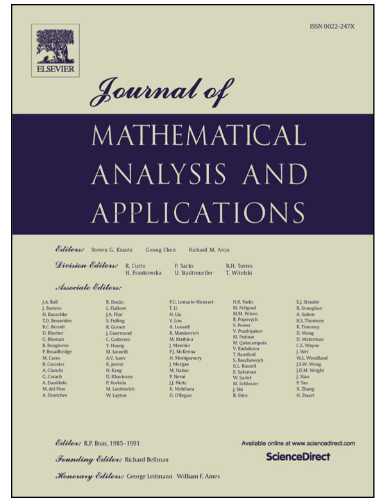
Accepted Manuscript



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PII: S0022-247X(17)30003-3
DOI: http://dx.doi.org/10.1016/j.jmaa.2016.12.078
Reference: YJMAA 21018

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 25 July 2016

INJECTIVITY AND ALMOST GLOBAL ASYMPTOTIC STABILITY OF HURWITZ VECTOR FIELDS

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ABSTRACT. We present, in dimension $n \geq 3$, a survey of examples to: the Jacobian conjecture, the weak Markus–Yamabe conjecture. Furthermore, we show and construct new examples of vector fields where the origin is almost globally asymptotic stable by using the novel concept of density functions introduced by Rantzer.

1. Introduction

One of the central problems on dynamical systems is to determine conditions under which certain points or sets are attractors for some dynamics, that is, the orbits through points in a neighborhood of the attractor converge to them. In the case of continuous-time, that is, flows associated to vector fields, an analytic condition ensuring that an equilibrium point x^* is a local attractor is given by the negativeness of the real part of the eigenvalues of the Jacobian matrix at x^*. Motivated by this simple observation, in [18], L. Markus and H. Yamabe establish their well known global stability conjecture

Markus–Yamabe Conjecture (MYC): Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1- vector field with $F(0) = 0$. If for any $x \in \mathbb{R}^n$ all the eigenvalues of $JF(x)$, the Jacobian matrix of F at x, have negative real part, then the origin is a global attractor of the system $\dot{x} = F(x)$.

Let us recall that the C^1- vector fields $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying that for any $x \in \mathbb{R}^n$ all the eigenvalues of $JF(x)$ have negative real part are called Hurwitz vector fields. It is known that the MYC is true when $n \leq 2$ and false when $n \geq 3$ (see [10] for a counterexample). The proofs in the planar context, both the polynomial case (G. Meisters and C. Olech in [19]) as the C^1- case (R. Feßler in [14], A.A. Glutsyuk in [16] and C. Gutiérrez in [17]) are based on a remarkable result of C. Olech [20], where the author showed that MYC (in dimension two) is equivalent to the injectivity of the map F. In $\mathbb{R}^n$, the problem of knowing if a Hurwitz vector field is injective is known as the Weak Markus–Yamabe conjecture.

Weak Markus–Yamabe Conjecture (WMYC): If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1- Hurwitz map, then F is injective.

The **WMYC** is true when $n \leq 2$ and, to the best of our knowledge, it has been proved in dimension $n \geq 3$ for C^1- Lipschitz Hurwitz maps by A. Fernandes, C.

1991 *Mathematics Subject Classification.* 14R15, 37C10, 37C75 .
Key words and phrases. Global injectivity, Jacobian conjecture, weak Markus-Yamabe conjecture, Almost global stability.
The both authors were funded by MATHAMSUD 16-MATH-04 STADE .

Gutiérrez and R. Rabanal in [13, Corollary 4]. Notice that this conjecture restricted to polynomial maps was introduced by A. van den Essen in [12].

For $n = 2$, the strong injectivity theorem of Gutiérrez [17] is the following: "A C^1-map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is injective if $[0, \infty) \cap \operatorname{Spec}(Jf(x)) = \emptyset$, for all $x \in \mathbb{R}^2$". A similar result also appears in [14], and it was recently improved in [23]. However, this results fail in high dimensions as shown B. Smyth and F. Xavier [26, Theorem 4]: "There exist integers $n > 2$ and non–injective polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^n$ with $[0, \infty) \cap \operatorname{Spec}(Jf(x)) = \emptyset$, for all $x \in \mathbb{R}^n$".

On the other hand, a new tool (density functions) introduced by A. Rantzer in [24], gives sufficient conditions ensuring almost global stability of an equilibrium point for a C^1-vector field in $\mathbb{R}^n$ (*i.e.*, all trajectories, except for a set of initial states with zero Lebesgue measure, converge to the equilibrium point). Recently in [21] B. Pires and R. Rabanal have studied, in dimension two, almost Hurtwitz vector fields, i.e. vector fields that satisfy the Hurwitz condition outsize of a zero measure set. Their main result is: almost Hurwitz vector fields with the origin a hyperbolic singular point are all topologically equivalent to the radial vector field.

Both concepts, density functions and almost Hurwitz vector fields, have been related in dimension three by R. Potrie and P. Monzón [22] to construct a vector field X where the origin is almost globally stable but is not a local attractor for the differential system generated for X.

This article is focused on two tasks: Firstly, we will construct polynomial maps $F = \lambda I + H : \mathbb{R}^3 \to \mathbb{R}^3$ with JH nilpotent, such that the **WMYC** and the **Jacobian Conjecture** (a formal description will be given later) are true, giving the inverse of F explicitly. The results obtained are strongly related with the works of L.A. Campbell [5] and M. Chamberland and A. van den Essen [8]. Secondly, we construct two families of three dimensional vector fields having the Rantzer's density functions stated above. The vector fields of the first family are a generalization of the Potrie–Monzón's example [22] in the sense that are almost Hurwitz and the vector field restricted to the invariant plane $z = 0$ is a centre. Moreover, perturbing these vector fields by λI, we obtain a new family a Hurwitz vector fields with the origin as global attractor which are not included in the examples of [15, Theorem 2.7] and [6, Theorem 2.5]. The vector fields of the second family are not almost Hurwitz and the the existence of a density function seems to be the only way to demonstrate the almost global stability of the origin.

The paper is organized as follows. The Section 2 is devoted to generalize the procedure of L.A. Campbell in order to obtain new examples to **WMYC** and the **Jacobian Conjecture** on $\mathbb{R}^n$. The Section 3, using density functions, shows two family of vector fields such that the origin is almost globally stable. Moreover, we construct new examples to **MYC**.

2. Examples to Jacobian conjecture and WMYC

In dimension three, all existing examples and counterexamples to MYC are maps of the form $\lambda I + H$ with $\lambda < 0$ and JH nilpotent (see [10], [9], [6]). This kind of maps, in any dimension, also are important in the study of the Jacobian conjecture since H. Bass, E. Connell and D. Wright showed in [1] that it suffices to solve this conjecture for such maps. Indeed, the Jacobian conjecture follows from the injectivity of these maps for all dimensions (see [2, 25]).

It is worth to emphasize that the result above triggered new questions and problems as the following one. Let κ be a field of characteristic zero.

(Homogeneous) dependence problem. Let $H = (H_1, \ldots, H_n) \in \kappa[x_1, \ldots, x_n]^n$ (homogeneous of degree $d \geq 1$) such that JH is nilpotent and $H(0) = 0$. Does it follow that $H_1, \ldots, H_n$ are linearly dependent over κ?

The homogeneous problem is true in dimension three [3] and false in dimensions bigger than five [4]. A counterexample for the inhomogeneous problem in dimension three was given by E. Hubbers in [11]. The map

$$H = (y - x^2, z + 2x(y - x^2), -(y - x^2)^2)$$

verifies that JH is nilpotent, $\mathrm{rank}(JH) = 2$ and H_1, H_2, H_3 are linearly independent over κ. Moreover, L.A. Campbell in [5] generalizes this counterexample obtaining

$$H = (\phi(y - x^2), z + 2x\phi(y - x^2), -(\phi(y - x^2))^2)$$

with the same properties. Here ϕ is a C^1- function of a single variable. Notice that the inverse of $F = I + H$ can be computed explicitly as follows

$$F^{-1} = (x - \phi(y - x^2 - z), y - z - 2x\phi(y - x^2 - z) + (\phi(y - x^2 - z))^2, z + (\phi(y - x^2 - z))^2)\,.$$

Therefore $F = I + H$ with $\phi \in \mathbb{R}[t]$ is an example in dimension three to problem enunciated by M. Chamberland in [7]:

Jacobian Conjecture on $\mathbb{R}^n$. Every polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det JF \equiv 1$ is a bijective map with a polynomial inverse.

Let us emphasize that the previous examples have the special form

$$H = (u(x, y, z), v(x, y, z), h(u, v)),$$

which is completely studied in [8], which deals with polynomial maps satisfying $H(0) = 0$, h has no linear part and the components of H are linearly independent over κ. Given $A = v_x u_z - u_x v_z$ and $B = v_y u_z - u_y v_z$, they show that if JH is nilpotent and $\deg_z uA \neq \deg_z vB$, then there exists $T \in GL_3(k)$ such that THT^{-1} takes the form

$$(g(t), v_1 z - (b_1 + 2v_1\alpha x)g(t), \alpha(g(t))^2)\,, \tag{1}$$

with $t = y + b_1 x + v_1\alpha x^2$, $v_1\alpha \neq 0$ and $g(t) \in k[t], g(0) = 0$ and $\deg_t g(t) \geq 1$.

The expressions A and B are very useful for determine if a map H is nilpotent [8, Proposition 3.1]. Also they are useful for decide if the rows of H are linearly independent over κ. In fact, we obtain the following result with $\kappa = \mathbb{R}$.

Proposition 2.1. *Let $H = (u(x, y, z), v(x, y, z), h(u(x, y, z), v(x, y, z)))$ be a polynomial map such that the components of H are linearly dependent over $\mathbb{R}$ and h has no linear part in u and v. Then $A = B = 0$.*

Proof. Let $\alpha, \beta \in \mathbb{R}^*$ such that

$$h_u \cdot (u_x, u_y, u_z) + h_v \cdot (v_x, v_y, v_z) \equiv \alpha \cdot (u_x, u_y, u_z) + \beta \cdot (v_x, v_y, v_z)$$

which is equivalent to

$$(h_u - \alpha) \cdot (u_x, u_y, u_z) + (h_v - \beta) \cdot (v_x, v_y, v_z) \equiv 0. \tag{2}$$

Thus, we have the following systems of equations

$$\left\{\begin{array}{rcl}(h_u-\alpha)u_x+(h_v-\beta)v_x &=& 0\\ (h_u-\alpha)u_y+(h_v-\beta)v_y &=& 0\\ (h_u-\alpha)u_z+(h_v-\beta)v_z &=& 0.\end{array}\right. \tag{3}$$

By using (3), we see that

$$\begin{aligned} u_zA-v_zB &= u_z^2v_x-u_zv_z(u_x+v_y)+v_z^2u_y\\ &= \tfrac{v_z^2}{(h_u-\alpha)^2}\{(h_v-\beta)^2v_x+(h_v-\beta)(h_u-\alpha)(u_x+v_y)+(h_u-\alpha)^2u_y\}\\ &= \tfrac{v_z^2}{(h_u-\alpha)^2}\{(h_v-\beta)^2v_x\\ &\qquad -(h_v-\beta)(h_u-\alpha)(\tfrac{(h_v-\beta)}{(h_u-\alpha)}v_x-v_y)-\tfrac{(h_u-\alpha)^2(h_v-\beta)}{(h_u-\alpha)}v_y\}\\ &= 0\end{aligned}$$

over the set of points where $h_u \neq \alpha$. Otherwise, if $h_u = \alpha$ then in (2) we have that $(v_x, v_y, v_z) = 0$ or $h_v = \beta$. The first case immediately implies that $A = B = 0$, the second case is not possible due to is a contradiction with no linearity of h with respect to u and v.

In similar way is proved that $u_zA + v_zB = 0$ and the result follows. □

Theorem 2.2. *Consider a polynomial map of the form $H = (u, v, h(u, v))$ such that $H(0) = 0$, h has no linear part and the components of H are linearly independent over k. Then if JH is nilpotent and $\deg_z uA \neq \deg_z vB$, for all $\lambda \neq 0$ the polynomial map $\lambda I + H$ is injective and has inverse polynomial.*

Proof. According to [8, Corollary 4.1], we can suppose that the components H_1, H_2, H_3 of H are as in (1). Namely,

$$\begin{array}{rcl} H_1(x,y,z) &=& g(y+b_1x+v_1\alpha x^2)\,,\\ H_2(x,y,z) &=& v_1z-(b_1+2v_1\alpha x)g(y+b_1x+v_1\alpha x^2)\,,\\ H_3(x,y,z) &=& \alpha(g(y+b_1x+v_1\alpha x^2))^2.\end{array} \tag{4}$$

If

$$\begin{array}{rcl} u_1 &=& \lambda x+g(y+b_1x+v_1\alpha x^2)\\ u_2 &=& \lambda y+v_1z-(b_1+2v_1\alpha x)g(y+b_1x+v_1\alpha x^2)\\ u_3 &=& \lambda z+\alpha(g(y+b_1x+v_1\alpha x^2))^2,\end{array}$$

it is easy to obtain

$$\gamma u_2+\gamma b_1u_1+\gamma^2v_1\alpha u_1^2-\gamma^2v_1u_3 = y+b_1x+v_1\alpha x^2, \tag{5}$$

where $\gamma = \frac{1}{\lambda}$. Now, put $\Phi = g(y+b_1x+v_1\alpha x^2) = g(\gamma u_2+\gamma b_1u_1+\gamma^2v_1\alpha u_1^2-\gamma^2v_1u_3)$ and observe that $\lambda I + H = (\lambda x+\Phi, \lambda y+v_1z-(b_1+2v_1\alpha x)\Phi, \lambda z+\alpha\Phi^2)$. By using (5), we obtain the inverse of this map which is

$$(\gamma I+P) = (\gamma x+P_1, \gamma y+P_2, \gamma z+P_3)$$

where

$$
\begin{aligned}
P_1(x,y,z) &= -\gamma g(\gamma y+\gamma b_1 x+\gamma^2 v_1\alpha x^2-\gamma^2 v_1 z)\\
P_2(x,y,z) &= -\gamma^2 v_1 z+\gamma(b_1+2\gamma v_1\alpha x)g(\gamma y+\gamma b_1 x+\gamma^2 v_1\alpha x^2-\gamma^2 v_1 z)\\
&\quad -\gamma^2 v_1\alpha(g(\gamma y+\gamma b_1 x+\gamma^2 v_1\alpha x^2-\gamma^2 v_1 z))^2\\
P_3(x,y,z) &= -\alpha\gamma(g(\gamma y+\gamma b_1 x+\gamma^2 v_1\alpha x^2-\gamma^2 v_1 z))^2.
\end{aligned} \tag{6}
$$

□

Remark 2.3. *For $\lambda<0$ (resp. $\lambda=1$), the polynomial map $F=\lambda I+H$ of the above Theorem is an example of the WMYC (resp. the Jacobian conjecture) on $\mathbb{R}^3$.*

Remark 2.4. *More examples for the both conjectures in dimension $n\geq 4$ can be constructed consider the following maps of* [11, Proposition 7.1.9]*:*

$$
\begin{aligned}
H_1(x_1,\ldots,x_n) &= g(x_2-a(x_1))\,,\\
H_i(x_1,\ldots,x_n) &= x_{i+1}+\frac{(-1)^i}{(i-1)!}\,a^{(i-1)}(x_1)\,g(x_2-a(x_1))^{i-1}\,,\ \text{if } 2\leq i\leq n-1\,,\\
H_n(x_1,\ldots,x_n) &= \frac{(-1)^n}{(n-1)!}\,a^{(n-1)}(x_1)\,g(x_2-a(x_1))^{n-1}
\end{aligned}
$$

where $a(x_1)\in\mathbb{R}[x_1]$ with $\deg a=n-1$ and $g(t)\in k[t], g(0)=0$ and $\deg_t g(t)\geq 1$. Following the lines of the proof of the Theorem (2.2), we must consider for each fixed $n\geq 4, u_i=\lambda x_i+H_i$, $i=1,\ldots,n$ and

$$\Phi=g(x_2-a(x_1))=g\Big(\frac{1}{\lambda}\,u_2-a(\frac{1}{\lambda}u)-\frac{1}{\lambda^2}\,u_3+(-1)^n\sum_{j=4}^{n}\frac{1}{\lambda^{j-1}}\,u_j\Big)$$

for obtain the inverse (polynomial) of $\lambda I+H$. In fact, the inverse is $\gamma I+P$ where

$$
\begin{aligned}
P_1(x_1,\ldots,x_n) &= -\gamma\Phi(x_1,\ldots,x_n)\\
P_i(x_1,\ldots,x_n) &= -\gamma(x_{i+1}-\frac{(-1)^i}{(i-1)!}a^{(i-1)}(\gamma(x_1-\Phi(x_1,\ldots,x_n)))(\Phi(x_1,\ldots,x_n))^{i-1}),\\
P_n(x_1,\ldots,x_n) &= -\gamma\frac{(-1)^n}{(n-1)!}\,a^{(n-1)}(\gamma(x_1-\Phi(x_1,\ldots,x_n))(\Phi(x_1,\ldots,x_n))^{n-1}.
\end{aligned}
$$

with $2\leq i\leq n-1$. Therefore, for $\lambda<0$ (resp. $\lambda=1$), the polynomial map $F=\lambda I+H$ is an example of the WMYC (resp. the Jacobian conjecture) on $\mathbb{R}^n$ with $n\geq 4$.

Example 2.5. *We consider $n=4$, the inverse of the map $\lambda I+(H_1,H_2,H_3,H_4)$ is $\gamma I+(P_1,P_2,P_3,P_4)$ where*

$$
\begin{aligned}
P_1(x_1,x_2,x_3,x_4) &= -\gamma\Phi,\\
P_2(x_1,x_2,x_3,x_4) &= -\gamma^2x_3+\gamma^3x_4-\gamma(a_1+\gamma 2a_2x_1+\gamma^2 3a_3x_1^2)\Phi+\gamma^2(a_2+3a_3x_1)\Phi^2,\\
P_3(x_1,x_2,x_3,x_4) &= -\gamma^2x_4+\gamma(a_2+\gamma 3a_3x_1)\Phi^2-\gamma 2a_3\Phi^3,\\
P_4(x_1,x_2,x_3,x_4) &= -\gamma a_3\Phi^3,
\end{aligned}
$$

where $\Phi=\Phi(x_1,x_2,x_3,x_4)=(\gamma x_2-\gamma x_1-\gamma^2x_1^2-\gamma^3x_1^3-\gamma^2x_3+\gamma^3x_4)$.

Remark 2.6. *A example to WMYC and Jacobian Conjecture, in dimension 4, which does not belong to the above family of maps is*

$$F(x,y,z,w)=(\lambda x+y,\lambda y+x^2-w,\lambda z+y^2,\lambda w+2xy-z).$$

Is easy to see that JH is nilpotent and the rows of JH are linearly independent over $\mathbb{R}$ and has inverse

$$F^{-1}(x,y,z,w)=(\gamma(x-\phi),\phi,\gamma(z-\phi^2),\gamma(w-2\gamma(x-\phi)\phi+\gamma(z-\phi^2))),$$

where $\phi=\gamma y-\gamma^3x^2+\gamma^3z+\gamma^2w$ and $\gamma=1/\lambda$. Therefore, for $\lambda<0$ (resp. $\lambda=1$), the polynomial map F is an example of the WMYC (resp. the Jacobian conjecture) on $\mathbb{R}^4$.

Remark 2.7. *In* [8, Corollary 4.2] *it is stated that a complete study of maps $H=(u,v,h(u,v))$ with JH is nilpotent will be achieved when considering the case $deg_z(uA)=deg_z(vB)$. In this context, the following result gives a partial progress to this study since we introduce a large family of maps satisfying $deg_z(uA)=deg_z(vB)$, which through a linear of change of coordinates have the form described in (1).*

Proposition 2.8. *The map $\gamma I+P$ with $P=(P_1,P_2,P_3)$ as in (6) has the following properties:*

- ***(P0)*** *$P(x,y,z)$ has the form $(u(x,y,z),v(x,y,z),h(u(x,y,z),v(x,y,z)))$.*
- ***(P1)*** *The Jacobian matrix JP is nilpotent and their rows are linearly independent over $\mathbb{R}$.*
- ***(P2)*** *$deg_z(uA)=deg_z(vB)$.*
- ***(P3)*** *Under the linear change of coordinates $(\widetilde{u},\widetilde{v},\widetilde{w})=(x,y-\gamma v_1z,z)$ the map $\gamma I+P$ is transformed into $\gamma I+Q$ where Q has the form $(u,v,h(u,v))$ and $deg_z(uA)\neq deg_z(vB)$.*

Proof. **(P0)** We have $P=(u,v,h(u,v))$ with $h(u,v)=-\frac{\alpha}{\gamma}\,u^2$.

(P1) By using an algebraic manipulator we see that $(JP)^3=0$. We have $P(0)=0$ and $P=(P_1,P_2,h(P_1,P_2))$ with $h(P_1,P_2)=-\frac{\alpha}{\gamma}\,P_1^2$. Moreover, if we consider $\omega(x)=b1+2\gamma v_1\alpha x$, we have

$$
\begin{aligned}
B &= \gamma^5v_1\omega(x)(g'(t))^2-2\gamma^6v_1^2\alpha g(t)(g'(t))^2-\gamma^4v_1g'(t)+\\
&\quad -\gamma^5v_1\omega(x)(g'(t))^2+2\gamma^6v_1^2\alpha g(t)(g'(t))^2\\
&= -\gamma^4v_1g'(t)\neq 0\,,
\end{aligned}
$$

and

$$A = -\gamma^4 v_1 g'(t)(\omega(x) - 2\gamma v_1 \alpha g(t)) \neq 0,$$

and the result follows from Proposition 2.1.

(P2) We have

$$u(x, y, z) \cdot A = -2\gamma^6 v_1^2 \alpha (g(t))^2 g'(t) + \gamma^5 v_1 \omega(x) g(t) g'(t)$$

and

$$v(x, y, z) \cdot B = \gamma^6 v_1^2 z g'(t) - \gamma^5 v_1 \omega(x) g(t) g'(t) + \gamma^6 v_1^2 \alpha (g(t))^2 g'(t).$$

Finally, $\deg_z(uA) = \max\{3k-1, 2k-1\} = 3k-1$ and $\deg_z(vB) = \max\{k, 2k-1, 3k-1\} = 3k-1$.

(P3) By considering the linear change of coordinate proposed, it is easy to see that the coordinates Q_1, Q_2, Q_3 of Q are

$$\begin{aligned} Q_1(x, y, z) &= g(\gamma y + \gamma b_1 x + \gamma^2 v_1 \alpha x^2) \\ Q_2(x, y, z) &= -\gamma^2 v_1 z + \gamma(b_1 + 2\gamma v_1 \alpha x) g(\gamma y + \gamma b_1 x + \gamma^2 v_1 \alpha x^2) \\ Q_3(x, y, z) &= -\alpha\gamma (g(\gamma y + \gamma b_1 x + \gamma^2 v_1 \alpha x^2))^2. \end{aligned}$$

This coordinates Q_1, Q_2, Q_3 are the same that (1) with $\gamma = 1$, thus $deg_z(uA) \neq deg_z(vB)$.

□

Now we consider, in dimension three, polynomial maps of the form $F = \lambda I + H$ with $H(0) = 0$ and JH nilpotent such that the components of H are linearly dependent over $\mathbb{R}$. As it was shown in [6, Proposition 2.1], for such vector field $F = \lambda I + H$, there exists $T \in Gl_3(\mathbb{R})$ such that $T^{-1} F T = \lambda I + (P, Q, 0)$, where

$$\begin{aligned} P(x, y, z) &= -b(z)\, f(a(z)\, x + b(z)\, y) \ + \ c(z) \quad \text{and} \\ Q(x, y, z) &= a(z)\, f(a(z)\, x + b(z)\, y) \ + \ d(z) \end{aligned} \tag{7}$$

with $a, b, c, d \in \mathbb{R}[z]$ and $f \in \mathbb{R}[z][t]$. Also it was pointed out in [6, Proposition 2.3] that if $\lambda \neq 0$, this polynomial map is injective and thus satisfies the conclusion of WMYC. In fact, if we consider $\gamma = \frac{1}{\lambda}$, we can find explicitly the inverse of $\lambda I + (P, Q, 0)$, with P and Q as in (7). Since $f \in \mathbb{R}[z][t]$, in what follows we replace $f(t)$ by $f(t, z)$.

Proposition 2.9. *For $\lambda \neq 0$, the inverse of $\lambda I + (P, Q, 0)$, with P and Q as in (7), is $\gamma I + (R, S, 0)$, where $\gamma = \frac{1}{\lambda}$ and*

$$\begin{aligned} R &= \gamma b(\gamma z)\, f\Big(\gamma\{a(\gamma z)\, x + b(\gamma z)\, y - a(\gamma z) c(\gamma z) - b(\gamma z) d(\gamma z)\}, \gamma z\Big) - \gamma c(\gamma z) \\ S &= -\gamma a(\gamma z)\, f\Big(\gamma\{a(\gamma z)\, x + b(\gamma z)\, y - a(\gamma z) c(\gamma z) - b(\gamma z) d(\gamma z)\}, \gamma z\Big) - \gamma d(\gamma z). \end{aligned}$$

Proof. Putting

$$\begin{aligned} u &= \lambda x - b(z)\, f(a(z)\,x + b(z)\,y, z) \; + \; c(z) \\ v &= \lambda y + a(z)\, f(a(z)\,x + b(z)\,y, z) \; + \; d(z) \\ w &= \lambda z\,, \end{aligned}$$

we obtain

$$a(z)u + b(z)v = \lambda(a(z)x + b(z)y) + a(z)c(z) + b(z)d(z)\,, \tag{8}$$

which implies

$$a(z)x + b(z)y = \gamma\big(a(z)u + b(z)v - a(z)c(z) - b(z)d(z)\big).$$

Therefore, by consider that $z = \gamma w$, we can deduce

$$\begin{aligned} u &= \lambda x - b(\gamma w)\, f(m(u,v,w), \gamma w) \; + \; c(\gamma w) \\ v &= \lambda y + a(\gamma w)\, f(m(u,v,w), \gamma w) \; + \; d(\gamma w), \end{aligned}$$

with $m(u,v,w) = \gamma\big\{a(\gamma w)u + b(\gamma w)v - a(\gamma w)c(\gamma w) - b(\gamma w)d(\gamma w)\big\}$. Finally,

$$\begin{aligned} x &= \gamma(u + b(\gamma w)\, f(m(u,v,w), \gamma w) - c(\gamma w)) \\ y &= \gamma(v - a(\gamma w)\, f(m(u,v,w), \gamma w) - d(\gamma w)). \end{aligned}$$

□

Remark 2.10. *Let $F = \lambda I + H : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map verifying $H(0) = 0$, JH nilpotent and their components are linearly dependent over $\mathbb{R}$. If $\lambda < 0$ (resp. $\lambda = 1$), then the map F is an example to WMYC (resp. the Jacobian conjecture).*

3. Examples of almost global stability

In [6, Theorem 3.2,Theorem 3.5] we prove that the vector fields in $\mathbb{R}^3$ of the form $F = \lambda I + H$, with $\lambda < 0$, H as in (4) and $g(t) = A_1\, t + A_2\, t^2$, are counterexamples to **MYC** since they have unbounded orbits. Moreover, by following the respective proofs, we can deduce the existence of an open set of initial states whose trajectories do not converge to the origin. Thus the origin for these vector fields is not almost global attractor in the following sense:

Definition 3.1. *Consider the differential equation*

$$\dot{x} = F(x) \tag{9}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-map and $F(0) = 0$. We say the origin is an almost global attractor if all trajectories, except for a set of initial states with zero Lebesgue measure, converge to the origin.

From Definition 3.1, it arises the following question:

Question 1. Do there exist counterexamples to the **MYC** with the origin almost global attractor ?

In [24], A. Rantzer introduces a new tool, namely, the density functions in order to obtain sufficient conditions for almost global stability of an equilibrium point for a C^1-vector field in $\mathbb{R}^n$.

Definition 3.2. *A density function of* (9) *is a* C^1 *map* $\rho\colon \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$, *integrable outside a ball centered at the origin that satisfies*

$$[\nabla \cdot \rho F](x) > 0$$

almost everywhere with respect to $\mathbb{R}^n$, *where*

$$\nabla \cdot [\rho F] = \nabla\rho \cdot F + \rho[\nabla \cdot F],$$

and $\nabla\rho$, $\nabla \cdot F$ *denote respectively the gradient of* ρ *and the divergence of* F.

The main result of A. Rantzer [24] is the following.

Theorem 3.3. *Given the differential system*

$$\dot{x} = F(x),$$

where $F \in C^1$, $F(0) = 0$, *suppose there exists a density function* $\rho : \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$ *such that* $\rho(x)F(x)/\parallel x\parallel$ *is integrable on* $\{x \in \mathbb{R}^n : \parallel x\parallel \geq 1\}$. *Then, almost all trajectories converge to the origin, i.e., the origin is almost globally stable.*

In this context, the next question arises naturally:
Question 2. Do there exist counterexamples to the **MYC** that support density functions ?

In this paper the best answer we give is the following family of almost Hurwitz vector fields; i.e. of vector fields for which the Hurwitz condition hold over $\mathbb{R}^n - A$ with A a zero Lebesgue measure set.

Theorem 3.4. *Consider the real numbers* $c \leq a < 0$, $b \in \mathbb{R}$, $k \geq 1$, *and the polynomial* $R(z) = \sum\limits_{i=1}^{k} a_{2i} z^{2i}$ *with* $a_{2i} > 0$ *for* $i = 1, \dots, k$. *Then*

$$F(x,y,z) = (y,-x,0) + ((ax+by)R(z), (-bx+cy)R(z), -zR(z)) \tag{10}$$

is an almost Hurwitz vector field. Moreover, this vector field has associated the density function $\rho(x,y,z) = (x^2+y^2+R(z))^{-\alpha}$ *with* $\alpha > \max\{2, \frac{a+c-1-2k}{2a}, \frac{3-a-c}{2}\}$.

Proof. The Jacobian matrix of F in a point (x,y,z) is

$$JF(x,y,z) = \begin{pmatrix} aR(z) & 1+bR(z) & * \\ -(1+bR(z)) & cR(z) & * \\ 0 & 0 & -(R(z)+zR'(z)) \end{pmatrix}.$$

Then $\lambda_3 = -(R(z)+zR'(z))$ is an eigenvalue. The others two eigenvalues λ_1, λ_2 verify

$$\lambda_1 + \lambda_2 = (a+c)\ R(z) \quad \text{and} \quad \lambda_1\lambda_2 = acR(z)^2 + (1+bR(z))^2\,.$$

Therefore, for $z \neq 0$ (resp. $z = 0$), we have $\lambda_3 < 0$ (resp. $\lambda_3 = 0$) and λ_1 and λ_2 have negative real part (resp. null real part). In addition, F verifies the Hurwitz condition except in the invariant plane $z = 0$.

In what follows we prove that the map

$$\rho(x,y,z) = (x^2+y^2+R(z))^{-\alpha}\,,$$

under the conditions of the theorem is a density function for the vector field F. The condition $\alpha > 2$ ensures the integrability of $\rho(x,y,z)$ outside the ball centered at the origin of radius one.

It remains to prove that $\nabla\cdot(\rho F)(x,y,z)$ is positive almost everywhere in $\mathbb{R}^3$. We have

$$\triangledown\rho(x,y,z)=\frac{-\alpha}{(x^2+y^2+R(z))^{\alpha+1}}\,(2x,2y,R'(z))\,,$$

and

$$[\triangledown\cdot F](x,y,z)=(a+c-1)R(z)-zR'(z)\,.$$

Then

$$\begin{aligned}
[\nabla\cdot\rho F](x,y,z) &= (\triangledown\rho\cdot F)(x,y,z)+\rho(x,y,z)\,[\triangledown\cdot F](x,y,z)\\
&= \frac{-\alpha\,R(z)}{(x^2+y^2+R(z))^{\alpha+1}}\,[2(ax^2+cy^2)-z\,R'(z)]\\
&\quad+\frac{1}{(x^2+y^2+R(z))^{\alpha}}\,[(a+c-1)\,R(z)-z\,R'(z)]\\
&= \frac{1}{(x^2+y^2+R(z))^{\alpha+1}}\,\big[-2\alpha(ax^2+cy^2)R(z)+\alpha zR(z)R'(z)\\
&\quad+(x^2+y^2+R(z))\,[(a+c-1)\,R(z)-z\,R'(z)]\big]\\
&= \frac{1}{(x^2+y^2+R(z))^{\alpha+1}}\,[((a+c-1)R(z)+(\alpha-1)zR'(z))R(z)\\
&\quad+((a+c-1-2\alpha a)R(z)-zR'(z))x^2\\
&\quad+((a+c-1-2\alpha c)R(z)-zR'(z))y^2\big]\,.
\end{aligned}$$

Since $R(z)=\sum_{i=1}^{k}a_{2i}z^{2i}$ and $zR'(z)=\sum_{i=1}^{k}2ia_{2i}z^{2i}$ with $a_{2i}>0$ for $i=1,\ldots,k$, we obtain $[\triangledown\cdot F](x,y,z)>0$ for $z\neq 0$ and $[\triangledown\cdot F](x,y,z)=0$ for $z=0$, if

$$a+c-1+2(\alpha-1)i>0\,,\quad a+c-1-2\alpha a-2i>0\,,\quad a+c-1-2\alpha c-2i>0\,,$$

for $i=1,\ldots,k$. Then, the proof is finished due to these inequalities are consequence of our hypothesis $\alpha>\max\{2,\frac{a+c-1-2k}{2a},\frac{3-a-c}{2}\}$. □

This family is a generalization of the following example of R. Potrie and P. Monzón ([22])

$$F(x,y,z)=(y-2xz^2,-x-2yz^2,-z^3)\,, \tag{11}$$

which has density function $\rho(x,y,z)=(x^2+y^2+z^2)^{-4}$.

Corollary 3.5. *The vector field F given by (10) under the conditions of Theorem 3.4 has the origin as an almost global attractor which is not locally asymptotic stable.*

Proof. We have $F(x,y,0)=(y,-x,0)$, then the origin is not locally asymptotic stable. On the other hand, to prove that the origin is almost global attractor we use Rantzer's result (Theorem 3.3). Then it is sufficient to show that the condition $\alpha>2$ ensures the integrability of $\rho(x,y,z)F(x,y,z)/\|(x,y,z)\|$ outside the ball centered at the origin of radius one. In fact, if we consider $k_0=\max\{1,a_1\}$, $k_1=\max\{1,b^2+c^2+|\,b|\,(a-c)\}$ and $k_2=2\,|\,b|+a-c$, we have that

$$\begin{aligned}
\| F(x,y,z)\|^2 &= x^2+y^2+R(z)^2\ \left[(a^2+b^2)x^2+(b^2+c^2)y^2+z^2+2b(a-c)xy\right]\\
&\quad +2R(z)\ \left[(a-c)xy+b(x^2+y^2)\right]\\
&\le x^2+y^2+R(z)^2\ \left[(b^2+c^2)(x^2+y^2)+z^2+|\,b|\,(a-c)(x^2+y^2)\right]\\
&\quad +R(z)(a-c+2\,|\,b|)(x^2+y^2)\\
&\le (x^2+y^2+z^2)+k_1(x^2+y^2+R(z))^2(x^2+y^2+z^2)\\
&\quad +k_2(x^2+y^2+R(z))(x^2+y^2+z^2).
\end{aligned}$$

and that $x^2+y^2+R(z)\ge k_0$. Thus, this facts combined with the assumption over α imply that

$$\begin{aligned}
\frac{\| F(x,y,z)\|^2\,\rho(x,y,z)^2}{\|\,(x,y,z)\|^2} &\le \frac{1}{(x^2+y^2+R(z))^{2\alpha}}+\frac{k_1}{(x^2+y^2+R(z))^{2\alpha-2}}\\
&\quad +\frac{k_2}{(x^2+y^2+R(z))^{2\alpha-1}}.
\end{aligned}$$

Therefore $\rho(x,y,z)F(x,y,z)/\,\|\,(x,y,z)\|$ is integrable outside the ball centered at the origin of radius one. □

Remark 3.6. *An alternative –and very simple– proof of Corollary 3.5 is obtained by considering the fact that* $\langle F(x,y,z),(x,y,z)\rangle=(ax^2+by^2-z^2)R(z)<0$ *for all* $z\neq 0$ *combined with the invariance of the plane* $z=0$.

Motivated by the last remark, we have the following result.

Proposition 3.7. *Let* $F:\mathbb{R}^3\to\mathbb{R}^3$ *be a* C^1-*vector field with* $F(0)=0$ *such that the plane* $z=0$ *is invariant and* $[\nabla\cdot F](x,y,z)<0$ *for all* $z\neq 0$. *Suppose that there exists a positive* C^1-*function* $\rho:\mathbb{R}^3-\{0\}\to\mathbb{R}$ *satisfying:*

a) $[\nabla\cdot\rho F](x,y,z)>0$ *for all* $z\neq 0$,
b) $\lim_{p\to 0}\ \rho(p)=\infty$.
c) $\lim_{\|p\|\to\infty}\ \rho(p)=0$.

Then, for any initial state $\alpha(0)=(x(0),y(0),z(0))$ *verifying* $z(0)\neq 0$, *the trajectory* $\alpha(t)$ *exists for all* $t\in[0,\infty[$ *and tends to zero as* $t\to\infty$.

Proof. Condition a) implies $[\nabla\cdot\rho F](x,y,z)\ge 0$ everywhere, and $(\nabla\rho\cdot F)(x,y,z)>-\rho(x,y,z)\ [\nabla\cdot F](x,y,z)>0$ for all (x,y,z) with $z\neq 0$.
Let us consider the function $V:\mathbb{R}^3\to\mathbb{R}$ defined by $V=\rho^{-1}$ outside the origin and $V(0)=0$. Then V is continuous by b), it is C^1 outside the origin and

1) $V(p)\ge 0$ for all p and $V(p)=0$ if and only if $p=0$.
2) $(\nabla V\cdot F)(x,y,z)=-\rho(x,y,z)^{-2}\,(\nabla\rho\cdot F)(x,y,z)<0$, for $z\neq 0$.
3) $\lim_{\|p\|\to\infty}\ V(p)=\infty$.

Then, if $\alpha(0)=(x(0),y(0),z(0))$ is a initial state with $z(0)\neq 0$, we have $\frac{d}{dt}V(\alpha(t))<0$ by 2). Then the trajectory $\alpha(t)$ remains over the sublevel $V(\alpha(0))$, which is bounded by 3). Therefore, $\alpha(t)$ exists for all $t\in[0,\infty[$ and tend to 0 as $t\to\infty$. □

Corollary 3.8. *If in Proposition 3.7 we have* $[\nabla\cdot\rho F](x,y,z)>0$ *for all* $z\in\mathbb{R}$, *then the origin is globally asymptotic stable.*

If we perturb the vector field F defined by (10) with λI and $\lambda<0$, then $G=\lambda I+F$ is a Hurwitz vector field, which is another example of **MYC**, by the following

Proposition 3.9. *Let $G = \lambda I + F$ with $\lambda < 0$ and F as in (10) that verify the conditions of Theorem 3.4. Then G is Hurwitz, admits the same density function $\rho(x,y,z) = (x^2+y^2+R(z))^\alpha$ but with $\alpha > 3$, and the origin is globally asymptotic stable.*

Proof. G is Hurwitz due to the fact that F is almost Hurwitz and $JG = \lambda I + JF$. Since

$$[\nabla \cdot \rho G](x,y,z) = [\nabla \cdot \rho F](x,y,z) + \lambda \frac{(3-2\alpha)(x^2+y^2)+3R(z)-\alpha z R'(z)}{(x^2+y^2+R(z))^{\alpha+1}} ,$$

under the additional condition $\alpha > 3$, ρ is a density function for G. Finally, the global asymptotic stability of the origin follows from Corollary 3.8. □

The previous results imply that if we are interested in finding vector fields F with the plane $z=0$ invariant supporting density functions (but without obvious Lyapunov functions), these fields must verify $\nabla \cdot F > 0$ over some open set. The following family of vector fields satisfy these conditions.

Proposition 3.10. *Consider the vector field $F = (P,Q,R)$ defined by*

$$\begin{aligned} P(x,y,z) &= -x + A_2\,y + a_1\,x^3 + 3\,a_2\,x^2\,y + 3\,a_3\,x\,y^2 + a_4\,y^3 + 3\,a_5\,x^2\,z + \\ &\quad 6\,a_6\,x\,y\,z + 3\,a_7\,y^2\,z + 3\,a_8\,x\,z^2 + 3\,a_9\,y\,z^2 + a_{10}\,z^3 , \\ Q(x,y,z) &= -A_2\,x - y + b_1\,x^3 + 3\,b_2\,x^2\,y + 3\,b_3\,x\,y^2 + b_4\,y^3 + 3\,b_5\,x^2\,z + \\ &\quad 6\,b_6\,x\,y\,z + 3\,b_7\,y^2\,z + 3\,b_8\,x\,z^2 + 3\,b_9\,y\,z^2 + b_{10}\,z^3 , \\ R(x,y,z) &= z\,(-1 + 3\,c_5\,x^2 + 6\,c_6\,x\,y + 3\,c_7\,y^2 + 3\,c_8\,x\,z + 3\,c_9\,y\,z + c_{10}\,z^2) , \end{aligned}$$

with

$$\begin{aligned} a_8 &= \frac{2}{3} - a_3 ,\ a_{10} = \frac{3}{2}(a_5 - a_7) , \\ b_1 &= -3\,a_2 + a_4 + 3\,b_3 ,\ b_2 = 1 - a_3 - \frac{c_{10}}{3} ,\ b_4 = 3 - a_1 - c_{10} , \\ b_6 &= \frac{1}{2}(a_5 - a_7) ,\ b_7 = 2\,a_6 + b_5 ,\ b_8 = 2\,a_2 - a_4 - a_9 - b_3 , \\ b_9 &= -\frac{1}{3} + a_3 + \frac{c_{10}}{3} ,\ b_{10} = 3\,a_6 , \\ c_5 &= -\frac{2}{3} + \frac{a_1}{3} + a_3 + \frac{c_{10}}{3} ,\ c_6 = \frac{2a_4}{3} - a_2 + b_3 ,\ c_7 = \frac{4}{3} - \frac{a_1}{3} - a_3 - \frac{c_{10}}{3} , \\ c_8 &= \frac{1}{2}(a_5 + a_7) ,\ c_9 = a_6 + b_5 . \end{aligned}$$

Then F has the density function

$$\rho(x,y,z) = \frac{1}{(x^2+y^2+z^2)^2} .$$

Moreover the plane $z=0$ is invariant and the coefficients of F can be chosen such that $\nabla \cdot F$ is positive over some open set.

Proof. Clearly the plane $z=0$ is invariant. After straightforward computations we obtain

$$[\nabla \cdot \rho F](x,y,z) = \frac{1+x^2+y^2+z^2}{(x^2+y^2+z^2)^2} ,$$

and that the integrability condition of ρ holds. Moreover

$$\begin{aligned} \nabla \cdot F\,(x,y,z) \quad &= \quad -3+(1+4a_1)x^2+(13-4a_1-4c_{10})y^2+(24a_6+12b_5)yz \\ &\quad +(1+4c_{10})z^2+x((4a_4+12b_3)y+12a_5 z)\,, \end{aligned}$$

assume positive values. For example, if $y=0$, we have $\nabla \cdot F\,(x,0,z) = -3+(1+4a_1)x^2+(1+4c_{10})z^2+12a_5xz$ which is positive for x,z sufficient large and $a_1>0$ and $a_5<0$. □

Remark 3.11. *The coefficients of the vector field $F=(P,Q,R)$ of Proposition 3.10 were obtained from the equation*

$$[\nabla \cdot \rho F](x,y,z) \;=\; \frac{1+x^2+y^2+z^2}{(x^2+y^2+z^2)^2}\,,$$

which is equivalent to

$$-4\,(xP+yQ+zR)\;+\;(x^2+y^2+z^2)\,[\nabla\cdot F]\;=\;(1+x^2+y^2+z^2)\,(x^2+y^2+z^2)\,.$$

4. Acknowledgements

The authors acknowledge to the anonymous referee for her/his careful and deep reading this manuscript, contributing to improve it.

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