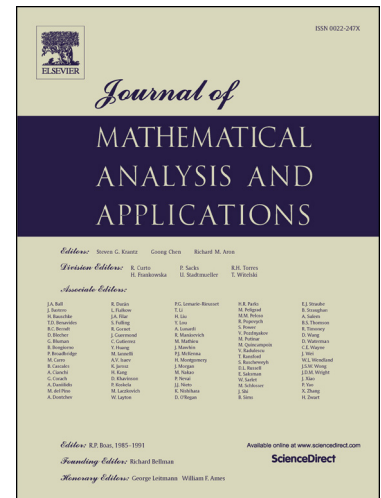


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## Highlights

- Shape derivatives of solutions to elasto-acoustic coupled system are proposed.
- The characterization of shape derivatives is derived for both the differential form and the Euclidean form.
- Shape derivative for stochastic elasto-acoustic equations is firstly studied.
- The variational approach and perturbation characterized by the velocity method are applied.
- Different boundary regularities are discussed.

# Variational approach to shape derivatives for elasto-acoustic coupled scattering fields and an application with random interfaces

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## Abstract

We establish the theoretical results, governed by Helmholtz equation and Lamé system, of shape derivatives of solutions to the elasto-acoustic coupled scattering problem. The primary techniques use the variational approach and the admissible perturbation characterized by the velocity method. Unlike perturbations of the boundary in the normal direction, the velocity method is introduced to conduct sensitivity analysis for an arbitrary domain with the least smooth conditions on a geometric boundary. In view of different boundary regularities, shape derivatives are investigated only in suitable Sobolev spaces. As a further application of our results, we derive the first order shape derivatives of solutions to stochastic elasto-acoustic equations with random interfaces, which can be used to obtain the approximation expectation, variance, and high order moments through Taylor shape expansion.

**Keywords:** shape derivative, elasto-acoustic coupled scattering problem, differential forms, stochastic interface problem.

**2010 MSC:** 49Q10, 49Q12, 74J20, 74J25.

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## 1. Introduction

We are concerned with reconstructing the shape of a solid elastic obstacle immersed in a homogeneous isotropic inviscid fluid with the measurements of a scattered field when time-harmonic incident plane waves are imposed upon the obstacle. Interested readers may refer to [1, 2, 3, 4, 14, 15, 16, 22, 23, 24, 29]. This process can be reduced to inverse mapping from far-field or near-field data into an unknown obstacle boundary represented by parameters. In the past half century, a number of associated applications and technologies such as radar, geophysical exploration, sonar and nondestructive testing have been widely studied by both mathematicians and engineers. However, inverse scattering problems are severely nonlinear and ill-posed and especially depend strongly on the accuracy of the measurements (cf. [2, 4, 12, 13, 14, 15, 16, 27]). In the mathematical framework, the dependence with respect to the domain has been investigated in depth in inverse scattering problems (cf. [2, 4, 12, 13, 14, 15, 16, 23, 24, 27]). The information given by shape derivatives of scattered fields, especially for regularized iterative or Newton-type methods, is an important step when solving these inverse problems. Moreover, the characterization of shape derivatives serves to ensuring the stability, convergence rate and numerical efficiency of these iterative methods. The corresponding extensive study of Fréchet derivatives for the elasto-acoustic case is first given in [2] with a

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$C^2$  boundary domain. Later, in [4], based on the implicit function theorem and the standard trace theorems, the authors establish that the scattered field is continuously Fréchet differentiable with only the Lipschitz boundary. However, the explicit description of shape derivatives is still open; the main drawback of the Lipschitz case is the lack of surjection of the trace operators, particularly for the elastic field.

Delfour and Zolésio extend the velocity(speed) method to define and study the shape gradient and structure theorems from  $C^k$  to non-smooth domains in [11, 34]. The velocity method neatly fits into the Lie derivative, after which shape derivatives in differential forms are deduced instead of the classical vector calculus in [19, 20]. Compared to the classical vector calculus, differential forms recursively deduce even higher-order shape derivatives to avoid tedious manipulations. In previous papers [2, 4], the authors use perturbations of the boundary along the normal direction to study the Fréchet derivatives, which is equivalent to the velocity method when the boundary is sufficient smooth. The existence of the Fréchet derivative can be drawn from the implicit function theorem in [2] (the same technique is also used in [12, 13, 14] for acoustic scattering problems). However, the explicit characterization of shape derivatives allows the rigorous justification, especially for the trace terms of transmission conditions.

The goal of this paper is two-fold. First, we derive the characterization of shape derivatives of solutions to the elasto-acoustic equations, in both the exterior differential form and classical Euclidean form in  $\mathbb{R}^3$  by virtue of the velocity method in the variational sense. Shape derivatives can be viewed as solutions to a similar direct elasto-acoustic boundary value problem with vanishing source terms but with nonhomogeneous transmission conditions in the interface. In [29] and later in [15, 22], the well-posed nature of the direct elasto-acoustic equations is studied by reformulating it as an integro-differential system under the  $C^2$  class. Recently, in [1], for the Lipschitz boundary, the authors prove the existence and uniqueness of elasto-acoustic equations. Based on the existing results, we analyze the existence and uniqueness of the nonhomogeneous elasto-acoustic problems through a variational approach for both the original and shape-derivative cases under the different regularities. When the obstacle domain is regular, according to the standard trace theorem, the transmission conditions on the interference can be addressed in Sobolev spaces  $H^{-1/2}$ , which can guarantee the existence and uniqueness of elasto-acoustic equations. However, the cases of class  $C^{1,1}$  polyhedral domains are not well analyzed by the standard trace theorem, which is more useful for numerical computation and practical applications (see also [14]). Moreover, other tools based on variational methods such as an extension of the compliance operator, tangential Green formula on the surface, and tangential and normal decomposition are also introduced for deducing the corresponding items.

Second, as a random version of the elasto-acoustic coupled system, the governing equations of random shape derivatives with stochastic perturbed interfaces are also illustrated. Stochastic partial differential equations (SPDE) are powerful in illustrating physical behavior, especially for uncertainties of interfaces or geometries in highly heterogeneous media (cf. [18, 28], etc.). In [17], concerning Maxwell equations with random interfaces, we introduce a robust numerical method via shape calculus; the information from shape calculus is used to estimate the statistical moments of stochastic Maxwell equations in terms of perturbation magnitude. Consider the advantage of Taylor shape expansions; one can also obtain equations for the second moment with the tensor product of the stochastic boundary value problems. There are few studies in the literature about elasto-acoustic coupled equations with random interfaces. These equations can be applied to estimate the statistical moments based on the results of shape differentiability provided in this paper. One can also apply the Taylor shape-expansion to obtain the approximation expectation, variance, and high order moments. By plugging the expansion into the original random coupled equations and neglecting the high order terms, one can investigate the equations for even  $k$ -th moment discussed in [18, 28].

This paper is organized as follows. In Section 2, we introduce some concepts and reformulate the elasto-acoustic equations with different equivalent forms. Structure lemmas from shape calculus in differential

forms are described and applied to the elasto-acoustic coupled system. In Section 3, main theorems about the characterization of shape derivatives are provided and discussed with the regularities of the class  $C^2$  and polyhedral  $C^{1,1}$  scatterer. The shape derivatives of the random interface case are also demonstrated in Section 4. Finally, we conclude with a summary.

## 2. Preliminaries and model problem

### 2.1. Notations and useful spaces

Let  $\Omega$  denote a bounded, three dimensional open domain with Lipschitz boundary  $\partial\Omega = \Gamma$ ,  $\Omega^c = \mathbb{R}^3 \setminus \bar{\Omega}$  is a simply connected unbounded exterior domain.  $H^s(\Omega)$  stands for the standard complex-valued Sobolev space for a real number  $s$  with the norm  $\|\cdot\|_{H^s(\Omega)}$ . And spaces of vector functions will be denoted by boldface letters in this paper. The weighted Sobolev space  $H_\rho^1(\Omega^c)$  first introduced in [1, Sect. 2.2, p. 573] and [14, Sect. 1.2.2, p. 15] is defined by  $H_\rho^1(\Omega^c) = \overline{C_0^\infty(\Omega^c)}^{\|\cdot\|_{1,\rho}}$ , whose norm  $\|\cdot\|_{1,\rho}$  is deduced by the following inner product (that is,  $\|p\|_{1,\rho} = (p, p)_{1,\rho}$ ). For any  $p, q \in C_0^\infty(\Omega^c) = \{p|_{\Omega^c} : p \in C_0^\infty(\mathbb{R}^3)\}$ ,

$$(p, q)_{1,\rho} = \int_{\Omega^c} \left( \frac{\nabla p \cdot \nabla \bar{q} + p \bar{q}}{\rho(r)} + \left( \frac{\partial p}{\partial r} - ikp \right) \overline{\left( \frac{\partial q}{\partial r} - ikq \right)} \right) dx,$$

where  $\rho(r) = 1 + r^2$  with  $r = \|x\|_2$  for any  $x \in \Omega^c \subset \mathbb{R}^3$ . Likewise, the weighted Sobolev space  $H_{1/\rho}^1(\Omega^c)$  is defined by  $H_{1/\rho}^1(\Omega^c) = \overline{C_0^\infty(\Omega^c)}^{\|\cdot\|_{1,1/\rho}}$ .

The set of differential form  $\mathcal{D}\mathcal{F}^{l,m}(\bar{\Omega})$  of degree  $l$  ( $0 \leq l \leq d, l, m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ ), in some domain  $\Omega \subset \mathbb{R}^d$ , is defined that for any  $\omega \in \mathcal{D}\mathcal{F}^{l,m}(\bar{\Omega})$ ,  $\omega$  takes value in the space of alternating  $l$ -multilinear forms  $\wedge^l$  on  $\mathbb{R}^d$  given in [20, Sect. 2, p. 1097],

$$\omega = \sum_I \omega_I dx_I, \text{ with } \omega(x) \in \wedge^l \text{ for any } x \in \Omega \subset \mathbb{R}^d,$$

where  $\omega_I \in C^m(\bar{\Omega})$ , the summation is over all the increasing  $l$ -permutations  $I = (i_1, \dots, i_l)$  and  $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_l}$ ,  $1 \leq i_1 < \dots < i_l \leq d$ . In an analogous way, we define  $\mathcal{D}\mathcal{F}^{l,\infty}(\bar{\Omega})$  if all  $\omega_I \in C^\infty(\bar{\Omega})$ . Likewise,  $H^s(\Omega; \wedge^l(\mathbb{R}^d))$  with  $s \in \mathbb{R}_0^+$  denotes the space of all differential forms with each component in  $H^s(\Omega)$ .

Denote by  $(\Theta, \Sigma, P)$  a complete probability space, where  $\Theta$  is the set of outcomes,  $\Sigma$  is the  $\sigma$ -algebra of events and  $P$  is the probability measure. Let  $(X, \|\cdot\|_X)$  be a separable Hilbert space and the Bochner space  $L^p(\Theta; X)$  is defined to be the Kolmogorov quotient (by equality almost everywhere) of the space of all Bochner measurable functions  $u : \Theta \rightarrow X$  such that the corresponding norm is finite,

$$\|u\|_{L^p(X;\Theta)} = \left( \int_{\Theta} \|u(\cdot, \xi)\|_X^p d\mathbb{P}(\xi) \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.$$

### 2.2. Deterministic interaction between acoustic and elastic fields

Suppose  $p^i(x) = \exp(ikx \cdot d)$  is a time-harmonic incident wave, where  $d$  is an incident direction. It is imposed upon the elastic target  $\Omega$ , then it generates a scattering wave  $p^s$  in homogeneous inviscid fluid  $\Omega^c$  governed by the Helmholtz equation (1a) and a transmitted elastic wave  $u$  in the scatterer  $\Omega$ . In the

isotropic elastic body  $\Omega$ , the displacement field  $\mathbf{u}$  is governed by the reduced Navier equation (1b) with Lamé constants  $\lambda$  and  $\mu$  satisfying  $\mu > 0, 3\lambda + 2\mu > 0$ ,

$$\begin{cases} -\Delta p^s - k^2 p^s = 0 & \text{in } \Omega^c, & (1a) \\ -\text{div} \boldsymbol{\sigma}(\mathbf{u}) - \rho_s \omega^2 \mathbf{u} = \mu \nabla \times \nabla \times \mathbf{u} - (2\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}) - \rho_s \omega^2 \mathbf{u} = 0 & \text{in } \Omega, & (1b) \\ \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} = \frac{\partial p^s}{\partial \mathbf{n}} + \frac{\partial p^i}{\partial \mathbf{n}} & \text{on } \Gamma, & (1c) \\ T\mathbf{u} = -p^s \mathbf{n} - p^i \mathbf{n} & \text{on } \Gamma, & (1d) \\ \lim_{r \rightarrow \infty} r \left( \frac{\partial p^s}{\partial r} - ikp^s \right) = 0 & r = \|\mathbf{x}\|_2, & (1e) \end{cases}$$

where  $\rho_f$  and  $\rho_s$  are positive real numbers denoting the densities of fluid  $\Omega^c$  and scatterer  $\Omega$  respectively.  $\mathbf{n}$  is the outer unit normal vector on  $\Gamma$  pointing to  $\Omega^c$ .  $k$  is the wave number satisfying  $k^2 = \omega^2/c_f^2$ . The linear stress tensor  $\boldsymbol{\sigma}(\mathbf{u})$  is expressed by a positive symmetric fourth-order tensor operator  $\mathcal{A}^{-1}$ , and it acts on the linearized strain tensor  $\boldsymbol{\epsilon}(\mathbf{u}) = (\nabla \mathbf{u} + [\nabla \mathbf{u}]^T)/2$  with  $\nabla \mathbf{u} = (\partial u_i / \partial x_j)_{i,j=1}^3$  and  $T$  is the traction operator,

$$\begin{cases} \boldsymbol{\sigma}(\mathbf{u}) = \mathcal{A}^{-1} \boldsymbol{\epsilon}(\mathbf{u}) = 2\mu \boldsymbol{\epsilon}(\mathbf{u}) + \lambda \text{tr}(\boldsymbol{\epsilon}(\mathbf{u})) \mathbb{I}_3 = \lambda(\text{div} \mathbf{u}) \mathbb{I}_{3 \times 3} + \mu(\nabla \mathbf{u} + [\nabla \mathbf{u}]^T), \\ T\mathbf{u} = \mathcal{A}^{-1} \boldsymbol{\epsilon}(\mathbf{u}) \cdot \mathbf{n} = 2\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \lambda(\text{div} \mathbf{u}) \mathbf{n} + \mu \mathbf{n} \times (\nabla \times \mathbf{u}), \end{cases}$$

where  $\mathbb{I}_3 \in \mathbb{R}^{3 \times 3}$  is the identity matrix and  $\partial \mathbf{u} / \partial \mathbf{n} = (\mathbf{n} \cdot \nabla u_i)_{i=1}^3$  is the normal derivative of the displacement vector  $\mathbf{u} = (u_i)_{i=1}^3$ .

Next, we reformulate the BVP (1) in the truncated domain by constructing the DtN map given in [27, Sect. 2, p. 84] on an artificial spherical boundary  $\mathcal{B} = \partial \mathcal{B}_R$ .  $\mathcal{B}_R$  is an open ball with radius  $R$  centered at the origin containing  $\Omega$  (furthermore, choose  $R > 0$  such that  $\overline{\Omega} \subset \mathcal{B}_{R/2}$ ). The artificial boundary decomposes domain  $\Omega^c$  into two parts: a scattering bounded domain  $\Omega_R = \Omega^c \cap \mathcal{B}_R$  and an exterior domain  $\mathcal{B}_R^c = \{y \in \mathbb{R}^3 : \|y\|_2 > R\}$ . The Dirichlet-to-Neumann (DtN) operator  $\mathcal{T} : H^{1/2}(\mathcal{B}) \rightarrow H^{-1/2}(\mathcal{B})$ , which is an exact boundary condition at finite distance standing for the outgoing Sommerfeld condition. The explicit scattering solution in  $\mathcal{B}_R^c$  can be expanded by a series of form in polar coordinates using the separation of variables procedure for the analytical expression,

$$\begin{cases} \mathcal{T} : p|_{\mathcal{B}} = g \longrightarrow \frac{\partial p}{\partial \mathbf{n}}|_{\mathcal{B}}, \\ \mathcal{T} p(R, \hat{\mathbf{x}}) = \sum_{n=0}^{\infty} \sum_{|m| \leq n} k \frac{h_n^{(1)'}(kR)}{h_n^{(1)}(kR)} g_n^m Y_n^m(\hat{\mathbf{x}}), \quad \text{with } g_n^m = \int_{\mathcal{B}} g \overline{Y_n^m}(\hat{\mathbf{x}}) d\mathbf{s}, \end{cases}$$

where  $p$  solves the exterior Dirichlet problem of Helmholtz equation in  $\mathcal{B}_R^c$  with  $p|_{\mathcal{B}} = g$ .  $h_n^{(1)}(x)$  is the Hankel function of first kind and  $Y_n^m(\hat{\mathbf{x}}) = \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}} P_n^{|m|}(\cos \theta) e^{im\phi}$ ,  $n \geq 0, |m| \leq n$  denotes the  $n - m$  spherical harmonics. When  $k, \omega$  are real numbers, the equivalent elasto-acoustic equations read,

$$\begin{cases}
 -\Delta p^s - k^2 p^s = 0 & \text{in } \Omega_R, & (2a) \\
 \mu \nabla \times \nabla \times \mathbf{u} - (2\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}) - \rho_s \omega^2 \mathbf{u} = 0 & \text{in } \Omega, & (2b) \\
 \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} = \frac{\partial p^s}{\partial \mathbf{n}} + \frac{\partial p^i}{\partial \mathbf{n}} & \text{on } \Gamma, & (2c) \\
 T\mathbf{u} = -p^s \mathbf{n} - p^i \mathbf{n} & \text{on } \Gamma, & (2d) \\
 \frac{\partial p^s}{\partial r} = \mathcal{T} p^s & \text{on } \mathcal{B}, & (2e) \\
 \lim_{r \rightarrow \infty} r \left( \frac{\partial p^s}{\partial r} - ik p^s \right) = 0 & r = \|\mathbf{x}\|_2. & (2f)
 \end{cases}$$

**Remark 2.1.** The equivalence of BVP (1) and BVP (2) is proved in [1, Thm. 1, p. 575] and [14, Thm. 1.2.4.1, p. 22] even when  $\Gamma$  is only Lipschitz continuous, in the sense of restricting the solution  $(p^s, \mathbf{u}) \in H^1_\rho(\Omega^c) \times \mathbf{H}^1(\Omega)$  into  $\Omega_R \times \Omega$ , and extension of  $(p^s_R, \mathbf{u}) \in H^1(\Omega_R) \times \mathbf{H}^1(\Omega)$  with  $p^s = p^e_R$ , where  $p^e_R$  denotes the unique radiation solution to the Helmholtz equation in  $\mathcal{B}^c_R$ .

The variational form for BVP (2) reads,

$$(VF) \begin{cases} \text{Seek } (p^s, \mathbf{u}) \in H = H^1(\Omega_R) \times \mathbf{H}^1(\Omega), \text{ such that} & (3a) \\ \mathcal{A}((p^s, \mathbf{u}), (q, \mathbf{v})) = l(q, \mathbf{v}), \quad \text{for any } (q, \mathbf{v}) \in H = H^1(\Omega_R) \times \mathbf{H}^1(\Omega), & (3b) \end{cases}$$

where the sesquilinear form  $\mathcal{A}(\cdot, \cdot)$  in (3b) is given

$$\begin{aligned}
 \mathcal{A}((p^s, \mathbf{u}), (q, \mathbf{v})) &= \int_{\Omega_R} (\nabla p^s \cdot \nabla \bar{q} - k^2 p^s \bar{q}) dx + \langle \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n}, q \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} - \langle \mathcal{T} p^s, q \rangle_{H^{-\frac{1}{2}}(\mathcal{B}), H^{\frac{1}{2}}(\mathcal{B})} \\
 &\quad + \rho_f \omega^2 \left( a(\mathbf{u}, \mathbf{v}) - \int_{\Omega} \rho_s \omega^2 \mathbf{u} \cdot \bar{\mathbf{v}} dx + \langle p^s \mathbf{n}, \mathbf{v} \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} \right),
 \end{aligned}$$

$l(\cdot, \cdot)$  and  $a(\cdot, \cdot)$  are as follows,

$$\begin{aligned}
 l(q, \mathbf{v}) &= \left\langle \frac{\partial p^i}{\partial \mathbf{n}}, q \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} - \rho_f \omega^2 \langle p^i \mathbf{n}, \mathbf{v} \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}, \\
 a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \nabla \bar{\mathbf{v}} dx = \int_{\Omega} (\lambda (\mathbf{div} \mathbf{u})(\mathbf{div} \bar{\mathbf{v}}) + 2\mu \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\bar{\mathbf{v}})) dx.
 \end{aligned}$$

The stain tensor is related to the stress tensor by a positive symmetric fourth-order tensor operator  $\mathcal{A}^{-1} = \mathcal{C}$ ,

$$\begin{cases} \boldsymbol{\sigma}(\mathbf{u}) = \mathcal{A}^{-1} \boldsymbol{\epsilon}(\mathbf{u}) = \mathcal{C} \boldsymbol{\epsilon}(\mathbf{u}) = c_{ijkl} \epsilon_{kl}, \\ c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad i, j, k, l = 1, 2, 3. \end{cases}$$

**Lemma 2.2.**  $(p^s, \mathbf{u})$  is the solution of BVP (1) if and only if it is the solution of variational form (3) with the extension from  $\Omega \times \Omega_R$  to  $\Omega \times \Omega^c$  in the sense of Remark 2.1.

The equivalence in Lemma 2.2 is directly deduced based Remark 2.1 and the classical analysis of the variational form in the sense of distribution. We omit the proof and leave it to the reader. The sesquilinear form  $\mathcal{A}((p^s, \mathbf{u}), (q, \mathbf{v}))$  satisfies the continuity and Gårding's inequality in  $H = H^1(\Omega_R) \times \mathbf{H}^1(\Omega)$  given in [1, Sect. 3.2, p. 579] and [14, Sect. 1.4, p. 40], thus, according to the Fredholm alternative theorem, the existence and uniqueness can be stated in Lemma 2.3.

**Lemma 2.3.** (Barucq, Djellouli and Estecahandy [1, Cor. 8, p. 852]) *For any  $p^i \mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma)$  and  $\partial p^i / \partial \mathbf{n} \in H^{-1/2}(\Gamma)$ , there always exists a unique scattering solution  $p^s \in H^1_\rho(\Omega^c)$  and  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  is unique modulo Jones frequencies occurring for certain symmetry geometries of BVP (1).*

Jones modes, the associated Jones frequencies, are firstly being discussed by D.G. Jones. They are referred to the nontrivial solution of Navier equation,

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{u}) - \rho_s \omega^2 \mathbf{u} = 0 & \text{in } \Omega, \\ T\mathbf{u} = 0 & \text{on } \Gamma, \\ \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

Jones frequencies may exist for balls and other axisymmetric bodies analyzed in [25, 30], however, intuitively, are “rare” for an “arbitrary” domain in general. Moreover, when  $\omega, k$  have positive imaginary parts, (that is,  $p^s$  decays exponentially at infinity),  $(p^s, \mathbf{u})$  have the unique solution for BVP (1) given in [29, Sect. 3, p. 908]. For this case, the equivalence in Lemma 2.2 is also satisfied.

### 2.3. Shape calculus

The velocity(speed) method, that is, spatial vector fields generate flows, which govern the shape perturbation. Given a Lipschitz continuous velocity field  $V(t, x) \in C([0, \varepsilon); C^m_0(\bar{D}))$  for a nonempty bounded domain  $\Omega \subset D \subset \mathcal{B}_{R/2} \subset \mathbb{R}^d$  and suppose  $\Omega$  is class  $C^m$ , ( $m \geq 0$  is an integer,  $\varepsilon$  is a small real nonnegative number), and  $D$  is known as *hold-all* domain. The flow generates a family of  $C^m$ -diffeomorphism admissible deformed domains parameterized by the pseudo-time  $t$ ,

$$\begin{cases} T_t(V)\mathbf{X} = \mathbf{x}(t, \mathbf{X}) & t > 0, \mathbf{X} \in D, \\ \Omega_t(V) = T_t(V)(\Omega) = \{T_t(V)(\mathbf{X}) : \forall \mathbf{X} \in \Omega\}. \end{cases}$$

The outward unit normal field  $\mathbf{n}_t$  on  $\Gamma_t = \partial\Omega_t$  belongs to  $C^{m-1}(\Gamma_t, \mathbb{R}^d)$  (see [11, 34] for more details). We denote  $dp$  the first-order shape derivative of scattering field  $p^s$ , which is defined formally by the pointwise limit,

$$dp = \lim_{t \rightarrow 0} \frac{p^s_t - p^s}{t}, \quad x \in \Omega^c \cap \Omega^c_t.$$

It is a local definition in the strong (or weak) sense of corresponding Sobolev spaces. Likewise,  $d\mathbf{u}$  represents the first-order shape derivative of the elastic field.

The Lie derivative  $\mathcal{L}_V$  of a  $l$ -form  $\omega$  in the direction  $V$  is defined if the following limit exists,

$$\mathcal{L}_V \omega = \lim_{t \rightarrow 0} \frac{T_t(V)^* \omega - \omega}{t}.$$

The domain functional  $J(\Omega)$  with a density form  $\omega \in \mathcal{D}\mathcal{F}^{d,m}(\mathbb{R}^d)$ , and the boundary functional  $I(\Gamma)$  of a surface density form  $\eta \in \mathcal{D}\mathcal{F}^{d-1,m}(\mathbb{R}^d)$  are globally defined in (4),

$$\begin{cases} J(\Omega) = \int_{\Omega} \omega, \\ I(\Gamma) = \int_{\Gamma} \eta. \end{cases} \quad (4)$$

If  $\omega = \sum_I \omega_I dx_I \in \mathcal{D}\mathcal{F}^{l,\infty}(\bar{\Omega})$ , through the exterior differential operator  $d$ ,  $d\omega = \sum_{i=1}^d \sum_I \frac{\partial \omega_I}{\partial x_i} dx_i \wedge dx_I \in \mathcal{D}\mathcal{F}^{l+1,\infty}(\bar{\Omega})$ , and  $d\omega = 0$  when  $l \geq d$ . The incarnation of  $d$  is  $\nabla$ , **curl** and **div** when  $l = 0, 1$  and  $2$ ,



respectively, in  $\mathbb{R}^3$  (i.e.  $d = 3$ ). We denote the contraction of  $\omega \in \mathcal{D}\mathcal{F}^{l,m}(\bar{\Omega})$  with a vector field  $V \in C^m(\mathbb{R}^d)$ , and the exterior product of differential forms  $\omega \in \mathcal{D}\mathcal{F}^{l,m}(\bar{\Omega})$  and  $\eta \in \mathcal{D}\mathcal{F}^{k,m}(\bar{\Omega})$ , respectively (more details see [19, 20, 21]),

$$i_V \omega \in \mathcal{D}\mathcal{F}^{l-1,m}(\bar{\Omega}), \quad \omega \wedge \eta \in \mathcal{D}\mathcal{F}^{l+k,m}(\bar{\Omega}).$$

**Lemma 2.4.** (Hiptmair and Li [20, Thm. 1, p. 1083, Cor. 2, p. 1085]) *The domain functional  $J(\Omega)$  and boundary functional  $I(\Gamma)$  in (4) are shape differentiable under suitable smoothness conditions on the domain and the velocity fields, with the shape gradients*

$$\begin{cases} \langle dJ(\Omega), V \rangle &= \int_{\Omega} \mathcal{L}_V \omega = \int_{\Omega} d i_V \omega = \int_{\Gamma} i_V \omega, \\ \langle dI(\Gamma), V \rangle &= \int_{\Omega} \mathcal{L}_V d\eta = \int_{\Gamma} i_V d\eta. \end{cases}$$

The smooth scalar function  $f$  is understood as a differential form  $\omega$  with the degree  $d$ ,

$$\begin{cases} J(\Omega) = \int_{\Omega} f dx = \int_{\Omega} \omega, \\ J(\Omega; V(t)) = \int_{\Omega_t(V)} f dx = \int_{\Omega_t(V)} \omega = \int_{\Omega} T_t(V)^* \omega. \end{cases}$$

The exterior derivative  $d$  is nothing but the **div** operator in Lamme 2.4 when  $d = 3$ , thus, the shape gradient  $\langle dJ(\Omega), V \rangle$  is derived by

$$\langle dJ(\Omega), V \rangle = \int_{\Omega} d i_V \omega = \int_{\Omega} \mathbf{div}(fV) dx = \int_{\Gamma} f(V \cdot \mathbf{n}) ds.$$

The smooth scalar function  $g = (g\mathbf{n}) \cdot \mathbf{n}$ , and the boundary functional  $\int_{\Gamma} g ds$  can be rewritten in the differential forms with integrand  $\eta$  of degree  $d - 1$ , thus,

$$I(\Gamma) = \int_{\Gamma} g ds = \int_{\Gamma} (g\mathbf{n}) d\vec{s} = \int_{\Gamma} \eta = \int_{\Omega} d\eta.$$

By applying the formula in Lemma 2.4, we can derive the shape gradient  $\langle dI(\Gamma), V \rangle$ ,

$$\langle dI(\Gamma), V \rangle = \int_{\Gamma} i_V d\eta = \int_{\Gamma} \mathbf{div}(g\mathbf{n})(V \cdot \mathbf{n}) ds = \int_{\Gamma} \left( \frac{\partial g}{\partial n} + \mathfrak{H}g \right) (V \cdot \mathbf{n}) ds,$$

where  $\mathfrak{H} := (d - 1)H$  is the additive curvature,  $H$  is the mean curvature. This formula agrees with [11, Thm. 4.3, p. 486], but arriving is more easily.

#### 2.4. Reformulate the coupled system in differential forms and the abstract shape derivatives

We consider the Helmholtz equation for an  $l$ -form  $\omega$ , so its differential form (see also [19, 20, 21])

$$\begin{cases} (-1)^{d-l} d * d\omega - *_\alpha \omega = 0 & \text{in } \Omega_R, \\ \text{Tr}(*d\omega) = (-1)^{d-l} \text{Tr}\Phi(\mathbf{u}) & \text{on } \Gamma, \\ \text{Tr}(*d\omega) = (-1)^{d-l} \text{Tr}\Psi & \text{on } \mathcal{B}. \end{cases} \quad \begin{matrix} (5a) \\ (5b) \\ (5c) \end{matrix}$$

where  $*_\alpha$  is a fixed Hodge operators in  $\Omega_R$ , it is viewed as a multiplication with the coefficient function  $k^2$ ,  $\text{Tr}$  is the trace operator on the boundary.  $\Phi, \Psi$  are corresponding smooth  $d - l - 1$  differential forms on the

boundary  $\partial\Omega_R = \Gamma \cup \mathcal{B}$ . The weak version of abstract form (5) is obtained through the formula of integration by parts given in [20, Eq. (59), p. 1093] and [19], reads:

Find  $\omega \in \{\boldsymbol{\varsigma} | \boldsymbol{\varsigma} \in \mathbf{H}(\mathbf{d}, \Omega_R, \wedge^l(\mathbb{R}^d)), \text{Tr}(\boldsymbol{\varsigma}) \in L^2(\partial\Omega_R, \wedge^l(\mathbb{R}^d))\}$ , such that

$$\mathcal{B}_1(\omega, \boldsymbol{\eta}, \mathbf{u}) = \int_{\Omega_R} (*d\omega \wedge d\bar{\boldsymbol{\eta}} - *_\alpha \omega \wedge \bar{\boldsymbol{\eta}}) + \int_{\mathcal{B}} \text{Tr}(\boldsymbol{\Psi} \wedge \bar{\boldsymbol{\eta}}) + \int_{\Gamma} \text{Tr}(\boldsymbol{\Phi}(\mathbf{u}) \wedge \bar{\boldsymbol{\eta}}) = 0,$$

for all smooth test  $l$ -forms  $\boldsymbol{\eta}$ .

Compared to BVP (2), the equivalence abstract form (5) can be obtained if we set  $d = 3$  and  $l = 0$ , and functions  $\rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} - \frac{\partial p^i}{\partial n}$  and  $\mathcal{T} p^s$  are viewed as vector proxies of differential forms  $\boldsymbol{\Phi}$  and  $\boldsymbol{\Psi}$ .

Unlike Helmholtz equation and Maxwell equation, the covariant exterior derivative of Navier equation [35], however, depends on the metric of the bundle. Here, we show two ways to express the exterior differential forms. First,  $\mathbf{u} = \vec{u}^b = g(\vec{u})$  is viewed as the musical isomorphism 1-form of a vector field  $\vec{u}$  [33, p. 1], which is used the same letter and  $g$  is a metric tensor.

$$\begin{cases} -*_\beta \mathbf{d} * \mathbf{d} * \mathbf{u} + \mathbf{d} *_\mu \mathbf{du} - *_\Pi \mathbf{u} = 0 & \text{in } \Omega, \\ \mathfrak{T} = * \vec{t}^b = \text{Tr}(\Upsilon) & \text{on } \Gamma. \end{cases} \quad (6a)$$

$\mathfrak{T}$  is the 2-form traction field, and  $\text{Tr}(\Upsilon)$  is related to  $\omega$  (based on (2d) on the interface  $\Gamma$ ).  $*_\beta, *_\mu, *_\Pi$  are fixed Hodge operators as the multiplications with coefficient functions denoted by  $2\mu + \lambda, \mu$  and  $\rho_s \omega^2$ . The weak formula of abstract form (6) reads:

Seek  $\mathbf{u} \in \{\boldsymbol{\varsigma} | \boldsymbol{\varsigma} \in \mathbf{H}(\mathbf{d}, \Omega, \wedge^1(\mathbb{R}^3)), \text{Tr}(\boldsymbol{\varsigma}) \in L^2(\partial\Omega, \wedge^1(\mathbb{R}^3))\}$ , such that

$$\mathcal{B}_2(\mathbf{u}, \mathbf{v}, \omega) = \int_{\Omega} (*_\beta \mathbf{d} * \mathbf{u} \wedge \mathbf{d} * \bar{\mathbf{v}} - *_\mu \mathbf{du} \wedge \bar{\mathbf{dv}} - *_\Pi \mathbf{u} \wedge \bar{\mathbf{v}}) - \int_{\Gamma} \text{Tr}(\Upsilon \wedge \bar{\mathbf{v}}) + \int_{\Gamma} \text{Tr}(\mathcal{P} \wedge \bar{\mathbf{v}}) = 0,$$

for all smooth test 1-forms  $\mathbf{v}$ , where  $\mathcal{P}$  is the difference 2-form between  $\Upsilon$  and the boundary integral term of (6) through the formula of integration by parts given in [20, eq. (59), p. 1093]. This extra term  $\mathcal{P}$  is used to ensure the integrity of the structure 2-form of Navier equation.

Second,  $\mathbf{u}$  can also be viewed as a vector-valued 0-form, refer to [35, Sect. C, p. 022901-13], the stress can be thought of as a covector-valued 2-form. Thus, governing equations can also read in terms of bundle-valued forms. The existence of stress form  $\mathbb{T}$  is assumed to be  $\mathbb{T} = *_2 \boldsymbol{\sigma} = \sigma_{ab} dx^a \otimes (*dx^b)$ , where  $\boldsymbol{\sigma}$  is the cauchy stress and  $*_2$  means a Hodge star operator acting on the area form of stress form (i.e. on the second form). Specially, in this case of linear Navier equation,  $\mathbb{T} = dx^a \otimes [g_{ak} g_{bl} g_{mc} \epsilon^{klcd} \epsilon_d^m] (*dx^b) = *_E \mathbf{e}^{b1}$  given in [35, Eq. (3.61), p. 022901-20].

$$\begin{cases} \mathfrak{d}\mathbb{T} + *_\mathbf{u} \otimes \rho_s = 0 & \text{in } \Omega, \\ \text{Tr}(\mathbb{T}) = \text{Tr}(\Upsilon^\circ) & \text{on } \Gamma. \end{cases}$$

$\mathbb{F}$  is a bundle and an element of  $\mathbb{F} \otimes \mathfrak{d} : TS \otimes \wedge^{l-1} \mapsto TS \otimes \wedge^l$  such that  $\langle \mathbf{v}, \mathfrak{d}\mathbb{T} \rangle = \mathbf{d}(\langle \mathbf{v}, \mathbb{T} \rangle) - \nabla \mathbf{v} \wedge \mathbb{T}$  (more details see [35]). We can similarly write the weak formula of the above abstract form.

Combine the two abstract weak forms  $\mathcal{B}_1(\omega, \boldsymbol{\eta}, \mathbf{u})$  and  $\mathcal{B}_2(\mathbf{u}, \mathbf{v}, \omega)$ , one can reformulate the corresponding differential form of classical variational form  $\mathcal{A}((p^s, \mathbf{u}), (q, \mathbf{v})) = l(q, \mathbf{v})$  in (3), that is,

$$\mathcal{B}_1(\omega, \boldsymbol{\eta}, \mathbf{u}) + \rho_f \omega^2 \mathcal{B}_2(\mathbf{u}, \mathbf{v}, \omega) = 0.$$

We characterize the shape derivatives of abstract way in Lemma 2.5, the shape derivatives of differential forms are denoted by  $\delta\omega, \delta\mathbf{u}$ . We suppose the well-posed conditions for abstract form, and examine the conditions rigorously in the following lemmas and theorems in next section when  $d = 3$ .

**Lemma 2.5.** *Under suitable smooth conditions on  $\Omega$  and the velocity field  $V$ , the shape derivatives  $\delta\omega, \delta\mathbf{u}$  of the abstract solutions  $\omega$  and  $\mathbf{u}$  exist and satisfy the following variational problem,*

$$\begin{aligned} & \int_{\Omega_R} (*d\delta\omega \wedge d\bar{\eta} - *_\alpha d\delta\omega \wedge \bar{\eta}) + \int_{\Gamma} i_V (*d\omega \wedge d\bar{\eta} - *_\alpha \omega \wedge \bar{\eta}) \\ & + \int_{\mathcal{B}} Tr(\Psi(\delta\omega) \wedge \bar{\eta}) + \int_{\Gamma} Tr(\Phi(\delta\mathbf{u}) \wedge \bar{\eta}) + \int_{\Gamma} i_V dTr(\Phi \wedge \bar{\eta}) \\ & + \rho_f \omega^2 \int_{\Omega} (*_\beta d * \delta\mathbf{u} \wedge d * \bar{\mathbf{v}} - *_\mu d\delta\mathbf{u} \wedge d\bar{\mathbf{v}} - *_\Pi \delta\mathbf{u} \wedge \bar{\mathbf{v}}) \\ & + \rho_f \omega^2 \int_{\Gamma} i_V (*_\beta d * \mathbf{u} \wedge d * \bar{\mathbf{v}} - *_\mu d\mathbf{u} \wedge d\bar{\mathbf{v}} - *_\Pi \mathbf{u} \wedge \bar{\mathbf{v}}) \\ & + \rho_f \omega^2 \int_{\Gamma} Tr((\mathcal{P} - \Upsilon)(\delta\omega) \wedge \bar{\mathbf{v}}) + \rho_f \omega^2 \int_{\Gamma} i_V dTr(\mathcal{P} - \Upsilon) \wedge \bar{\mathbf{v}} = 0, \end{aligned}$$

for all smooth test form  $\eta \in \mathcal{D}\mathcal{F}^{0,\infty}(\mathbb{R}^3)$  and  $\mathbf{v} \in \mathcal{D}\mathcal{F}^{1,\infty}(\mathbb{R}^3)$ .

### 3. The characterization of shape derivatives for the elasto-acoustic coupled fields

For the classical Euclidean form in  $\mathbb{R}^3$ , the explicit characterization of shape derivatives  $(dp, d\mathbf{u})$  of solutions to elasto-acoustic fields will be deduced in Lemma 3.1. It slightly differs from the original BVP (1) with nonhomogeneous conditions on the interface. To make the readability of the proof easier, we have decide to postpone the regularity issues and address them later in Theorem 3.4 and Theorem 3.7.

**Lemma 3.1.** *Let the Lipschitz continuous velocity field  $V \in C([0, \varepsilon); C_0^2(\bar{D}, \mathbb{R}^d))$  and  $p^i = e^{ikx \cdot d}$ , then there always exists a unique scattering solution  $p^s(x) \in H_{\text{loc}}^1(\Omega^c)$  and  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  is unique modulo Jones frequencies for the BVP (1). And their shape derivatives  $(dp, d\mathbf{u})$  can be characterized as solutions to the following system,*

$$\begin{cases} -\Delta dp - k^2 dp = 0 & \text{in } \Omega^c, & (7a) \\ -\text{div} \sigma(d\mathbf{u}) - \rho_s \omega^2 d\mathbf{u} = 0 & \text{in } \Omega, & (7b) \\ \rho_f \omega^2 d\mathbf{u} \cdot \mathbf{n} = \frac{\partial dp}{\partial \mathbf{n}} - \Xi(\mathbf{u}, p^s, p^i) & \text{on } \Gamma, & (7c) \\ T d\mathbf{u} = -dp \mathbf{n} + \Lambda(\mathbf{u}, p^s, p^i) & \text{on } \Gamma, & (7d) \\ \lim_{r \rightarrow \infty} r \left( \frac{\partial dp}{\partial r} - ik dp \right) = 0 & r = |\mathbf{x}|, & (7e) \end{cases}$$

where functions  $\Xi(\mathbf{u}, p^s, p^i)$  in (7c) and  $\Lambda(\mathbf{u}, p^s, p^i)$  in (7d) are given by,

$$\Xi(\mathbf{u}, p^s, p^i) = \text{div}_{\Gamma}(\nabla_{\Gamma} p^s(V(0) \cdot \mathbf{n})) + \left( \Im \frac{\partial p^s}{\partial \mathbf{n}} + k^2 p^s + \frac{\partial}{\partial \mathbf{n}}(\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} - \frac{\partial p^i}{\partial \mathbf{n}}) \right) (V(0) \cdot \mathbf{n}), \quad (8a)$$

$$\Lambda(\mathbf{u}, p^s, p^i) = \text{div}_{\Gamma}(\sigma_{\Gamma}(\mathbf{u})(V(0) \cdot \mathbf{n})) + \left( \rho_s \omega^2 \mathbf{u} - \frac{\partial}{\partial \mathbf{n}}([p^s + p^i] \mathbf{n}) + \Im T \mathbf{u} \right) (V(0) \cdot \mathbf{n}), \quad (8b)$$

and column vector  $\mathbf{n}$  is a unit normal vector on  $\Gamma$  pointing to  $\Omega^c$ ,  $\sigma_{\Gamma}(\mathbf{u}) = (\mathcal{A}^{-1} \epsilon(\mathbf{u}))_{\Gamma}$  is defined by

$$\sigma_{\Gamma} = \sigma - \sigma \mathbf{n} \mathbf{n}^T \in \mathbb{C}^{3 \times 3}.$$

Moreover, the far field pattern  $dp_\infty$  of the shape derivative  $dp$  can be represented by

$$dp_\infty(\hat{x}) = - \left\langle ikdp(\hat{x} \cdot \mathbf{n}) + \frac{\partial dp}{\partial \mathbf{n}}, e^{ik\hat{x} \cdot \mathbf{y}} \right\rangle_{H^{-\frac{1}{2}}(\mathcal{B})H^{\frac{1}{2}}(\mathcal{B})}.$$

**Proof.** Based on the equivalence form given in Lemma 2.2, we apply the variational form (3) of BVP (2) in  $\Omega_R \times \Omega$ , that is,  $\mathcal{A}((p^s, \mathbf{u})(q, \mathbf{v})) = l(q, \mathbf{v})$  for any  $(q, \mathbf{v}) \in H^1(\Omega_R) \times \mathbf{H}^1(\Omega)$ . And the proof falls naturally into four steps.

Step 1. Extend the symmetric operator  $\sigma(\mathbf{u}) = \mathcal{A}^{-1}\epsilon(\mathbf{u})$ . We start to extend the compliance tensor  $\mathcal{C} = \mathcal{A}^{-1}$  to a symmetric and positive definite operator  $\tilde{\mathcal{C}}$  mapping from  $\mathbb{M}$  into  $\mathbb{M}$ , where  $\mathbb{M}$  is the space of  $\mathbb{C}^{3 \times 3}$  matrices, such that

$$\tilde{\mathcal{C}}B = \begin{cases} CB, & \forall B \in \mathbb{S}, \\ 0, & \forall B \in \mathbb{K}, \end{cases}$$

where  $\mathbb{K}$  is the set of all anti-symmetric matrices in  $\mathbb{C}^{3 \times 3}$  (i.e.,  $\forall B \in \mathbb{K}, (B)_{ij} = -(B)_{ji}, 1 \leq i, j \leq 3$ ). So for any  $\mathbb{C}^{3 \times 3}$  matrix function  $M \in \mathbb{M}$ ,

$$M = \frac{M + M^T}{2} + \frac{M - M^T}{2} \quad \text{and} \quad \tilde{\mathcal{C}}M = \mathcal{C} \left( \frac{M + M^T}{2} \right).$$

Then the variational form (3) can be rewritten as, for any  $(q, \mathbf{v}) \in H^1(\Omega_R) \times \mathbf{H}^1(\Omega)$ , such that

$$\begin{aligned} & \int_{\Omega_R} (\nabla p^s \cdot \nabla \bar{q} - k^2 p^s \bar{q}) dx + \langle \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n}, q \rangle_\Gamma - \langle \mathcal{T} p^s, q \rangle_{\mathcal{B}} + \rho_f \omega^2 \int_\Gamma p^s \mathbf{n} \cdot \bar{\mathbf{v}} ds \\ & + \rho_f \omega^2 \int_\Omega (\tilde{\mathcal{C}}(\nabla \mathbf{u}) : \nabla \bar{\mathbf{v}} - \rho_s \omega^2 \mathbf{u} \cdot \bar{\mathbf{v}}) dx = \left\langle \frac{\partial p^i}{\partial \mathbf{n}}, q \right\rangle_\Gamma - \rho_f \omega^2 \langle p^i \mathbf{n}, \mathbf{v} \rangle_\Gamma. \end{aligned}$$

For the velocity field  $V(0) \in C_0^2(\bar{D}; \mathbb{R}^3)$ , it will cause no confusing if we use the same notation  $V(0) \in C^2(\mathbb{R}^3; \mathbb{R}^3)$  as a continuous extension with  $V = 0$  for exterior domain  $\bar{D}^c$ .

Step 2.  $\mathbf{u}$  is viewed as a vector-valued 0-form or musical isomorphism 1-form,  $\tilde{\mathcal{C}}$  is treated as an operator which maps from vector-valued 1-form to vector-valued 2-form. Denoting the derivatives  $(dp, d\mathbf{u})$ , in view of Lemma 2.4, yields,

$$\begin{aligned} & \int_{\Omega_R} (\nabla dp \cdot \nabla \bar{q} - k^2 dp \bar{q}) dx - \int_\Gamma (\nabla p^s \cdot \nabla \bar{q} - k^2 p^s \bar{q}) (V(0) \cdot \mathbf{n}) ds + \rho_f \omega^2 \langle d\mathbf{u} \cdot \mathbf{n}, q \rangle_\Gamma \\ & + \rho_f \omega^2 \int_\Gamma \left( \frac{\partial(\mathbf{u} \cdot \mathbf{n} \bar{q})}{\partial \mathbf{n}} + \mathfrak{S} \mathbf{u} \cdot \mathbf{n} \bar{q} \right) (V(0) \cdot \mathbf{n}) ds - \int_\Gamma \left( \frac{\partial}{\partial \mathbf{n}} \left( \frac{\partial p^i}{\partial \mathbf{n}} \bar{q} \right) + \mathfrak{S} \frac{\partial p^i}{\partial \mathbf{n}} \bar{q} \right) (V(0) \cdot \mathbf{n}) ds \\ & - \langle \mathcal{T} dp, q \rangle_{\mathcal{B}} + \rho_f \omega^2 \int_\Omega (\tilde{\mathcal{C}}(d\mathbf{u}) : \nabla \bar{\mathbf{v}} - \rho_s \omega^2 d\mathbf{u} \cdot \bar{\mathbf{v}}) dx \\ & + \rho_f \omega^2 \langle dp \mathbf{n}, \mathbf{v} \rangle_\Gamma + \rho_f \omega^2 \int_\Gamma (\tilde{\mathcal{C}}(\nabla \mathbf{u}) : \nabla \bar{\mathbf{v}} - \rho_s \omega^2 \mathbf{u} \cdot \bar{\mathbf{v}}) (V(0) \cdot \mathbf{n}) ds \\ & + \rho_f \omega^2 \int_\Gamma \left( \frac{\partial((p^s + p^i) \mathbf{n} \cdot \bar{\mathbf{v}})}{\partial \mathbf{n}} + \mathfrak{S}((p^i + p^s) \mathbf{n} \cdot \bar{\mathbf{v}}) \right) (V(0) \cdot \mathbf{n}) ds = 0. \end{aligned} \tag{9}$$

By the definition of  $\tilde{C}$ , we replace it by  $\sigma$  again. Recall the tangential components of stress tensor  $\sigma_\Gamma(\mathbf{u}) = \sigma(\mathbf{I} - \mathbf{nn}^T) \in \mathbb{C}^{3 \times 3}$ , split the normal and tangential directions of the tensor row-wise (10a)-(10b),

$$\begin{cases} \sigma(\mathbf{u}) : \nabla \bar{\mathbf{v}} = \sigma_\Gamma(\mathbf{u}) : \nabla_\Gamma \bar{\mathbf{v}} + (\sigma(\mathbf{u}) \cdot \mathbf{n}) \cdot \frac{\partial \bar{\mathbf{v}}}{\partial \mathbf{n}}, \end{cases} \quad (10a)$$

$$\begin{cases} \nabla p^s \cdot \nabla \bar{q} = \nabla_\Gamma p^s \cdot \nabla_\Gamma \bar{q} + \frac{\partial p^s}{\partial \mathbf{n}} \frac{\partial \bar{q}}{\partial \mathbf{n}}, \end{cases} \quad (10b)$$

$$\sigma_\Gamma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{0}, \quad (10c)$$

$$\nabla_\Gamma p^s \cdot \mathbf{n} = 0. \quad (10d)$$

Moreover, apply the corresponding tangential Green or Stokes formulas in [11, Eq.(5.27), p. 498] into the concrete terms of (9), that is,

$$\begin{aligned} \int_\Gamma (\mathbf{div}_\Gamma (\sigma_\Gamma(\mathbf{u})(V(0) \cdot \mathbf{n})) \cdot \bar{\mathbf{v}} + \sigma_\Gamma(\mathbf{u}) : \nabla_\Gamma \bar{\mathbf{v}}(V(0) \cdot \mathbf{n})) \, ds &= \int_\Gamma \mathfrak{S}(\sigma_\Gamma(\mathbf{u}) \cdot \mathbf{n}) \cdot \bar{\mathbf{v}}(V(0) \cdot \mathbf{n}) \, ds = 0, \\ \int_\Gamma (\mathbf{div}_\Gamma (\nabla_\Gamma p^s(V(0) \cdot \mathbf{n})) \bar{q} + \nabla_\Gamma p^s \cdot \nabla_\Gamma \bar{q}(V(0) \cdot \mathbf{n})) \, ds &= \int_\Gamma \mathfrak{S}(\nabla_\Gamma p^s \cdot \mathbf{n} \bar{q})(V(0) \cdot \mathbf{n}) \, ds = 0. \end{aligned}$$

Then the integrand terms  $\sigma_\Gamma(\mathbf{u}) : \nabla_\Gamma \bar{\mathbf{v}}(V(0) \cdot \mathbf{n})$  and  $\nabla_\Gamma p^s \cdot \nabla_\Gamma \bar{q}(V(0) \cdot \mathbf{n})$  in (10a)-(10b) can be replaced by  $-\mathbf{div}_\Gamma (\sigma_\Gamma(V(0) \cdot \mathbf{n})) \cdot \bar{\mathbf{v}}$  and  $-\mathbf{div}_\Gamma (\nabla_\Gamma p^s(V(0) \cdot \mathbf{n})) \bar{q}$ . Combine the integration by parts in the sense of Green formula and the interface conditions (1c)-(1d),

$$\begin{aligned} & \int_{\Omega_R} (-\Delta - k^2) \, dp \bar{q} \, dx - \int_\Gamma \left( \frac{\partial dp}{\partial \mathbf{n}} - \mathbf{div}_\Gamma (\nabla_\Gamma p^s(V(0) \cdot \mathbf{n})) - k^2 p^s(V(0) \cdot \mathbf{n}) - \rho_f \omega^2 \mathbf{du} \cdot \mathbf{n} \right) \bar{q} \, ds \\ & + \int_\Gamma \left\{ \frac{\partial}{\partial \mathbf{n}} \left( \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} - \frac{\partial p^i}{\partial \mathbf{n}} \right) + \mathfrak{S} \frac{\partial p^s}{\partial \mathbf{n}} \right\} \bar{q}(V(0) \cdot \mathbf{n}) \, ds + \left\langle \frac{\partial dp}{\partial \mathbf{n}} - \mathcal{T} dp, q \right\rangle_{\mathcal{B}} \\ & + \rho_f \omega^2 \int_\Omega (-\mathbf{div} \sigma(\mathbf{du}) - \rho_s \omega^2 \mathbf{du}) \cdot \bar{\mathbf{v}} \, dx + \rho_f \omega^2 \int_\Gamma (T(\mathbf{du}) + dp \mathbf{n} - \mathbf{div}_\Gamma (\sigma_\Gamma(\mathbf{u})(V(0) \cdot \mathbf{n}))) \cdot \bar{\mathbf{v}} \, ds \\ & + \rho_f \omega^2 \int_\Gamma \left( \frac{\partial}{\partial \mathbf{n}} ([p^s + p^i] \mathbf{n}) - \rho_s \omega^2 \mathbf{u} - \mathfrak{S} T \mathbf{u} \right) (V(0) \cdot \mathbf{n}) \cdot \bar{\mathbf{v}} \, ds = 0. \end{aligned}$$

Step 3. One can choose the smooth dense subspaces of testing functions  $(q, \mathbf{v})$ . Indeed, let  $(q, \mathbf{v}) \in \mathcal{D}(\Omega_R) \times \mathcal{D}(\Omega)$ . If  $(q, \mathbf{v}) = (q, \mathbf{0})$ , therefore, we deduce (11a) in the distribution sense  $\mathcal{D}'(\Omega_R)$ . Moreover, let  $q \in \mathcal{D}(\Omega_R)$ , and take  $q$  that vanishes on  $\mathcal{B}$ , we can obtain the transmission condition (11b) on the fluid-structure interface  $\Gamma$ ,

$$\begin{cases} -\Delta dp - k^2 dp = 0 & \text{in } \mathcal{D}'(\Omega_R), \end{cases} \quad (11a)$$

$$\begin{cases} \frac{\partial dp}{\partial \mathbf{n}} = \rho_f \omega^2 \mathbf{du} \cdot \mathbf{n} + \Xi(\mathbf{u}, p^s, p^i) & \text{in } H^{-\frac{1}{2}}(\Gamma). \end{cases} \quad (11b)$$

Similarly (resp.  $\mathbf{du}$ ),  $\mathbf{du}$  satisfies the Navier equation (7b) and another interface condition (7d). Since  $q$  vanishes in a neighborhood of  $\Gamma$ , and obtain the boundary condition on  $\mathcal{B}$ ,

$$\frac{\partial dp}{\partial \mathbf{n}}|_+ = \mathcal{T}(dp) = \frac{\partial dp}{\partial \mathbf{n}}|_- \quad \text{in } H^{-\frac{1}{2}}(\mathcal{B}).$$

Step 4. For the radiation condition (7e), it results from the asymptotic behavior of the scattered field with  $M_t$  is differentiable at  $t = 0$ , because  $p_t^s$  and  $p_{t,\infty}^s$  are differentiable in [12, Lem. 5, p. 274], that is ,

$$\begin{cases} p_t^s(\mathbf{x}) = \frac{e^{ikr}}{r} \left( p_{t,\infty}^s(\hat{\mathbf{x}}) + M_t \mathcal{O}\left(\frac{1}{r}\right) \right) & \text{in } \Omega_t^c, \end{cases} \quad (12a)$$

$$\begin{cases} p_\infty^s(\hat{\mathbf{x}}) = \frac{1}{4\pi} \int_{\mathcal{B}} p^s(\mathbf{y}) \frac{\partial}{\partial \mathbf{n}} e^{-ik\hat{\mathbf{x}} \cdot \mathbf{y}} \, ds - \langle \mathcal{T} p^s, e^{ik\hat{\mathbf{x}} \cdot \mathbf{y}} \rangle & \hat{\mathbf{x}} \in \mathbb{S}^2. \end{cases} \quad (12b)$$

Thus, based on the far-field expression (12b) and asymptotic pattern (12a), then take shape derivative,

$$\begin{cases} dp_\infty(\hat{\mathbf{x}}) = -\left\langle ikdp(\hat{\mathbf{x}} \cdot \mathbf{n}) + \frac{\partial dp}{\partial \mathbf{n}}, e^{ik\hat{\mathbf{x}} \cdot \mathbf{y}} \right\rangle_{H^{-\frac{1}{2}}(\mathcal{B}; \mathbb{C}) H^{\frac{1}{2}}(\mathcal{B}; \mathbb{C})}, \\ dp(\mathbf{x}) = \frac{e^{ikr}}{r} \left( \frac{\partial p_{t,\infty}^s(\hat{\mathbf{x}})}{\partial t} \Big|_{t=0} + O\left(\frac{1}{r}\right) \right). \end{cases} \quad (13a)$$

$$(13b)$$

Therefore, the radiation condition (7e) follows from (14), that is,

$$r \left( \frac{\partial dp}{\partial r} - ikdp \right) = -\frac{e^{ikr}}{r} \left( \frac{\partial p_{t,\infty}^s(\hat{\mathbf{x}})}{\partial t} \Big|_{t=0} + O\left(\frac{1}{r}\right) \right) + O\left(\frac{1}{r^2}\right). \quad (14)$$

□

**Remark 3.2.** The equivalent proof for Lemma 3.1 can be generalized by reformulating variational forms on the weighted Sobolev spaces. Here, we just clarify the main idea and difference with proof of Lemma 3.1. The direct variational form of original BVP (1) given in [1, Eq.(12), p. 576]:

Seek  $(p^s, \mathbf{u}) \in H_p^1(\Omega^c) \times \mathbf{H}^1(\Omega)$ , for any  $(q, \mathbf{v}) \in H_{1/p}^1(\Omega^c) \times \mathbf{H}^1(\Omega)$  such that

$$\mathcal{A}^*((p^s, \mathbf{u}), (q, \mathbf{v})) = l(q, \mathbf{v}),$$

where

$$\begin{aligned} \mathcal{A}^*((p^s, \mathbf{u}), (q, \mathbf{v})) &= \int_{\Omega^c} (\nabla p^s \cdot \nabla \bar{q} - k^2 p^s \bar{q}) dx + \langle \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n}, q \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} \\ &\quad + \rho_f \omega^2 \left( \int_{\Omega} (\boldsymbol{\sigma}(\mathbf{u}) : \nabla \bar{\mathbf{v}} - \omega^2 \rho_s \mathbf{u} \cdot \bar{\mathbf{v}}) dx + \langle p^s \mathbf{n}, \mathbf{v} \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} \right). \end{aligned}$$

If we consider  $\mathcal{A}^*((p^s, \mathbf{u}), (0, \mathbf{v})) = l(0, \mathbf{v})$ , follow the standard steps in the proof of Lemma 3.1, one can derive the shape derivative of solution  $p^s$  in  $\Omega^c$ , which satisfies the following variational formula,

$$\begin{aligned} &\int_{\Omega^c} (-\Delta dp - k^2 dp) \bar{q} dx + \int_{\Gamma} \left( \frac{\partial}{\partial \mathbf{n}} (\rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} - \frac{\partial p^i}{\partial \mathbf{n}}) - \mathfrak{S} \frac{\partial p^s}{\partial \mathbf{n}} \right) \bar{q} (V(0) \cdot \mathbf{n}) ds \\ &\quad - \int_{\Gamma} \left( \frac{\partial dp}{\partial \mathbf{n}} - \mathbf{div}_{\Gamma} (\nabla_{\Gamma} p^s (V(0) \cdot \mathbf{n})) - k^2 p^s (V(0) \cdot \mathbf{n}) - \rho_f \omega^2 d\mathbf{u} \cdot \mathbf{n} \right) \bar{q} ds = 0. \end{aligned}$$

For any bounded domain  $K \subset \Omega^c$ , the weighted sobolev Space in  $H_p(K)$  coincides with the classical Hilbert space  $H^1(K)$  and their norms are equivalent, which are proved in [1, p. 574] and [14, p. 18]. So if we restrict in the bounded domain  $K = \Omega_R$ , then the above variational formula is just one part of Lemma 3.1. Similarly, if  $\mathcal{A}^*((p^s, \mathbf{u}), (q, \mathbf{0})) = l(q, \mathbf{0})$ , thus, we can attain another part.

**Remark 3.3.** Lemma 3.1 shows the explicit characterization of shape derivatives  $(dp, d\mathbf{u})$ , nevertheless, the proof is not really complete since we omit analyzing the sense of trace terms in (7c)-(7d) on  $\Gamma$ . To illustrate the existence and regularity of the solutions to BVP (7), it requires to analyze the regularity conditions, and it's in general not true for only Lipschitz boundary.

The suitable Sobolev spaces for the above characterization is omitted, because it requires distinguishing different cases depending on the regularities of  $\Omega$ . So, for a high regularity, we suppose  $\Gamma$  is of  $C^2$ , the following Theorem 3.4 can be obtained.

**Theorem 3.4.** *Let  $\Gamma$  be of class  $C^2$  with the same conditions given in Lemma 3.1, for the shape derivatives of solutions to BVP (1), there exists a unique  $dp \in H_{\text{loc}}^1(\Omega^c)$  and  $du \in \mathbf{H}^1(\Omega)$  is unique modulo Jones frequencies.*

**Proof.** The previous existence and uniqueness conclusion in Lemma 2.3 or 3.1 can be applied for the BVP (7) about the shape derivatives. So we just make sense of the nonhomogeneous transmission conditions, that is, we just need to prove  $\Xi(\mathbf{u}, p^s, p^i) \in H^{-1/2}(\Gamma)$ ,  $\Lambda(\mathbf{u}, p^s, p^i) \in \mathbf{H}^{-1/2}(\Gamma)$ .

If  $\Gamma$  is of class  $C^2$ ,  $\mathbf{n} \in \mathbf{C}^1(\Gamma, \mathbb{R}^3)$  with an unitary extension vector field  $\mathbf{N}_0 \in \mathbf{C}^1(\mathbb{R}^3, \mathbb{R}^3)$ , which can also guarantee the continuity of additive curvature  $\mathfrak{S} = 2H = -\text{Tr}(D\mathbf{n})|_{\Gamma}$ . For the original boundary value problem (1), the regularity of data on the interface impacts on the regularity of the solutions. Since the incident field  $p^i$  is regular as a plane wave and  $p^s \in H_{\text{loc}}^1(\Omega^c)$ , based on the standard trace theorem,  $p^i \mathbf{n} \in \mathbf{H}^{1/2}(\Gamma)$ ,  $\partial p^i / \partial \mathbf{n} \in H^{1/2}(\Gamma)$  and  $p^s \mathbf{n} \in \mathbf{H}^{1/2}(\Gamma)$ , thus,  $\rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} \in H^{1/2}(\Gamma)$  and  $T\mathbf{u} \in \mathbf{H}^{1/2}(\Gamma)$  on  $\Gamma$ . Consider the Navier equation with the traction boundary condition, applying the classical regularity analysis, we can check that  $\mathbf{u} \in \mathbf{H}^2(\Omega)$ . Thus, similarly,  $p^s \in H_{\text{loc}}^2(\Omega^c)$ .

$\mathbf{u} \in \mathbf{H}^2(\Omega)$  implies that  $\sigma(\mathbf{u}) \in [H^1(\Omega)]^{3 \times 3}$ , according to  $\mathbf{n} \in \mathbf{C}^1(\Gamma)$ ,  $T\mathbf{u} = \sigma(\mathbf{u})\mathbf{n} \in \mathbf{H}^{1/2}(\Gamma)$ , which guarantee that the multiplier  $T\mathbf{u}(\mathbf{V}(0) \cdot \mathbf{n})$  is of  $\mathbf{H}^{1/2}(\Gamma)$ . On the other hand,  $\sigma_{\Gamma}(\mathbf{u}) = \sigma(\mathbf{I} - \mathbf{n}\mathbf{n}^T) \in [H^{1/2}(\Gamma)]^{3 \times 3}$ , which can similarly deduce that the multiplier  $\sigma_{\Gamma}(\mathbf{u})(\mathbf{V}(0) \cdot \mathbf{n})$  is of  $[H^{1/2}(\Gamma)]^{3 \times 3}$ . If the surface is regular, then  $V_{\gamma} = \gamma_{\tau}(\mathbf{H}^{1/2}(\Gamma))$  in [5, prop. 3.6, p. 855] with the “tangential trace” mapping  $\gamma_{\tau} : \mathcal{D}(\bar{\Omega}) \rightarrow L^2(\Gamma)$ , is just the standard Hilbertian Sobolev space of tangential vector field of order 1/2 :  $\mathbf{TH}^{1/2}(\Gamma) = \{\mathbf{v} | \mathbf{v} \in \mathbf{H}^{1/2}(\Gamma), \mathbf{v} \cdot \mathbf{n} = 0\}$ , that is,

$$\text{div}_{\Gamma} : \mathbf{TH}^{1/2}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma). \quad (15)$$

Since  $\sigma_{\Gamma}(\mathbf{u}) \cdot \mathbf{n}(\mathbf{V}(0) \cdot \mathbf{n}) = 0$ , then  $\sigma_{\Gamma}(\mathbf{u})(\mathbf{V}(0) \cdot \mathbf{n})|_{\Gamma} = (\sigma_{\Gamma}^1(\mathbf{u}), \sigma_{\Gamma}^2(\mathbf{u}), \sigma_{\Gamma}^3(\mathbf{u}))^T (\mathbf{V} \cdot \mathbf{n})|_{\Gamma} \in [\mathbf{TH}^{1/2}(\Gamma)]^3$ , according to (15) or [5, prop. 3.6, p. 855],  $\text{div}_{\Gamma}(\sigma_{\Gamma}^j(\mathbf{u})(\mathbf{V}(0) \cdot \mathbf{n})) \in H^{-\frac{1}{2}}(\Gamma)$ ,  $1 \leq j \leq 3$ . That is,  $\text{div}_{\Gamma}(\sigma_{\Gamma}(\mathbf{u})(\mathbf{V}(0) \cdot \mathbf{n})) \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ . Another term  $\frac{\partial}{\partial \mathbf{n}}([p^s + p^i]\mathbf{n})|_{\Gamma} = (\mathbf{N}_0 \cdot \nabla[(p^s + p^i)(\mathbf{N}_0)_k])_{k=1}^3|_{\Gamma} = (\mathbf{N}_0 \cdot \nabla(p^s + p^i))\mathbf{N}_0|_{\Gamma} \in \mathbf{H}^{1/2}(\Gamma) \subset \mathbf{H}^{-1/2}(\Gamma)$ . So collect the previous regularities of terms in  $\Lambda(\mathbf{u}, p^s, p^i)$ , the expression  $\Lambda(\mathbf{u}, p^s, p^i)$  in (8b) is well-defined in  $\mathbf{H}^{-1/2}(\Gamma)$ .

$p^s \in H_{\text{loc}}^2(\Omega^c)$  implies  $\nabla p^s \in \mathbf{H}_{\text{loc}}^1(\Omega^c)$ , thus,  $\nabla p^s|_{\Gamma} \in \mathbf{H}^{1/2}(\Gamma)$ . Hence, it follows that the multiplier  $\frac{\partial p^s}{\partial \mathbf{n}}(\mathbf{V}(0) \cdot \mathbf{n})|_{\Gamma} = \mathbf{N}_0 \cdot \nabla p^s(\mathbf{V}(0) \cdot \mathbf{N}_0)|_{\Gamma} \in \mathbf{H}^{1/2}(\Gamma)$ . Consequently, knowing that  $\nabla_{\Gamma} p^s|_{\Gamma} = \nabla p^s|_{\Gamma} - \frac{\partial p^s}{\partial \mathbf{n}}\mathbf{n}|_{\Gamma} \in \mathbf{H}^{1/2}(\Gamma)$ , then  $\nabla_{\Gamma} p^s(\mathbf{V}(0) \cdot \mathbf{n}) \in \mathbf{TH}^{1/2}(\Gamma)$ . As the operator  $\text{div}_{\Gamma}$  given in (15), we conclude that  $\text{div}_{\Gamma}(\nabla_{\Gamma} p^s(\mathbf{V}(0) \cdot \mathbf{n})) \in H^{-1/2}(\Gamma)$ . Obviously, according to the trace theorem in [26, Thm. 3.1, p. 266],  $\mathbf{u} \cdot \mathbf{n}|_{\Gamma} \in H^{3/2}(\Gamma)$  and normal vector  $\frac{\partial}{\partial \mathbf{n}}$  on  $\Gamma$  mapping  $H^{3/2}(\Gamma)$  to  $H(\Gamma)$ , therefore,  $\frac{\partial}{\partial \mathbf{n}}(\mathbf{u} \cdot \mathbf{n}) \in H^{-1/2}(\Gamma)$  and then  $\frac{\partial}{\partial \mathbf{n}}(\mathbf{u} \cdot \mathbf{n})(\mathbf{V}(0) \cdot \mathbf{n}) \in \mathbf{H}^{-1/2}(\Gamma)$ . Using the embedding theorem and the previous regularities of each term of  $\Xi(\mathbf{u}, p^s, p^i)$  in (8a), then we can obtain  $\Xi(\mathbf{u}, p^s, p^i) \in H^{-1/2}(\Gamma)$ .  $\square$

**Remark 3.5.** *The condition that  $\Gamma$  is of class  $C^2$  can be replaced by Lipschitz continuous with piecewise  $C^{2,1}$ , the analysis will be focused locally on each regular piece of  $\partial\Omega$ . Consequently,  $\mathfrak{S}$  and the transmission conditions are well-defined on regular piece of boundary surface. Moreover, in the previous Theorem 3.4 and Lemma 3.1, we assume that velocity field  $\mathbf{V}$  belongs to  $\mathbf{C}^2$ , actually,  $\mathbf{C}^1$  is sufficient because it only needs the multiplier of  $H^{1/2}$  in the proof.*



**Corollary 3.6.** Let  $dp_\infty$  be the far-field pattern of the solution  $dp$  to the BVP(7),  $\frac{\partial p_{t,\infty}^s}{\partial t}|_{t=0}$  is the derivative of the far-field pattern  $p_{t,\infty}^s$  at  $t = 0$ ,

$$dp_\infty = \frac{\partial p_{t,\infty}^s}{\partial t}|_{t=0}.$$

Corollary 3.6 is an immediate consequence of Eq.(13a)-(13b) and the uniqueness of the far-field pattern of the acoustic scattered field is satisfied when  $\Gamma$  is of  $C^2$ .

If the polyhedral boundary is of class  $C^{1,1}$  given in [14, Sec. III. 3.3, p. 234], we discuss later in the Corollary 3.7. Notice that  $\mathfrak{H}$  is not necessary exist, interface conditions (8) are not employed for this case.

Assume that  $\Gamma = \bigcup_{j=1}^N \Gamma_j$  defines the set of  $C^{1,1}$  faces  $\{\Gamma_j, 1 \leq j \leq N\}$ ,  $\mathbf{n}_j$  is the normal vector on  $\Gamma_j$  and  $\mathbf{n}_j = \mathbf{n}$  a.e. on  $\Gamma_j$ . For abbreviation, the space  $\tilde{H}^s(\Gamma_j) \subset H^s(\Gamma_j)$ ,  $1 \leq j \leq N$  with continuation by zero to  $\Gamma$  belongs to  $H^s(\Gamma)$ . And we introduce two spaces denoted by  $\mathbf{H}(\nabla \cdot \boldsymbol{\sigma}, \Omega) = \{\mathbf{u} \in L^2(\Omega), \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) \in L^2(\Omega)\}$ ,  $H(\Delta, \Omega^c) = \{p \in L_{\text{loc}}^2(\Omega^c), \Delta p \in L_{\text{loc}}^2(\Omega^c)\}$ . The generalized Green-like formula for the extended spaces  $H(\Delta, \Omega^c)$ ,  $\mathbf{H}(\nabla \cdot \boldsymbol{\sigma}, \Omega)$  is given in [14, p. 238], the mappings  $p \rightarrow (p|_{\Gamma_j}, \frac{\partial p}{\partial \mathbf{n}}|_{\Gamma_j})$  and  $\mathbf{u} \rightarrow (\mathbf{u}|_{\Gamma_j}, \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}|_{\Gamma_j})$  have continuous from  $H(\Delta, \Omega^c)$  and  $\mathbf{H}(\nabla \cdot \boldsymbol{\sigma}, \Omega)$  into  $(\tilde{H}^{-1/2}(\Gamma_j), \tilde{H}^{-3/2}(\Gamma_j))$  and  $(\tilde{H}^{-1/2}(\Gamma_j), \tilde{H}^{-3/2}(\Gamma_j))$  respectively, for all  $1 \leq j \leq N$ .

**Corollary 3.7.** Let  $\Gamma$  be of polyhedron of class  $C^{1,1}$ , there exist shape derivatives  $dp \in H_{\text{loc}}^{1/2}(\Omega^c)$  and  $d\mathbf{u} \in \mathbf{H}^{1/2}(\Omega)$  characterized as solutions to elasto-acoustic equations in (7a)-(7b) with the following transmission conditions (16) and radiation condition (7e),

$$\begin{cases} \tilde{\Xi}(\mathbf{u}, p^s, p^i) = (\rho_f \omega^2 \nabla \mathbf{u} - \nabla(\nabla(p^s + p^i))(V(0) \cdot \mathbf{n}) - (\rho_f \omega^2 \mathbf{u} - \nabla(p^s + p^i)) \cdot DV^T(0)\mathbf{n}, & (16a) \\ \tilde{\Lambda}(\mathbf{u}, p^s, p^i) = \boldsymbol{\sigma}(\mathbf{u})DV^T(0)\mathbf{n} - (V^T \nabla \boldsymbol{\sigma}(\mathbf{u}))_{1 \leq i \leq 3} \mathbf{n} - \nabla(p^s + p^i) \cdot V(0)\mathbf{n} + (p^s + p^i)DV^T(0)\mathbf{n}. & (16b) \end{cases}$$

We omit the proof due to the partial similarity with analysis in [14]. Two considerations are contributed to the proof of Corollary 3.7, one is the transmission conditions (16a)-(16b); they can be derived through shape derivatives of functions defined by composition. Take shape derivatives with respect to pseudo-time  $t$  on a hold-on domain containing the perturbation boundary  $\Gamma_t \cup \Gamma$ , and one can deduce through the standard steps by repeating calculation given in the proof. We leave it to the reader.

Another is making sense of transmission terms in corresponding Sobolev spaces, one can check the interface functions  $\tilde{\Xi}(\mathbf{u}, p^s, p^i) \in \tilde{H}^{-3/2}(\Gamma_j)$ ,  $\tilde{\Lambda}(\mathbf{u}, p^s, p^i) \in \tilde{H}^{-3/2}(\Gamma_j)$ , which agree with the traces of traction operator and normal derivative of  $dp$  and  $d\mathbf{u}$  in (16a)-(16b).

#### 4. Shape derivatives of solutions to the Elasto-acoustic equations with random interfaces

In this section, we first collect some concepts of random interface, and based on them we propose the corresponding random version of stochastic elasto-acoustic equations. According to the theoretical results in this article, we can derive first order shape derivatives for the random version, they are formulated in Theorem 4.1. In addition, we focus on the expectation and variance of random solutions in two lemmas.

Let

$$\mathcal{E} := \{v \in C^{2,1}(\Gamma, \mathbb{R}^3) : \|v\|_{C^{2,1}(\Gamma, \mathbb{R}^3)} \leq 1\},$$

where the space  $C^{2,1}(\Gamma, \mathbb{R}^3)$  is introduced by Lipschitzian mapping (see [11, 34]). We denote the random interface  $\Gamma_\delta$  characterized in (17) with the random field  $\kappa \in L^2(C^{2,1}(\Gamma, \mathbb{R}^3); \Theta)$  for each realization  $\xi \mapsto$



$\kappa(\cdot, \xi) \in \mathcal{E}$ . The perturbation amplitude  $\delta$  satisfies  $0 \leq \delta < \delta_0$  with some small positive constant  $\delta_0 > 0$ . The random interface  $\Gamma_\delta$  is given by

$$\Gamma_\delta(\xi) := \{\mathbf{x} + \delta\kappa(\mathbf{x}, \xi)\mathbf{n}(\mathbf{x}) : \mathbf{x} \in \Gamma\}. \quad (17)$$

For simplicity of notation, let  $\tilde{\kappa}$  and  $\tilde{\mathbf{n}}$  be any smoothness-preserving extensions of  $\kappa$  and  $\mathbf{n}$  into  $\mathbb{R}^3$ . That is, for any  $\xi \in \Theta$ , the so-called velocity  $\mathbf{V}(\cdot, \xi)$  generates the transformation mapping  $T_\delta(\cdot, \xi)$ .  $T_\delta(\cdot, \xi) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  transforms  $\Gamma$  into  $\Gamma_\delta(\xi)$ ,

$$T_\delta(\mathbf{x}, \xi) := \mathbf{x} + \delta\tilde{\kappa}(\mathbf{x}, \xi)\tilde{\mathbf{n}}(\mathbf{x}) := \mathbf{x} + \delta\mathbf{V}(\mathbf{x}, \xi), \quad \mathbf{x} \in \mathbb{R}^3.$$

Thus, compared to the deterministic elasto-acoustic equations (1), the corresponding random interface system is given by adding random variables, i.e.  $p^s = p^s(\mathbf{x}, \xi)$ ,  $\mathbf{u} = \mathbf{u}(\mathbf{x}, \xi)$ ,

$$\begin{cases} -\Delta p^s(\mathbf{x}, \xi) - k^2 p^s(\mathbf{x}, \xi) = 0 & \text{in } \Omega_\delta^c(\xi) \times \Theta, \end{cases} \quad (18a)$$

$$\begin{cases} -\operatorname{div} \sigma(\mathbf{u}(\mathbf{x}, \xi)) - \rho_s \omega^2 \mathbf{u}(\mathbf{x}, \xi) = 0 & \text{in } \Omega_\delta(\xi) \times \Theta, \end{cases} \quad (18b)$$

$$\begin{cases} \rho_f \omega^2 \mathbf{u}(\mathbf{x}, \xi) \cdot \mathbf{n} = \frac{\partial p^s(\mathbf{x}, \xi)}{\partial \mathbf{n}} + \frac{\partial p^i}{\partial \mathbf{n}} & \text{on } \Gamma_\delta(\xi) \times \Theta, \end{cases} \quad (18c)$$

$$\begin{cases} T\mathbf{u}(\mathbf{x}, \xi) = -p^s(\mathbf{x}, \xi)\mathbf{n} - p^i\mathbf{n} & \text{on } \Gamma_\delta(\xi) \times \Theta, \end{cases} \quad (18d)$$

$$\begin{cases} \lim_{r \rightarrow \infty} r \left( \frac{\partial p^s(\mathbf{x}, \xi)}{\partial r} - ikp^s(\mathbf{x}, \xi) \right) = 0 & r = \|\mathbf{x}\|_2, \end{cases} \quad (18e)$$

where  $\Omega_\delta^c = \mathbb{R}^3 \setminus \overline{\Omega_\delta}$ . Here, we require the nominal interface  $\Gamma \in C^{3,1}$ , which guarantees the stochastic interface satisfies  $\Gamma_\delta \in C^{2,1}(\Gamma, \mathbb{R}^3)$  almost sure for  $\xi \in \Theta$ .

For any random field  $\kappa \in L^2(C^{2,1}(\Gamma, \mathbb{R}^3); \Theta)$ , the mean, two-point correlation function, and covariance function of  $\kappa$  are defined by

$$\mathbb{E}_\kappa(\mathbf{x}) = \mathbb{E}[\kappa(\mathbf{x}, \xi)] = \int_\Theta \kappa(\mathbf{x}, \xi) dP(\xi), \quad \mathbf{x} \in \Gamma,$$

$$\operatorname{Cor}_\kappa(\mathbf{x}, \mathbf{y}) = \mathbb{E}[\kappa(\mathbf{x}, \xi)\kappa(\mathbf{y}, \xi)] = \int_\Theta \kappa(\mathbf{x}, \xi)\kappa(\mathbf{y}, \xi) dP(\xi), \quad \mathbf{x}, \mathbf{y} \in \Gamma,$$

$$\operatorname{Covar}_\kappa(\mathbf{x}, \mathbf{y}) = \operatorname{Cor}_\kappa(\mathbf{x}, \mathbf{y}) - \mathbb{E}_\kappa(\mathbf{x})\mathbb{E}_\kappa(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \Gamma,$$

where  $\mathbb{E}$  is the *expectation* with respect to the probability measure  $P$ . Without loss of generality, we may assume

$$\mathbb{E}_\kappa(\mathbf{x}) = 0,$$

which implies

$$\mathbb{E}(\Gamma_\delta(\xi)) = \{\mathbf{x} + \delta\mathbb{E}_\kappa(\mathbf{x})\mathbf{n}(\mathbf{x}) : \mathbf{x} \in \Gamma\} = \Gamma,$$

and

$$\operatorname{Covar}_\kappa(\mathbf{x}, \mathbf{y}) = \operatorname{Cor}_\kappa(\mathbf{x}, \mathbf{y}). \quad (19)$$

For any random vector field  $\mathbf{E} = (E_1, E_2, E_3)^T \in [L^2(C^{2,1}(\Omega, \mathbb{R}^3); \Theta)]^3$ , the mean vector, two-point correlation matrix, and covariance matrix of  $\mathbf{E}$  are defined by

$$\begin{aligned}\mathbb{E}_{\mathbf{E}}(\mathbf{x}) &= \mathbb{E}[\mathbf{E}(\mathbf{x}, \xi)] = \int_{\Theta} \mathbf{E}(\mathbf{x}, \xi) dP(\xi), & \mathbf{x} \in \Omega, \\ \mathbf{Corr}_{\mathbf{E}}(\mathbf{x}, \mathbf{y}) &= (\mathbb{E}[E_i(\mathbf{x}, \xi)E_j(\mathbf{y}, \xi)])_{i,j}, & \mathbf{x}, \mathbf{y} \in \Omega, \\ \mathbf{Covar}_{\mathbf{E}}(\mathbf{x}, \mathbf{y}) &= \mathbf{Corr}_{\mathbf{E}}(\mathbf{x}, \mathbf{y}) - (\mathbb{E}(E_i(\mathbf{x}, \xi))\mathbb{E}(E_j(\mathbf{y}, \xi)))_{i,j}, & \mathbf{x}, \mathbf{y} \in \Omega.\end{aligned}$$

Consider a second velocity field  $V'$  defined by the analogous manner of  $V(\cdot, \xi)$  in (17), that is,  $V'(\mathbf{x}) = \kappa'(\mathbf{x})\mathbf{n}(\mathbf{x})$  with  $\kappa' \in \mathcal{E}$ . Then the second order shape derivative is a bilinear form on pairs of velocity fields  $(V, V')$ , denoted by  $d^2\mathbf{u} = d^2\mathbf{u}[\kappa, \kappa']$  and  $d^2p = d^2p[\kappa, \kappa']$  (see shape Hessian in [11, 34]).

**Theorem 4.1.** *Assume the stochastic perturbed interface satisfies  $\Gamma_{\delta}(\cdot) \in L^2(\Theta, C^{2,1}(\Gamma, \mathbb{R}))$ , and  $\delta_0$  is sufficiently small to ensure that the stochastic interface  $\Gamma_{\delta}(\xi)$  is not degenerate and lies still inside the hold-on domain  $D$ . Then the first order shape derivatives  $d\mathbf{u}(\mathbf{x}, \xi)$ ,  $dp(\mathbf{x}, \xi)$  exist and satisfy the following interface problem,*

$$\begin{cases} -\Delta dp(\mathbf{x}, \xi) - k^2 dp(\mathbf{x}, \xi) = 0 & \text{in } \Omega^c \times \Theta, \end{cases} \quad (20a)$$

$$\begin{cases} -\mathbf{div}\sigma(d\mathbf{u}(\mathbf{x}, \xi)) - \rho_s \omega^2 d\mathbf{u}(\mathbf{x}, \xi) = 0 & \text{in } \Omega \times \Theta, \end{cases} \quad (20b)$$

$$\begin{cases} \rho_f \omega^2 d\mathbf{u}(\mathbf{x}, \xi) = \frac{\partial dp(\mathbf{x}, \xi)}{\partial \mathbf{n}} - \Xi^*(\mathbf{u}, p^s, p^i) & \text{on } \Gamma \times \Theta, \end{cases} \quad (20c)$$

$$\begin{cases} T d\mathbf{u}(\mathbf{x}, \xi) = -dp\mathbf{n} + \Lambda^*(\mathbf{u}, p^s, p^i) & \text{on } \Gamma \times \Theta, \end{cases} \quad (20d)$$

$$\begin{cases} \lim_{r \rightarrow \infty} r \left( \frac{\partial dp}{\partial r} - ik dp \right) = 0 & r = |\mathbf{x}|, \end{cases} \quad (20e)$$

where the functions  $\Xi^*(\mathbf{u}, p^s, p^i)$  in (20c) and  $\Lambda^*(\mathbf{u}, p^s, p^i)$  (20d) are given by

$$\begin{cases} \Xi^*(\mathbf{u}, p^s, p^i) = \mathbf{div}_{\Gamma}(\kappa(\mathbf{x}, \xi)\nabla_{\Gamma} p^s) + \kappa(\mathbf{x}, \xi) \left( \xi \frac{\partial p^s}{\partial \mathbf{n}} + k^2 p^s + \frac{\partial}{\partial \mathbf{n}}(\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} - \frac{\partial p^i}{\partial \mathbf{n}}) \right), \end{cases} \quad (21a)$$

$$\begin{cases} \Lambda^*(\mathbf{u}, p^s, p^i) = \mathbf{div}_{\Gamma}(\kappa(\mathbf{x}, \xi)\sigma_{\Gamma}(\mathbf{u})) + \kappa(\mathbf{x}, \xi) \left( \rho_s \omega^2 \mathbf{u} - \frac{\partial}{\partial \mathbf{n}}([p^s + p^i]\mathbf{n}) + \xi T \mathbf{u} \right). \end{cases} \quad (21b)$$

Here, the quantities of  $p^s, \mathbf{u}$  on the right hand side of (21a) and (21b) are determined by interface elasto-acoustic BVP (2).

Under the results in Theorem 4.1, we shall study statistics for solutions of the random interface equations (20), and we focus on the expectation and variance of the random solutions (denoted by  $\mathbb{E}_{p^s}(\mathbf{x})$ ,  $\mathbb{E}_{\mathbf{u}}(\mathbf{x})$  and  $\mathbb{Var}_{p^s}(\mathbf{x})$ ,  $\mathbb{Var}_{\mathbf{u}}(\mathbf{x})$  respectively), and the results can be generalized to high order moments (cf. [17, 18]).

Set  $D(\Gamma, \delta_0)$  be

$$D(\Gamma, \delta_0) = \{\mathbf{y} = \mathbf{x} + t\delta_0\mathbf{n}(\mathbf{x}), t \in [-1, 1], \mathbf{x} \in \Gamma_0\},$$

and  $K_1, K_2$  be compact subsets such that  $K_1 \subset \Omega_R \setminus D(\Gamma, \delta_0)$  and  $K_2 \subset \Omega \setminus D(\Gamma, \delta_0)$ . Let  $(p^s(\mathbf{x}, \xi), \mathbf{u}(\mathbf{x}, \xi))$  and  $(p^{s*}(\mathbf{x}), \mathbf{u}^*(\mathbf{x}))$  be the solutions of the stochastic interface elasto-acoustic equations (20) and the deterministic coupled equations (1) respectively. Using the standard shape Taylor expansion technique (cf. [11, 18]), one can obtain the following expansions

$$\begin{cases} p^s(\mathbf{x}, \xi) = p^{s*}(\mathbf{x}) + dp[\kappa](\mathbf{x})\delta + \frac{1}{2}d^2p[\kappa, \kappa](\mathbf{x})\delta^2 + O(\delta^3), & \mathbf{x} \in K_1, \\ \mathbf{u}(\mathbf{x}, \xi) = \mathbf{u}^*(\mathbf{x}) + d\mathbf{u}[\kappa](\mathbf{x})\delta + \frac{1}{2}d^2\mathbf{u}[\kappa, \kappa](\mathbf{x})\delta^2 + O(\delta^3), & \mathbf{x} \in K_2, \end{cases}$$

a.s.  $\xi \in \Theta$ . Moreover, noting the linearity of expectation operator  $\mathbb{E}$  and the zero mean filed assumption  $\mathbb{E}_k(x) = 0$ , we have the following lemma for approximating  $\mathbb{E}_{p^s}$  and  $\mathbb{E}_u$ .

**Lemma 4.2.** *Under the assumptions of Theorem 4.1, then the following approximations hold*

$$\begin{cases} \mathbb{E}_{p^s}(\mathbf{x}) = p^{s*}(\mathbf{x}) + O(\delta^2), & \mathbf{x} \in K_1, \text{ a.s. } \xi \in \Theta, \\ \mathbb{E}_u(\mathbf{x}) = \mathbf{u}^*(\mathbf{x}) + O(\delta^2), & \mathbf{x} \in K_2, \text{ a.s. } \xi \in \Theta. \end{cases}$$

By using the relation (19) and simple calculations, we can derive following identity:

$$\begin{aligned} \mathbb{V}\text{ar}_{dp}(\mathbf{x}) &= \mathbb{V}\text{ar}(dp[\kappa(\mathbf{x}, \xi)](\mathbf{x})) \\ &= \text{Cor}(dp[\kappa(\mathbf{x}, \xi)](\mathbf{x}), dp[\kappa(\mathbf{y}, \xi)](\mathbf{y})|_{\mathbf{y}=\mathbf{x}}) \\ &= \text{Cor}_{dp}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}}. \end{aligned}$$

Once with the two-point correlation matrix  $\text{Cor}_{dp}(\mathbf{x}, \mathbf{y})$  at hand, using the above identity, we can approximate the variance  $\mathbb{V}\text{ar}_{p^s}(\mathbf{x})$  by the following Theorem.

**Lemma 4.3.** *Under the assumptions of Theorem 4.1, then the following estimate holds for the variance  $\mathbb{V}\text{ar}_{p^s}(\mathbf{x})$  of random solutions of (18a)-(18e) as*

$$\begin{aligned} \mathbb{V}\text{ar}_{p^s}(\mathbf{x}) &= [\mathbb{V}\text{ar}_{dp}(\mathbf{x})]\delta^2 + O(\delta^3) \\ &= \left[ \text{Cor}_{dp}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} \right] \delta^2 + O(\delta^3), \quad \mathbf{x} \in K_1, \text{ a.s. } \xi \in \Theta. \end{aligned}$$

The proof follows easily from the some properties of variance, the Cauchy-Schwarz inequality and the shape Taylor expansion, and we refer to the monographs [11] for the details. One can also deduce the corresponding result of  $\mathbb{V}\text{ar}_u(\mathbf{x})$  when  $\mathbf{x} \in K_2$ ,

$$\begin{aligned} \mathbb{V}\text{ar}_u(\mathbf{x}) &= [\mathbb{V}\text{ar}_{du}(\mathbf{x})]\delta^2 + O(\delta^3) \\ &= \left[ \text{Cor}_{du}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} \right] \delta^2 + O(\delta^3), \quad \mathbf{x} \in K_2, \text{ a.s. } \xi \in \Theta. \end{aligned}$$

Generally speaking, the quantity  $\mathbb{V}\text{ar}_{dp}(\mathbf{x})$  is unpredictable. However, consider the tensor product of shape derivatives equations (20), and take the expectation on both sides, one can obtain the tensor product interface problem for  $\text{Cor}_{dp}(\mathbf{x}, \mathbf{y})$  (more details see [17, 18]), which is greatly important for numerical computation. Numerical methods for the expectation and variance is another arduous tasks, efficient and effective computation of the stochastic interface problems will be proposed in the forthcoming paper, especially for the HDG methods given in [6, 7, 8, 9, 10, 31, 32]

## 5. Conclusions

In this paper, in terms of the time-harmonic elasto-acoustic transmission scattering problem, we present the characterization of shape derivatives from the perspective of both differential and classical Euclidean forms. This description is a necessary theoretical framework for estimating the stability and convergence for geometric optimization in inverse obstacle problems. Instead of discussing the shape derivatives with integral equation methods, we deduce the description of shape derivatives based on variational weak forms with shape perturbations governed by the velocity(speed) method. The DtN map on the artificial boundary is employed analytically, which helps construct the weak variational form in the truncated domain and is

independent of the choice of artificial boundary. As solutions to the abstract PDEs for shape derivatives, they are easily applied to iterative methods of inverse problems to provide the necessary theoretical analysis. To understand the transmission condition, we take different regularities of domains into consideration.

Finally, we derive the first order shape derivatives of solutions to the stochastic elasto-acoustic coupled equations. We study the statistics for solutions to the random interface equations in terms of perturbation amplitude, especially for the expectation and the variance of the random solutions, and the results can be generalized to high order moments.

**Acknowledgements** The author would like to thank Prof. Li, Jingzhi from Southern University of Science and Technology (SUSTech) for providing helpful discussion and academic reference [20] (cooperate with Prof. Ralf Hiptmair, SAM, ETH Zurich) during my visit at SUSTech in June 2014. At last, We thank the anonymous referee for his careful reading of the manuscript. His comments and suggestion improve the paper significantly.

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