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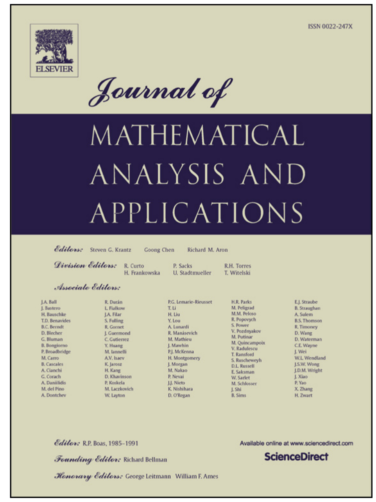

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q-DIFFERENCE EQUATIONS FOR ASKEY-WILSON TYPE INTEGRALS VIA q-POLYNOMIALS

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Abstract. In this paper, we show how to deduce Askey-Wilson type integrals by the method of q-difference equation. In addition, we construct the relation between bilinear generating functions and solutions of q-difference equation. More over, we generalize Bailey's ${}_6\psi_6$ summation from the perspective of q-integral by the method of q-difference equation. At last, we deduce $U(n+1)$ type generating functions for Al-Salam-Carlitz polynomials by the method of q-difference equation.

1. Introduction

In this paper, we follow the notations and terminology in [21] and suppose that $0 < q < 1$. The q-series and its compact factorials are defined respectively by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{1.1}$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$, where $m \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The q-difference operators D_a and θ_a are defined by [14]

$$D_a\{f(a)\} = \frac{f(a) - f(aq)}{a} \quad \text{and} \quad \theta_a\{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \tag{1.2}$$

The Rogers-Szegö polynomials [15]

$$h_n(b,c|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} b^k c^{n-k} \quad \text{and} \quad g_n(b,c|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} b^k c^{n-k} \tag{1.3}$$

play important roles in the theory of orthogonal polynomials. The Al-Salam-Carlitz polynomials [14, Eq. (4.4)]

$$\Phi_n^{(a)}(b,c|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k b^k c^{n-k} \quad \text{and} \quad \Psi_n^{(a)}(b,c|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k+1}{2}-nk}\Big(\frac{1}{a};q\Big)_k (ab)^k c^{n-k}. \tag{1.4}$$

are generalizations of Rogers-Szegö polynomials. For more information about q-polynomials, please refer to [8, 24, 26, 27].

Liu [29] and [30] obtained several important results by the following q-difference equations. In [31], Liu and Zeng studied relations between q-difference equations and q-orthogonal polynomials.

Proposition 1 ([31, Eq. (2.2) and (2.3)]). *Let $f(a,b)$ be a two-variable analytic function at $(0,0) \in \mathbb{C}^2$. Then*

(A) *f can be expanded in terms of $h_n(a,b|q)$ if and only if f satisfies the functional equation*

$$bf(aq,b) - af(a,bq) = (b-a)f(a,b). \tag{1.5}$$

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(B) *f can be expanded in terms of $g_n(a, b|q)$ if and only if f satisfies the functional equation*

$$af(aq, b) - bf(a, bq) = (a - b)f(aq, bq). \tag{1.6}$$

The method of q-difference equation shows itself an effective way to deduce many important results in q-series. For more information, please refer to [14, 29, 30].

In this paper, we generalize Liu-Zeng's results (1.5) and (1.6) as follows.

Theorem 2. *Let $f(a, b, c)$ be a three-variable analytic function at $(0, 0, 0) \in \mathbb{C}^3$. Then*

(C) *f can be expanded in terms of $\Phi_n^{(a)}(b, c|q)$ if and only if f satisfies the functional equation*

$$abf(a, bq, cq) - bf(a, b, cq) + (c - ab)f(a, bq, c) = (c - b)f(a, b, c). \tag{1.7}$$

(D) *f can be expanded in terms of $\Psi_n^{(a)}(b, c|q)$ if and only if f satisfies the functional equation*

$$abf(a, b, cq^{-1}) - bf(a, qb, cq^{-1}) + \left(b - cq^{-1}\right)f(a, qb, c) = \left(ab - cq^{-1}\right)f(a, b, c). \tag{1.8}$$

Remark 3. *For $a = 0$ in Theorem 2, equations (1.7) and (1.8) reduce to (1.5) and (1.6) respectively.*

Proof of Theorem 2. From the theory of several complex variables [34], we assume that

$$f(a, b, c) = \sum_{k=0}^{\infty} A_k(a, c)b^k. \tag{1.9}$$

On one hand, substituting equation (1.9) into (1.7) yields

$$ab\sum_{k=0}^{\infty} A_k(a, cq)(bq)^k - b\sum_{k=0}^{\infty} A_k(a, cq)b^k + (c - ab)\sum_{k=0}^{\infty} A_k(a, c)(bq)^k = (c - b)\sum_{k=0}^{\infty} A_k(a, c)b^k. \tag{1.10}$$

Equating coefficients of b^k on both sides of equation (1.10) gives

$$A_k(a, c) = \frac{1 - aq^{k-1}}{1 - q^k} D_c\{A_{k-1}(a, c)\}. \tag{1.11}$$

Repeating the process, we obtain

$$A_k(a, c) = \frac{(a; q)_k}{(q; q)_k} D_c^k\{A_0(a, c)\}. \tag{1.12}$$

Letting $f(a, 0, c) = A_0(a, c) = \sum_{n=0}^{\infty} \mu_n \cdot c^n$, we have

$$A_k(a, c) = \frac{(a; q)_k}{(q; q)_k} \sum_{n=0}^{\infty} \mu_n \frac{(q; q)_n}{(q; q)_{n-k}} c^{n-k}. \tag{1.13}$$

Using equation (1.9), we have

$$\begin{aligned} f(a, b, c) &= \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} \sum_{n=k}^{\infty} \mu_n \frac{(q; q)_n}{(q; q)_{n-k}} c^{n-k} b^k \\ &= \sum_{n=0}^{\infty} \mu_n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k b^k c^{n-k} \\ &= \sum_{n=0}^{\infty} \mu_n \cdot \Phi_n^{(a)}(b, c|q). \end{aligned}$$

On the other hand, if $f(a, b, c)$ can be expanded in terms of $\Phi_n^{(a)}(b, c|q)$, we can verify that $f(a, b, c)$ satisfies equation (1.7). Similarly, we assume that

$$f(a, b, c) = \sum_{k=0}^{\infty} B_k(a, c)b^k. \tag{1.14}$$

Substituting equation (1.14) into (1.8) and comparing the coefficients of b^n, we have

$$B_k(a,c) = \frac{(1-q^{k-1}/a)(-ab)}{1-q^k}\theta_c\{B_{k-1}(a,c)\}. \tag{1.15}$$

Repeating the process and letting $f(a,0,c) = B_0(a,c) = \sum_{n=0}^{\infty} \eta_n \cdot c^n$, we have

$$B_k(a,c) = \frac{(1/a;q)_k(-ab)^k}{(q;q)_k}\theta_c^k\{B_0(a,c)\} = \frac{(1/a;q)_k(-ab)^k}{(q;q)_k}\sum_{n=0}^{\infty}\eta_n \cdot q^{\binom{k+1}{2}-kn}\frac{(q;q)_n}{(q;q)_{n-k}}c^{n-k}. \tag{1.16}$$

Using equation (1.14), we get

$$f(a,b,c) = \sum_{n=0}^{\infty}\eta_n \cdot \Psi_n^{(a)}(b,c|q). \tag{1.17}$$

Conversely, if $f(a,b,c)$ can be expanded in terms of $\Psi_n^{(a)}(b,c|q)$, we can verify that $f(a,b,c)$ satisfies equation (1.8). The proof is complete. □

The rest part of this paper is organized as follows. In Section 2, we give a new proof of generating functions for Al-Salam-Carlitz polynomials by the method of q-difference equation. In Section 3, we deduce Ismail-Stanton-Viennot type Askey-Wilson integral by the method of q-difference equation. In Section 4, we obtain reversal type Askey-Wilson integral by the method of q-difference equation. In Section 5, we gain Ramanujan type Askey-Wilson integral by the method of q-difference equation. In Section 6, we achieve the generalization of Bailey's ${}_6\psi_6$ summation by the method of q-difference equation. In Section 7, we acquire $U(n+1)$ type generating functions for Al-Salam-Carlitz polynomials by the method of q-difference equation.

2. Generating functions for Al-Salam-Carlitz polynomials

The following generating functions for Rogers-Szegö and Al-Salam-Carlitz polynomials are given respectively.

Lemma 4 ([11, Eq. (29)]). *We have*

$$\sum_{n=0}^{\infty} h_n(b,c|q)\frac{t^n}{(q;q)_n} = \frac{1}{(bt,ct;q)_\infty}, \quad \max\{|bt|,|ct|\} < 1, \tag{2.1}$$

$$\sum_{n=0}^{\infty} g_n(b,c|q)\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = (bt,ct;q)_\infty. \tag{2.2}$$

Lemma 5 ([9, Eq. (1.14) and (1.15)]). *We have*

$$\sum_{n=0}^{\infty} \Phi_n^{(a)}(b,c|q)\frac{t^n}{(q;q)_n} = \frac{(abt;q)_\infty}{(bt,ct;q)_\infty}, \quad \max\{|bt|,|ct|\} < 1, \tag{2.3}$$

$$\sum_{n=0}^{\infty} \Psi_n^{(a)}(b,c|q)\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = \frac{(bt,ct;q)_\infty}{(abt;q)_\infty}, \quad |abt| < 1. \tag{2.4}$$

Lemma 6 ([10, Eq. (2.10) and (2.11)]). *For* $\max\{|btz|,|ctz|,|cty|,|bty|\} < 1$*, we have*

$$\sum_{n=0}^{\infty} \Phi_n^{(a)}(b,c|q)\Phi_n^{(x)}(y,z|q)\frac{t^n}{(q;q)_n} = \frac{(abtz,ctxy;q)_\infty}{(btz,ctz,cty;q)_\infty}{}_3\phi_2\left[\begin{matrix} a, ctz, x \\ abtz, ctxy \end{matrix}; q, bty\right]. \tag{2.5}$$

For $F \in \mathbb{N}_0$*, if* $a = q^F$ *(or* $b = q^F$ *) and* $\max\left\{\left|ctxyq^{1-F}\right|, \left|abtzq^{1-F}\right|, \left|bcyzt^2q\right|\right\} < 1$*, we have*

$$\sum_{n=0}^{\infty} \Psi_n^{(a)}(b,c|q)\Psi_n^{(x)}(y,z|q)\frac{(-1)^n q^{\binom{n+1}{2}} t^n}{(q;q)_n} = \frac{(btzq,ctyq,ctzq;q)_\infty}{(ctxyq,abtzq;q)_\infty}{}_3\phi_2\left[\begin{matrix} 1/a, 1/x, 1/(ctz) \\ 1/(abtz), 1/(ctxy) \end{matrix}; q, q\right]. \tag{2.6}$$

There are many clever methods to deduce generating functions for Al-Salam-Carlitz polynomials (2.3)-(2.6). For example, Al-Salam and Carlitz [1] obtained bilinear generating functions (or Poisson kernel) for Al-Salam-Carlitz polynomials (2.5) and (2.6) with the help of certain identities. Verma and Jain [39] obtained Carlitz type bilinear generating function for Al-Salam-Carlitz polynomials by the q-Chu-Vandermonde formula. Srivastava and Jain [36] derived multilinear generating functions involving Al-Salam-Carlitz and Rogers-Szegö polynomials by the transformation theory. For more information, please refer to [1, 9, 36, 39].

In this section, we deduce generating functions for Al-Salam-Carlitz polynomials by the method of q-difference equation.

Proof of lemma 5. Denoting the right-hand side (RHS) of the formula (2.3) by $f(a,b,c)$, we check that $f(a,b,c)$ satisfies equation (1.7), so we have

$$f(a,b,c)=\sum_{n=0}^{\infty}\mu_n\cdot\Phi_n^{(a)}(b,c|q). \tag{2.7}$$

Using equation (2.1), we have

$$f(0,b,c)=\sum_{n=0}^{\infty}\mu_n\cdot h_n(b,c|q)=\frac{1}{(bt,ct;q)_\infty}=\sum_{n=0}^{\infty}h_n(b,c|q)\frac{t^n}{(q;q)_n}, \tag{2.8}$$

which yields the left-hand side (LHS) of the formula (2.3). Similarly, we obtain the formula (2.4). The proof of lemma 5 is complete. □

Proof of lemma 6. We denote the RHS of equation (2.5) by $f(a,b,c)$, and check that $f(a,b,c)$ satisfies equation (1.7), so we have

$$f(a,b,c)=\sum_{n=0}^{\infty}\mu_n\cdot\Phi_n^{(a)}(b,c|q). \tag{2.9}$$

Taking $b=0$ in equation (2.9), then using (2.3) gives

$$f(a,0,c)=\frac{(ctxy;q)_\infty}{(ctz,cty;q)_\infty}=\sum_{n=0}^{\infty}\mu_n\cdot c^n=\sum_{n=0}^{\infty}\Phi_n^{(x)}(y,z|q)\frac{(ct)^n}{(q;q)_n}, \tag{2.10}$$

that is, $f(a,b,c)$ equals the LHS of (2.5). Similarly, we can deduce equation (2.6). The proof is complete. □

3. Ismail-Stanton-Viennot type Askey-Wilson integral

For $x=\cos\theta$, we define the notation $h(x;a)$ and $h(x;a_1,a_2,\ldots,a_m)$ as

$$h(\cos\theta;a)=(ae^{i\theta},ae^{-i\theta};q)_\infty,\qquad h(\cos\theta;a_1,a_2,\ldots,a_m)=h(\cos\theta;a_1)h(\cos\theta;a_2)\cdots h(\cos\theta;a_m). \tag{3.1}$$

The following famous integral due to Askey and Wilson [3, Theorem 2.1].

Proposition 7 (Askey-Wilson integral)**.** *If* $\max\{|a|,|b|,|c|,|d|\}<1$*, we have*

$$\int_0^\pi\frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}\,\mathrm{d}\theta=\frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}. \tag{3.2}$$

The Ismail-Stanton-Viennot integral [22, Theorem 3.5] is a generalization of Askey-Wilson integral.

Proposition 8 (Ismail-Stanton-Viennot integral)**.** *If* $\max\{|a|,|b|,|c|,|d|,|f|\}<1$*, we have*

$$\int_0^\pi\frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d,f)}\,\mathrm{d}\theta=\frac{2\pi(abcf,bcdf,ad;q)_\infty}{(q,ab,ac,ad,af,bc,bd,bf,cd,cf,df;q)_\infty}{}_3\phi_2\left[\begin{matrix}bc,bf,cf\\ abcf,bcdf\end{matrix};q,ad\right]. \tag{3.3}$$

Chen and Gu [16] deduced Ismail-Stanton-Viennot type Askey-Wilson integral by the method of Cauchy operator.

Proposition 9 ([16, Eq. (3.3)]). *We have*

$$\int_0^\pi \frac{h(\cos 2\theta;1)\,\mathrm{d}\theta}{h(\cos\theta;a,b,c,d)}\frac{(fge^{i\theta};q)_\infty}{(ge^{i\theta};q)_\infty}{}_3\phi_2\left[\begin{matrix} f, ae^{i\theta}, be^{i\theta} \\ fge^{i\theta}, ab\end{matrix};q, ge^{-i\theta}\right]\mathrm{d}\theta$$
$$= \frac{2\pi(cfg, abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd,cg;q)_\infty}{}_3\phi_2\left[\begin{matrix} f, ac, bc \\ cfg, abcd\end{matrix};q,dg\right], \tag{3.4}$$

where $\max\{|a|,|b|,|c|,|d|,|g|\}<1$.

Those integrals play central roles in the theory of orthogonal polynomials. Different proofs can be found in [4, 7, 16, 18, 22, 23, 28, 29, 33].

In this section, we obtain the following Ismail-Stanton-Viennot type integral by the method of q-difference equation.

Theorem 10. *We have*

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}\frac{(uve^{i\theta}, rse^{i\theta};q)_\infty}{(ve^{i\theta}, se^{i\theta};q)_\infty}\sum_{j=0}^\infty \frac{(u, be^{i\theta}, se^{i\theta}, ae^{i\theta};q)_j (ve^{-i\theta})^j}{(q, uve^{i\theta}, rse^{i\theta}, ab;q)_j}{}_3\phi_2\left[\begin{matrix} r, ae^{i\theta}q^j, be^{i\theta}q^j \\ rse^{i\theta}q^j, abq^j\end{matrix};q, se^{-i\theta}\right]\mathrm{d}\theta$$
$$= \frac{2\pi(cuv, crs, abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd,cv,cs;q)_\infty}\sum_{k=0}^\infty \frac{(u,bc,cs,ac;q)_k(dv)^k}{(q,cuv,crs,abcd;q)_k}{}_3\phi_2\left[\begin{matrix} r, acq^k, bcq^k \\ crsq^k, abcdq^k\end{matrix};q,sd\right], \tag{3.5}$$

where $\max\{|a|,|b|,|c|,|d|,|s|,|v|\}<1$.

Remark 11. *For* $s=v=0$ *in Theorem 10, equation* (3.5) *reduces to* (3.2). *For* $s=0$ *in Theorem 10, equation* (3.5) *reduces to* (3.4).

Before we prove the main results, the following lemma is necessary.

Lemma 12. *For* $n\in\mathbb{N}_0$, *we have*

$$ab\Phi_n^{(a)}(bq,cq|q) - b\Phi_n^{(a)}(b,cq|q) + (c-ab)\Phi_n^{(a)}(bq,c|q) = (c-b)\Phi_n^{(a)}(b,c|q). \tag{3.6}$$

Proof of lemma 12. we can rewrite equation (3.6) by

$$ab\Phi_n^{(a)}(bq,cq|q) - b\Phi_n^{(a)}(b,cq|q) - ab\Phi_n^{(a)}(bq,c|q) + b\Phi_n^{(a)}(b,c|q) = c\Phi_n^{(a)}(b,c|q) - c\Phi_n^{(a)}(bq,c|q). \tag{3.7}$$

On one hand, we have

$$\begin{aligned} LHS\,of(3.7) &= \sum_{k=0}^n \begin{bmatrix} n \\ k\end{bmatrix}(a;q)_k b^k c^{n-k}\left[abq^n - bq^{n-k} - abq^k + b\right] \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k\end{bmatrix}(a;q)_k b^k c^{n-k} b\left(1-aq^k\right)\left(1-q^{n-k}\right) \\ &= (1-q^n)\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k\end{bmatrix}(a;q)_{k+1}b^{k+1}c^{n-k} \\ &= (1-q^n)\sum_{k=1}^{n}\begin{bmatrix} n-1 \\ k-1\end{bmatrix}(a;q)_k b^k c^{n-k+1}. \end{aligned} \tag{3.8}$$

On the other hand

$$\begin{aligned} RHS\,of(3.7) &= \sum_{k=0}^n \begin{bmatrix} n \\ k\end{bmatrix}(a;q)_k b^k c^{n-k}\left[c - cq^k\right] \\ &= (1-q^n)\sum_{k=1}^{n}\begin{bmatrix} n-1 \\ k-1\end{bmatrix}(a;q)_k b^k c^{n-k+1}. \end{aligned} \tag{3.9}$$

Comparing equations (3.8) with (3.9) yields (3.7). The proof of lemma 12 is complete. □

Now we begin to prove Theorem 10.

Proof of Theorem 10. First, we denote $f(u,v,a)$ by

$$f(u,v,a) = \frac{2\pi}{(q,bc,bd,cd;q)_\infty}\frac{(cuv,abcd;q)_\infty}{(cv,ac,ad;q)_\infty}{}_3\phi_2\left[\begin{matrix} u,ac,bc \\ cuv,abcd \end{matrix};q,dv\right] \quad by\,(2.5)$$
$$= \frac{2\pi}{(q,bc,bd,cd;q)_\infty}\sum_{n=0}^{\infty}\Phi_n^{(u)}(v,a|q)\Phi_n^{(bc)}(d,c|q)\frac{1}{(q;q)_n}. \tag{3.10}$$

Using lemmas 6 and 12, we check that $f(u,v,a)$ satisfies equation (1.7), so we get

$$f(u,v,a) = \sum_{n=0}^{\infty}\mu_n\cdot\Phi_n^{(u)}(v,a|q) \tag{3.11}$$

and

$$f(u,0,a) = \frac{2\pi(abcd;q)_\infty}{(q,ac,ad,bc,bd,cd;q)_\infty} = \int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)}\frac{(ab;q)_\infty}{(ae^{i\theta},ae^{-i\theta};q)_\infty}\,\mathrm{d}\theta \quad by\,(3.2)$$
$$= \int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)}\left\{\sum_{n=0}^{\infty}\Phi_n^{(be^{i\theta})}(e^{-i\theta},e^{i\theta}|q)\frac{a^n}{(q;q)_n}\right\}\mathrm{d}\theta.$$

We have

$$f(u,v,a) = \int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)}\left\{\sum_{n=0}^{\infty}\Phi_n^{(u)}(v,a|q)\Phi_n^{(be^{i\theta})}(e^{-i\theta},e^{i\theta}|q)\frac{1}{(q;q)_n}\right\}\mathrm{d}\theta \quad by\,(2.5)$$
$$= \int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}\frac{(uve^{i\theta},ab;q)_\infty}{(ve^{i\theta};q)_\infty}{}_3\phi_2\left[\begin{matrix} u,ae^{i\theta},be^{i\theta} \\ uve^{i\theta},ab \end{matrix};q,ve^{-i\theta}\right]\mathrm{d}\theta.$$

Repeat this process, we rewrite equation (3.5) equivalently by

$$\int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}\frac{(ab,uve^{i\theta},rse^{i\theta};q)_\infty}{(ve^{i\theta},se^{i\theta};q)_\infty}\sum_{j=0}^{\infty}\frac{(u,be^{i\theta},se^{i\theta},ae^{i\theta};q)_j(ve^{-i\theta})^j}{(q,uve^{i\theta},rse^{i\theta},ab;q)_j}{}_3\phi_2\left[\begin{matrix} r,ae^{i\theta}q^j,be^{i\theta}q^j \\ rse^{i\theta}q^j,abq^j \end{matrix};q,se^{-i\theta}\right]\mathrm{d}\theta$$
$$= \frac{2\pi(cuv;q)_\infty}{(q,bc,bd,cd,cv;q)_\infty}\sum_{k=0}^{\infty}\frac{(u,bc;q)_k(dv)^k}{(q,cuv;q)_k}\frac{(crsq^k,abcdq^k;q)_\infty}{(csq^k,acq^k,ad;q)_\infty}{}_3\phi_2\left[\begin{matrix} r,acq^k,bcq^k \\ crsq^k,abcdq^k \end{matrix};q,sd\right]. \tag{3.12}$$

We denote the RHS of equation (3.12) by $F(r,s,a)$ and verify that $F(r,s,a)$ satisfies equation (1.7), so we have

$$F(r,s,a) = \sum_{n=0}^{\infty}\mu_n\cdot\Phi_n^{(r)}(s,a|q) \tag{3.13}$$

and

$$F(r,0,a) = \frac{2\pi}{(q,bc,bd,cd;q)_\infty}\frac{(cuv,abcd;q)_\infty}{(cv,ac,ad;q)_\infty}{}_3\phi_2\left[\begin{matrix} u,ac,bc \\ cuv,abcd \end{matrix};q,dv\right]$$
$$= \int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;a,b,c,d)}\frac{(uve^{i\theta},ab;q)_\infty}{(ve^{i\theta};q)_\infty}{}_3\phi_2\left[\begin{matrix} u,ae^{i\theta},be^{i\theta} \\ uve^{i\theta},ab \end{matrix};q,ve^{-i\theta}\right]\mathrm{d}\theta$$
$$= \int_0^\pi \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)}\frac{(uve^{i\theta};q)_\infty}{(ve^{i\theta};q)_\infty}\sum_{j=0}^{\infty}\frac{(u,be^{i\theta};q)_j(ve^{-i\theta})^j}{(q,uve^{i\theta};q)_j}\frac{(abq^j;q)_\infty}{(ae^{i\theta}q^j,ae^{-i\theta};q)_\infty}\,\mathrm{d}\theta$$

$$= \int_0^{\pi} \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)} \frac{(uve^{i\theta};q)_\infty}{(ve^{i\theta};q)_\infty} \sum_{j=0}^{\infty} \frac{(u,be^{i\theta};q)_j (ve^{-i\theta})^j}{(q,uve^{i\theta};q)_j} \left\{ \sum_{n=0}^{\infty} \Phi_n^{(be^{-i\theta})}(q^j e^{i\theta}, e^{-i\theta}|q) \frac{a^n}{(q;q)_n} \right\} \mathrm{d}\theta.$$

Thus

$$F(r,s,a) = \int_0^{\pi} \frac{h(\cos 2\theta;1)}{h(\cos\theta;b,c,d)} \frac{(uve^{i\theta};q)_\infty}{(ve^{i\theta};q)_\infty} \sum_{j=0}^{\infty} \frac{(u,be^{i\theta};q)_j (ve^{-i\theta})^j}{(q,uve^{i\theta};q)_j} \left\{ \sum_{n=0}^{\infty} \Phi_n^{(r)}(s,a|q) \Phi_n^{(be^{-i\theta})}(q^j e^{i\theta}, e^{-i\theta}|q) \frac{1}{(q;q)_n} \right\} \mathrm{d}\theta,$$

which equals the LHS of equation (3.12) by equation (2.5). The proof is complete. □

4. Reversal type Askey-Wilson integral

By iterating functional equations, Askey [4, Eq. (3.11)] proved another remarkable integral formula as follows, which may be considered as reversal Askey-Wilson integral.

Proposition 13 (Reversal Askey-Wilson integral)**.** *For $|qabcd| < 1$, there holds*

$$\int_{-\infty}^{\infty} \frac{h(i\sinh x;qa,qb,qc,qd)}{h(\cosh 2x;-q)} \mathrm{d}x = \frac{(q,qab,qac,qad,qbc,qbd,qcd;q)_\infty}{(qabcd;q)_\infty} \log(q^{-1}), \tag{4.1}$$

where

$$h(i\sinh\alpha x;t) = \prod_{k=0}^{\infty} \left(1 - 2iq^k t \sinh\alpha x + q^{2k}t^2\right) = \left(ite^{\alpha x}, -ite^{-\alpha x};q\right)_\infty. \tag{4.2}$$

For more information about reversal type Askey-Wilson integral, please refer to [4, 18, 31].

In this section, we generalize reversal type Askey-Wilson integral by the method of q-difference equation.

Theorem 14. *For $M \in \mathbb{N}_0$ and $f = q^M$, we have*

$$\int_{-\infty}^{\infty} \frac{h(i\sinh x;qa,qb,qc,qd)}{h(\cosh 2x;-q)} \frac{(iqge^x;q)_\infty}{(iqfge^x;q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, e^{-x}/(ib), e^{-x}/(ia) \\ e^{-x}/(ifg), 1/(ab) \end{matrix}; q, q\right] \mathrm{d}x$$
$$= \frac{(q,qab,qac,qad,qbc,qbd,qcd,qcg;q)_\infty \log(q^{-1})}{(qabcd,qcfg;q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, 1/(bc), 1/(ac) \\ 1/(cfg), 1/(abcd) \end{matrix}; q, q\right], \tag{4.3}$$

where $\max\left\{\left|q^{1-M}abcd\right|, \left|q^{1-M}cfg\right|, |acdgq|\right\} < 1$.

Corollary 15. *We have*

$$\int_{-\infty}^{\infty} \frac{h(i\sinh x;qa,qb,qc,qd,qg)}{h(\cosh 2x;-q)} \frac{1}{(iqge^x;q)_\infty} {}_2\phi_1\left[\begin{matrix} -e^x/(ia), -e^x/(ib) \\ 1/(ab) \end{matrix}; q, -iqge^{-x}\right] \mathrm{d}x$$
$$= \frac{(q,qab,qac,qad,qbc,qbd,qcd,qcg;q)_\infty \log(q^{-1})}{(qabcd;q)_\infty} {}_2\phi_1\left[\begin{matrix} 1/(bc), 1/(ac) \\ 1/(abcd) \end{matrix}; q, qcg\right], \tag{4.4}$$

where $\max\{|qabcd|, |qcg|, |acdgq|\} < 1$.

Remark 16. *For $f \to 0$ in Theorem 14, equation* (4.3) *reduces to* (4.4)*. For $f \to 1$ in Theorem 14, equation* (4.3) *reduces to* (4.1).

Proof of Theorem 14. We can rewrite equation (4.3) equivalently as

$$\int_{-\infty}^{\infty} \frac{h(i\sinh x;qb,qc,qd)}{h(\cosh 2x;-q)} \frac{(iqge^x,-iqae^{-x},iqae^x;q)_\infty}{(qab,iqfge^x;q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, e^{-x}/(ib), e^{-x}/(ia) \\ e^{-x}/(ifg), 1/(ab) \end{matrix}; q, q\right] \mathrm{d}x$$
$$= \log(q^{-1})(q,qbc,qbd,qcd;q)_\infty \frac{(qcg,qad,qac;q)_\infty}{(qabcd,qcfg;q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, 1/(bc), 1/(ac) \\ 1/(cfg), 1/(abcd) \end{matrix}; q, q\right]. \tag{4.5}$$

We check that the RHS of equation (4.5) satisfies equation (1.8), so we have

$$F(f,g,a)=\sum_{n=0}^{\infty}\mu_n\cdot\Psi_n^{(f)}(g,a|q) \tag{4.6}$$

and

$$\begin{aligned}F(f,0,a)&=\frac{(q,qac,qad,qbc,qbd,qcd;q)_\infty}{(qabcd;q)_\infty}\log(q^{-1})\quad by\,(4.1)\\&=\int_{-\infty}^{\infty}\frac{h(i\sinh x;qb,qc,qd)}{h(\cosh 2x;-q)}\left\{\frac{(iqae^{x},-iqae^{-x};q)_\infty}{(qab;q)_\infty}\right\}\mathrm{d}x\\&=\int_{-\infty}^{\infty}\frac{h(i\sinh x;qb,qc,qd)}{h(\cosh 2x;-q)}\left\{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\Psi_n^{(ibe^{x})}(-iqe^{-x},iqe^{x}|q)\frac{a^n}{(q;q)_n}\right\}\mathrm{d}x.\end{aligned}$$

Thus

$$F(f,g,a)=\int_{-\infty}^{\infty}\frac{h(i\sinh x;qb,qc,qd)}{h(\cosh 2x;-q)}\left\{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\Psi_n^{(f)}(g,a|q)\Psi_n^{(ibe^{x})}(-iqe^{-x},iqe^{x}|q)\frac{1}{(q;q)_n}\right\}\mathrm{d}x,$$

which is equal to the LHS of equation (4.5) by equation (2.6). The proof is complete. □

Proof of Corollary 15. Taking $f\to 0$ in Theorem 14, equation (4.3) becomes

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{h(i\sinh x;qa,qb,qc,qd)}{h(\cosh 2x;-q)}(iqge^{x};q)_\infty{}_2\phi_1\left[\begin{matrix}e^{-x}/(ib),e^{-x}/(ia)\\1/(ab)\end{matrix};q,iqge^{x}\right]\mathrm{d}x\\&\qquad=\frac{(q,qab,qac,qad,qbc,qbd,qcd,qcg;q)_\infty\log(q^{-1})}{(qabcd;q)_\infty}{}_2\phi_1\left[\begin{matrix}1/(bc),1/(ac)\\1/(abcd)\end{matrix};q,qcg\right].\end{aligned} \tag{4.7}$$

Using Heine's ${}_2\phi_1$ formula [21, Eq. (Ⅲ.3)]

$${}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,z\right]=\frac{(abz/c;q)_\infty}{(z;q)_\infty}{}_2\phi_1\left[\begin{matrix}c/a,c/b\\c\end{matrix};q,\frac{abz}{c}\right], \tag{4.8}$$

we obtain

$${}_2\phi_1\left[\begin{matrix}e^{-x}/(ib),e^{-x}/(ia)\\1/(ab)\end{matrix};q,iqge^{x}\right]=\frac{(-iqge^{-x};q)_\infty}{(iqge^{x};q)_\infty}{}_2\phi_1\left[\begin{matrix}-e^{x}/(ia),-e^{x}/(ib)\\1/(ab)\end{matrix};q,-iqge^{-x}\right] \tag{4.9}$$

and deduce equation (4.4). The proof of corollary 15 is complete. □

5. Ramanujan type Askey-Wilson integral

With the help of Jacobi theta functions, N.M. Atakishiyev [6, Eq. (19)] discovered the following Ramanujan type representation for the Askey-Wilson integral by the transformation $q\to q^{-1}$.

Proposition 17 (Atakishiyev integral)**.** *If α is a real number and $q=\exp(-2\alpha^2)$, then we have*

$$\int_{-\infty}^{\infty}h(i\sinh\alpha x;a,b,c,d)e^{-x^2}\cosh\alpha x\,\mathrm{d}x=\sqrt{\pi}q^{-\frac{1}{8}}\frac{(ab/q,ac/q,ad/q,bc/q,bd/q,cd/q;q)_\infty}{(abcd/q^3;q)_\infty}. \tag{5.1}$$

For more information about Ramanujan type Askey-Wilson integral, please refer to [6, 19, 20, 31, 38].

In this section, we generalize the following Ramanujan type Askey-Wilson integral by the method of q-difference equation.

Theorem 18. *For $M \in \mathbb{N}_0$ and $f = q^M$, if α is a real number and $q = \exp(-2\alpha^2)$, then we have*

$$\begin{aligned}\int_{-\infty}^{\infty} h(i\sinh\alpha x; a,b,c,d)\frac{e^{-x^2}\cosh\alpha x (ige^{\alpha x};q)_\infty}{(ifge^{\alpha x};q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, qe^{-\alpha x}/(ib), qe^{-\alpha x}/(ia) \\ qe^{-\alpha x}/(ifg), q^2/(ab)\end{matrix}; q, q\right] \mathrm{d}x \\ = \sqrt{\pi}q^{-\frac{1}{8}}\frac{(ab/q, ac/q, ad/q, bc/q, bd/q, cd/q, cg/q;q)_\infty}{(abcd/q^3, cfg/q;q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, q^2/(bc), q^2/(ac) \\ q^2/(cfg), q^4/(abcd)\end{matrix}; q, q\right],\end{aligned} \tag{5.2}$$

where $\max\left\{\left|abcdq^{-M-3}\right|, \left|cfgq^{-M-1}\right|, \left|acdgq^{-3}\right|\right\} < 1$.

Corollary 19. *We have*

$$\begin{aligned}\int_{-\infty}^{\infty} h(i\sinh\alpha x; a,b,c,d,g)\frac{e^{-x^2}\cosh\alpha x}{(ige^{\alpha x};q)_\infty} {}_2\phi_1\left[\begin{matrix} -qe^{\alpha x}/(ia), -qe^{\alpha x}/(ib) \\ q^2/(ab)\end{matrix}; q, -ige^{-\alpha x}\right] \mathrm{d}x \\ = \sqrt{\pi}q^{-\frac{1}{8}}\frac{(ab/q, ac/q, ad/q, bc/q, bd/q, cd/q, cg/q;q)_\infty}{(abcd/q^3;q)_\infty} {}_2\phi_1\left[\begin{matrix} q^2/(bc), q^2/(ac) \\ q^4/(abcd)\end{matrix}; q, \frac{cg}{q}\right],\end{aligned} \tag{5.3}$$

where $\max\left\{\left|abcdq^{-M-3}\right|, \left|acdgq^{-3}\right|, \left|cgq^{-1}\right|\right\} < 1$.

Remark 20. *For $f \to 1$ in Theorem 18, equation* (5.2) *reduces to* (5.1). *For $f \to 0$ in Theorem 18, equation* (5.2) *reduces to* (5.3).

Proof of Theorem 18. The equation (5.2) can be written equivalently by

$$\begin{aligned}\int_{-\infty}^{\infty} h(i\sinh\alpha x; b,c,d)e^{-x^2}\cosh\alpha x\left\{\frac{(ige^{\alpha x}, -iae^{-\alpha x}, iae^{\alpha x};q)_\infty}{(ab/q, ifge^{\alpha x};q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, qe^{-\alpha x}/(ib), qe^{-\alpha x}/(ia) \\ qe^{-\alpha x}/(ifg), q^2/(ab)\end{matrix}; q, q\right]\right\} \mathrm{d}x \\ = \sqrt{\pi}q^{-\frac{1}{8}}(bc/q, bd/q, cd/q;q)_\infty\left\{\frac{(cg/q, ad/q, ac/q;q)_\infty}{(abcd/q^3, cfg/q;q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, q^2/(bc), q^2/(ac) \\ q^2/(cfg), q^4/(abcd)\end{matrix}; q, q\right]\right\}.\end{aligned} \tag{5.4}$$

We denote the RHS of equation (5.4) by $F(f,g,a)$, and check that $F(f,g,a)$ satisfies equation (1.8), so does the LHS of equation (1.8), thus

$$F(f,g,a) = \sum_{n=0}^{\infty}\mu_n \cdot \Psi_n^{(f)}(g,a|q) \tag{5.5}$$

and

$$\begin{aligned}F(f,0,a) &= \sum_{n=0}^{\infty}\mu_n a^n = \sqrt{\pi}q^{-\frac{1}{8}}(bc/q, bd/q, cd/q;q)_\infty\frac{(ad/q, ac/q;q)_\infty}{(abcd/q^3;q)_\infty} \quad \textit{by}\,(5.1) \\ &= \int_{-\infty}^{\infty} h(i\sinh\alpha x; b,c,d)e^{-x^2}\cosh\alpha x\left\{\frac{(iae^{\alpha x}, -iae^{-\alpha x};q)_\infty}{(ab/q;q)_\infty}\right\}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} h(i\sinh\alpha x; b,c,d)e^{-x^2}\cosh\alpha x\left\{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\Psi_n^{\left(ibe^{\alpha x}/q\right)}(-ie^{-\alpha x}, ie^{\alpha x}|q)\frac{a^n}{(q;q)_n}\right\}\mathrm{d}x,\end{aligned}$$

that is,

$$F(f,g,a) = \int_{-\infty}^{\infty} h(i\sinh\alpha x; b,c,d)e^{-x^2}\cosh\alpha x\left\{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\Psi_n^{(f)}(g,a|q)\Psi_n^{\left(ibe^{\alpha x}/q\right)}(-ie^{-\alpha x}, ie^{\alpha x}|q)\frac{1}{(q;q)_n}\right\}\mathrm{d}x,$$

which is equal to the LHS of (1.8) after using equation (2.6). The proof is complete. □

6. q-integral type Bailey's ${}_6\psi_6$ summation

The bilateral basic hypergeometric series ${}_r\psi_s$ is defined as [21]

$$ {}_r\psi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} (-1)^{(s-r)n} q^{(s-r)\binom{n}{2}} z^n. \tag{6.1} $$

The Jackson's q-integral notation [21]

$$ \int_0^{\infty} f(t)\, \mathrm{d}_q t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n. \tag{6.2} $$

Bailey's identity of bilateral well-poised ${}_6\psi_6$-series is one of the deepest results in the theory of basic hypergeometric series, which implies the Askey-Wilson integral [18]. The famous Bailey's ${}_6\psi_6$ have some important applications in number theory and combinatorics. For more information, please refer to [18, 21].

Proposition 21 (Bailey's ${}_6\psi_6$)**.** *For $a, b, c, d \in \mathbb{C}$ with $\left|\alpha^2 abcd/q^3\right| < 1$, we have*

$$ {}_6\psi_6\left[\begin{matrix} q\sqrt{\alpha}, -q\sqrt{\alpha}, q/a, q/b, q/c, q/d \\ \sqrt{\alpha}, -\sqrt{\alpha}, \alpha a, \alpha b, \alpha c, \alpha d \end{matrix}; q, \frac{\alpha^2 abcd}{q^3}\right] $$
$$ = \frac{(q, q\alpha, q/\alpha, \alpha ab/q, \alpha ac/q, \alpha ad/q, \alpha bc/q, \alpha bd/q, \alpha cd/q; q)_\infty}{(a, b, c, d, \alpha a, \alpha b, \alpha c, \alpha d, \alpha^2 abcd/q^3; q)_\infty}. \tag{6.3} $$

In fact, we can rewrite Bailey's ${}_6\psi_6$ as the q-integral form [21, Eq. (5.16)]

$$ \int_0^{\infty} \frac{(\alpha at, a/t, \alpha bt, b/t, \alpha ct, c/t, \alpha dt, d/t; q)_\infty}{(\alpha qt^2, q/(\alpha t^2); q)_\infty} \frac{\mathrm{d}_q t}{t} $$
$$ = \frac{(1-q)(q, \alpha ab/q, \alpha ac/q, \alpha ad/q, \alpha bc/q, \alpha bd/q, \alpha cd/q; q)_\infty}{(\alpha^2 abcd/q^3; q)_\infty}. \tag{6.4} $$

There are many proofs of Bailey's ${}_6\psi_6$ summation in the literature, see details in [2, 5, 17, 25, 29, 31, 35].

In this section, we generalize Bailey's ${}_6\psi_6$ summation as follows by the method of q-difference equation.

Theorem 22. *For $M \in \mathbb{N}_0$ and $f = q^M$, we have*

$$ \int_0^{\infty} \frac{(\alpha at, a/t, \alpha bt, b/t, \alpha ct, c/t, \alpha dt, d/t, \alpha gt; q)_\infty}{(\alpha qt^2, q/(\alpha t^2), \alpha fgt; q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, q/(\alpha bt), q/(\alpha at) \\ q/(\alpha fgt), q^2/(\alpha ab) \end{matrix}; q, q\right] \frac{\mathrm{d}_q t}{t} $$
$$ = \frac{(1-q)(q, \alpha ab/q, \alpha ac/q, \alpha ad/q, \alpha bc/q, \alpha bd/q, \alpha cd/q, \alpha cg/q; q)_\infty}{(\alpha^2 abcd/q^3, \alpha cfg/q; q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, q^2/(\alpha bc), q^2/(\alpha ac) \\ q^2/(\alpha cfg), q^4/(\alpha^2 abcd) \end{matrix}; q, q\right], \tag{6.5} $$

where $\max\left\{\left|\alpha^2 abcdq^{-M-3}\right|, \left|\alpha cfgq^{-M-1}\right|, \left|\alpha^2 acdgq^{-3}\right|\right\} < 1.$

Remark 23. *For $g = 0$ in Theorem 22, equation* (6.5) *reduces to* (6.4) *directly.*

Proof of Theorem 22. We rewrite equation (6.5) by

$$ \int_0^{\infty} \frac{(\alpha at, a/t, \alpha bt, b/t, \alpha ct, c/t, \alpha dt, d/t; q)_\infty}{(\alpha qt^2, q/(\alpha t^2); q)_\infty} \frac{(\alpha gt; q)_\infty}{(\alpha ab/q, \alpha fgt; q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, q/(\alpha bt), q/(\alpha at) \\ q/(\alpha fgt), q^2/(\alpha ab) \end{matrix}; q, q\right] \frac{\mathrm{d}_q t}{t} $$
$$ = (1-q)(q, \alpha bc/q, \alpha bd/q, \alpha cd/q; q)_\infty \frac{(\alpha cg/q, \alpha ad/q, \alpha ac/q; q)_\infty}{(\alpha^2 abcd/q^3, \alpha cfg/q; q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/f, q^2/(\alpha bc), q^2/(\alpha ac) \\ q^2/(\alpha cfg), q^4/(\alpha^2 abcd) \end{matrix}; q, q\right]. \tag{6.6} $$

Denote the RHS of equation (6.6) by $F(f, g, a)$, we can check that both $F(f, g, a)$ and the LHS of (6.6) satisfy equation (1.8) respectively, thus

$$ F(f, g, a) = \sum_{n=0}^{\infty} \mu_n \cdot \Psi_n^{(f)}(g, a|q), \tag{6.7} $$

and

$$
\begin{aligned}
F(f,0,a) &= \sum_{n=0}^{\infty} \mu_n a^n = \frac{(1-q)(q,\alpha ac/q,\alpha ad/q,\alpha bc/q,\alpha bd/q,\alpha cd/q;q)_\infty}{(\alpha^2 abcd/q^3;q)_\infty} \\
&= \int_0^\infty \frac{(\alpha bt,b/t,\alpha ct,c/t,\alpha dt,d/t;q)_\infty}{(\alpha qt^2,q/(\alpha t^2);q)_\infty} \left\{ \frac{(\alpha at,a/t;q)_\infty}{(\alpha ab/q;q)_\infty} \right\} \frac{\mathrm{d}_q t}{t} \\
&= \int_0^\infty \frac{(\alpha bt,b/t,\alpha ct,c/t,\alpha dt,d/t;q)_\infty}{(\alpha qt^2,q/(\alpha t^2);q)_\infty} \left\{ \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \Psi_n^{(b/(qt))}(\alpha t,1/t|q) \frac{a^n}{(q;q)_n} \right\} \frac{\mathrm{d}_q t}{t}. \qquad (6.8)
\end{aligned}
$$

By equation (6.7) and (6.8), we have

$$
F(f,g,a) = \int_0^\infty \frac{(\alpha bt,b/t,\alpha ct,c/t,\alpha dt,d/t;q)_\infty}{(\alpha qt^2,q/(\alpha t^2);q)_\infty} \left\{ \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \Psi_n^{(f)}(g,a|q) \Psi_n^{(b/(qt))}(\alpha t,1/t|q) \frac{1}{(q;q)_n} \right\} \frac{\mathrm{d}_q t}{t}, \qquad (6.9)
$$

which is equivalent to the LHS of (6.6) after using equation (2.6). The proof is complete. □

7. $U(n+1)$ type generating functions for Al-Salam-Carlitz polynomials

Multiple basic hypergeometric series associated to the unitary $U(n+1)$ group have been investigated by various authors, see [12, 13, 32, 37, 40, 41]. In [32], Milne initiated the theory and application of the $U(n+1)$ generalization of the classical Bailey transform and Bailey lemma, which involve the following nonterminating $U(n+1)$ generalizations of the q-binomial theorem.

Proposition 24 ([32, Theorem 5.42])**.** *Let b, z and $x_1, \ldots, x_n$ be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in the following identity vanishes, and that $0 < |q| < 1$ and $|z| < |x_1 \ldots x_n| \, |x_m|^{-n} \, |q|^{(n-1)/2}$, for $m = 1, 2 \ldots, n$. Then*

$$
\sum_{\substack{y_k \geq 0 \\ k=1,2\ldots,n}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s} q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q \right)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\ldots+y_n)} (-1)^{(n-1)(y_1+\ldots+y_n)} \right.
$$
$$
\times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)} (b;q)_{y_1+\ldots+y_n} z^{y_1+\ldots+y_n} = \frac{(bz;q)_\infty}{(z;q)_\infty}, \qquad (7.1)
$$

where $e_2(y_1, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

In this section, we generalize $U(n+1)$ type generating functions for Al-Salam-Carlitz polynomials by the method of q-difference equation.

Theorem 25. *Let u, v, r, s, x, y, z and $x_1, \ldots, x_n$ be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in the following identity vanishes, and that $0 < |q| < 1$ and $|z| < |x_1 \ldots x_n| \, |x_m|^{-n} \, |q|^{(n-1)/2}$, for $m = 1, 2 \ldots, n$. We have*

$$
\sum_{\substack{y_k \geq 0 \\ k=1,2\ldots,n}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s} q^{y_r-y_s}}{1-\frac{x_r}{x_s}} \right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q \right)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\ldots+y_n)} (-1)^{(n-1)(y_1+\ldots+y_n)} \right.
$$
$$
\times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)} \Phi^{(u)}_{y_1+\ldots+y_n}(v,x|q) \Phi^{(r)}_{y_1+\ldots+y_n}(s,y|q) z^{y_1+\ldots+y_n}
$$
$$
= \frac{(rsvz,uvyz;q)_\infty}{(svz,vyz,xyz;q)_\infty} {}_3\phi_2 \left[\begin{matrix} r, vyz, uv/x \\ rsvz, uvyz \end{matrix} ;q, sxz \right], \qquad (7.2)
$$

where $\max\{|svz|, |vyz|, |xyz|, |sxz|\} < 1$ and $e_2(y_1, \ldots, y_n)$ defined in formula (7.1).

Remark 26. *For $n = 1$ in Theorem 25, equation (7.3) reduces to (2.5). For $v = s = 0$ in Theorem 25, equation (7.3) reduces to (7.1) with $b = 0$.*

Proof of Theorem 25. First, we denote $f(u,v,x)$ by

$$f(u,v,x)=\sum_{\substack{y_k\geq 0\\ k=1,2\ldots,n}}\left\{\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1}\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\ldots+y_n)}(-1)^{(n-1)(y_1+\ldots+y_n)}\right.$$
$$\times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)}\Phi^{(u)}_{y_1+\ldots+y_n}(v,x|q)(zy)^{y_1+\ldots+y_n}, \quad (7.3)$$

we can check that $f(u,v,x)$ satisfies equation (1.7), so we have

$$f(u,v,x)=\sum_{n=0}^{\infty}\mu_n\cdot\Phi_n^{(u)}(v,x|q) \quad (7.4)$$

and

$$\begin{aligned}f(u,0,x)&=\sum_{n=0}^{\infty}\mu_n x^n=\frac{1}{(xyz;q)_\infty}\quad by\,(7.1)\\&=\sum_{n=0}^{\infty}\frac{(yz)^n x^n}{(q;q)_n}.\end{aligned}$$

Using equation (7.4) yields

$$f(u,v,x)=\sum_{n=0}^{\infty}\Phi_n^{(u)}(v,x|q)\frac{(yz)^n}{(q;q)_n}=\frac{(uvyz;q)_\infty}{(vyz,xyz;q)_\infty}. \quad (7.5)$$

Similarly, denote $F(r,s,y)$ by the LHS of equation (7.3), we can also check that $F(r,s,y)$ satisfies equation (1.7). By the same method, we have

$$F(r,s,y)=\sum_{n=0}^{\infty}\Phi_n^{(u)}(v,x|q)\Phi_n^{(r)}(s,y|q)\frac{z^n}{(q;q)_n}, \quad (7.6)$$

which is equal to the RHS of equation (7.3) by (2.5). The proof is complete. □

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