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Blow-up profiles and refined extensibility criteria in quasilinear Keller–Segel systems

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A R T I C L E   I N F O

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A B S T R A C T

In this work we consider the system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) & \text{in } \Omega \times (0,\infty) \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty)\,, \end{cases}$$

for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where the functions D and S behave similarly to power functions. We prove the existence of classical solutions under Neumann boundary conditions and for smooth initial data. Moreover, we characterise the maximum existence time $T_{\max}$ of such a solution depending chiefly on the relation between the functions D and S: We show that a finite maximum existence time also results in unboundedness in L^p-spaces for smaller $p \in [1,\infty)$.

1. Introduction

The Keller–Segel systems considered in this work attempt to describe the behaviour of certain slime molds. In particular, given a position x and a time t, by $u(x,t)$ we denote the density of a cell population whose movement is motivated by the concentration $v(x,t)$ of a signal substance.

In these systems, which were proposed by Keller and Segel [17] in 1970 and of which there are several modifications (cf. e.g. Hillen and Painter [14]), the cross-diffusion makes solutions prone to blow-up and indeed blow-up detection is one of the most challenging tasks; to this day results remain fragmented. Even with the original system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) & \text{in } \Omega \times (0,\infty) \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty) \end{cases}$$

there is no trivial answer on occurrences of blow-up, and if there is one, one often likes to know whether it arises in finite or infinite time. Beginning with a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary

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(and sufficiently regular initial data) we can state the following results: The case $n = 1$ has been studied (see [23]) with the result that there is no blow-up at all. For the two-dimensional setting we know that if the initial mass $\int_\Omega u_0$ is smaller than 4π, then solutions are bounded, for this we refer to [11] and [22], while for $n \geq 3$ a smallness condition on $\|u_0\|_{L^{\frac{n}{2}}(\Omega)} + \|v_0\|_{W^{1,n}(\Omega)}$ can be used to infer the existence of such a solution (see [4]). For larger initial data on the other hand we generally only know that there are blow-up solutions for which unboundedness can happen either in finite or infinite time [15].

In some cases the statements can be refined if we restrict ourselves to radially symmetric settings. For $\Omega = B_R(0) \subset \mathbb{R}^2$ and $\int_\Omega u_0 > 8\pi$ radially symmetric solutions that blow up in finite time have been found by [13] and [21] while in the case $\Omega = B_R(0) \subset \mathbb{R}^n$ and $n \geq 3$ even for small initial masses some solutions blow up in finite time (see [31]).

In this work we modify the first equation and for some bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we consider the system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) & \text{in } \Omega \times (0,\infty) \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0, \ v(\cdot,0) = v_0 & \text{in } \Omega \end{cases} \tag{KS}$$

with nonnegative functions D and S. For a helpful overview of many models arising out of this fundamental description we also refer to the survey [1].

Several choices for these functions have been proposed and studied in recent years. One suggestion is to couple them via some function Q and the relations $D(u) = Q(u) - uQ'(u)$ and $S(u) = uQ'(u)$ for all $u \geq 0$. Here, Q is intended to describe the probability of a cell at (x,t) to find space nearby, [3] considers a decreasing function with decay at large densities as the best fit. In [32] an overview of hydrodynamic approaches or those involving cellular Potts models is given.

There are also authors who propose a signal dependence in D or S, that is to write e.g. $S(u,v)$ as done in [29], [14] and [25] to incorporate saturation effects or a threshold for the activation of cross-diffusion. For similar changes to D we refer to the works [9], [19], [27] and [26].

One set of choices has been of particular interest, namely where D and S behave like powers of u, and the result heavily depends on the relation of these two quantities. Setting

$$D(s) = (s+1)^{m-1} \text{ for all } s \in [0,\infty)$$

and

$$S(s) = s(s+1)^{\kappa-1} \text{ for all } s \in [0,\infty)$$

for some $m \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ we find the following for $n \geq 2$: If $1 + \kappa - m < \frac{2}{n}$ and if the initial data are reasonably smooth, then we can find global classical solutions that are bounded [28] and this even remains true for general nonnegative functions D and S with

$$\frac{S(s)}{D(s)} \leq Cs^\alpha \text{ for all } s \geq 1$$

for some $C > 0$ and $\alpha < \frac{2}{n}$. On the other hand, if $1 + \kappa - m > \frac{2}{n}$ and if Ω is a ball, then for any $M > 0$ there are some $T \in (0,\infty]$ and a radially symmetric solution (u,v) in $\Omega \times (0,T)$ with $\int_\Omega u(\cdot,t) = M$ for all $t \in (0,T)$ such that u is not bounded in $\Omega \times (0,T)$ [30]. Once more there are also studies on more general choices of D and S (see [28], [30] as well as [18], [5], [24] and [16]) that find

$$\frac{S(s)}{D(s)} \geq C s^{\alpha} \text{ for all } s \geq 1$$

for some $C > 0$ and $\alpha > \frac{2}{n}$ to be enough to obtain the same result. In [6] and [7] the specific choice of D and S as powers of $u + 1$ has been examined in greater detail with respect to this blow-up phenomenon and the authors were able to prove that for $\kappa \geq 1$ or $m \geq 1$ a finite value of T is obtainable and that for $m < \frac{n-2}{n}$ and $\kappa < \frac{m}{2} - \frac{n-2}{2n}$ we have $T = \infty$; i.e. that the solution exists globally with $\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.

Here we want to refine the blow-up results in the case where $1 + \kappa - m < \frac{2}{n}$ does not necessarily hold. We consider twice differentiable D and S allowing for the inequalities

$$C_D(s+1)^{m-1} \leq D(s) \leq \hat{C}_D\,(s+1)^{\hat{m}-1} \tag{D}$$

with some $m, \hat{m} \in \mathbb{R}$ and $C_D, \hat{C}_D > 0$ for any $s \in [0,\infty)$ and

$$|S(s)| \leq C_S(s+1)^{\kappa} \tag{S}$$

with $\kappa \in \mathbb{R}$ and some $C_S > 0$, again for all $s \in [0,\infty)$.

We remark that the functions may even depend on the variables $(x,t) \in \Omega \times [0,\infty)$ and on the solution to the second equation in (KS), $\sigma \in [0,\infty)$, as long as the overall boundedness remains unaltered, e.g.

$$|S(x,t,s,\sigma)| \leq C_S(s+1)^{\kappa}\ \forall\,(x,t,s,\sigma) \in \Omega \times [0,\infty)^3$$

is actually good enough, but we omit the dependance on the other three quantities for the sake of clarity.

Furthermore, we even observe the following

Remark. From $\|u\|_{L^1(\Omega)} \equiv \|u_0\|_{L^1(\Omega)}$, lemma 2.6 in [10] deduces a uniform and global-in-time lower bound for v and therefore admissible choices of (S) include functions of the structure

$$S(u,v) = \frac{\tilde{S}(u)}{v^{\alpha}}$$

where $\tilde{S}$ has the properties of our previous S and where α can be any nonnegative constant.

We have already seen that the trivial observation of the boundedness of

$$t \mapsto \int_\Omega u(\cdot,t) \equiv \int_\Omega u_0$$

is certainly enough to prove a classical solution to be bounded and thereby global at least for $m > \kappa + \frac{n-2}{n}$ (see [28]). It is our endeavour to extend this result by considering any relation between κ and m and finding a $p_0 \geq 1$ such that boundedness of

$$t \mapsto \int_\Omega u(\cdot,t)^{p_0}$$

is sufficient for a solution to (KS) to be global and bounded.

In [12] the authors (in the case $\kappa > 1$ and $(n-2)\kappa < n+2$) have examined the semilinear heat equation

$$u_t = \Delta u + u^{\kappa}$$

in a convex domain Ω and concluded that a positive blow-up solution u with maximum existence time $T \in (0,\infty)$ for some $C > 0$ satisfies

$$u(\cdot,t) \leq C(T-t)^{\frac{1}{1-\kappa}} \text{ for all } t \in (0,T),$$

giving us a more precise idea on the manner in which u blows up. We will attempt to achieve similar knowledge for our problem (KS).

2. The main results

The centre of our computations and estimates is a threshold $\mathfrak{p}$ given by

$$\mathfrak{p} := \begin{cases} \frac{n}{2}(1+\kappa-m) & \text{if } \kappa < m+1 \\ n(\kappa-m) & \text{if } \kappa \geq m+1, \end{cases} \tag{$\mathfrak{p}$}$$

which gives us refined knowledge on the behaviour of blow-up solutions to (KS). Note that this is the same as setting $\mathfrak{p} := \max\left\{\frac{n}{2}(1+\kappa-m), n(\kappa-m)\right\}$. Locally, [28] gives us classical solutions to (KS):

Lemma 2.1. *Let $D \in \mathcal{C}^2([0,\infty))$ with (D) for some $m, \hat{m} \in \mathbb{R}$ and $C_D, \hat{C}_D > 0$ as well as $S \in \mathcal{C}^2([0,\infty))$ with $S(0) = 0$ and nonnegative initial data $u_0 \in \mathcal{C}^0\left(\overline{\Omega}\right)$ and $v_0 \in \mathcal{C}^1\left(\overline{\Omega}\right)$. Then there are $T_{max} \in (0,\infty]$ and a pair (u,v) of nonnegative functions in $\mathcal{C}^0\left(\overline{\Omega}\times[0,T_{max})\right) \cap \mathcal{C}^{2,1}\left(\overline{\Omega}\times(0,T_{max})\right)$ solving (KS) classically in $\Omega \times (0,T_{max})$. Additionally we either have*

$$T_{max} = \infty \ \text{or} \ \limsup_{t\nearrow T_{max}} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \right) = \infty.$$

Our main result asserts that such a solution can be extended to a solution in $\Omega \times (0,\infty)$ if we have more information on the $L^{p_0}(\Omega)$-norm of u for some $p_0 \geq 1$:

Theorem 2.2. *Let $D \in \mathcal{C}^2([0,\infty))$ with (D) for some $m, \hat{m} \in \mathbb{R}$ and $C_D, \hat{C}_D > 0$ as well as $S \in \mathcal{C}^2([0,\infty))$ with $S(0) = 0$ and (S) for some $\kappa \in \mathbb{R}$ and $C_S > 0$. Additionally, let nonnegative initial data $u_0 \in \mathcal{C}^0\left(\overline{\Omega}\right)$ and $v_0 \in \mathcal{C}^1\left(\overline{\Omega}\right)$ be given. If for a solution (u,v) to (KS) in $\Omega \times (0,T_{max})$ found in lemma 2.1 we then have $T_{max} < \infty$, we automatically know*

$$\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^{p_0}(\Omega)} = \infty$$

for any $p_0 > \mathfrak{p}$ with $\mathfrak{p}$ as in ($\mathfrak{p}$) such that $p_0 \geq 1$.

This indeed gives us an interpretation for the behaviour of u in comparison to some negative powers of the space variable x:

Corollary 2.3. *Let $D \in \mathcal{C}^2([0,\infty))$ with (D) for some $m, \hat{m} \in \mathbb{R}$ and $C_D, \hat{C}_D > 0$ as well as $S \in \mathcal{C}^2([0,\infty))$ with $S(0) = 0$ and (S) for some $\kappa \in \mathbb{R}$ and $C_S > 0$. Additionally, let nonnegative initial data $u_0 \in \mathcal{C}^0\left(\overline{\Omega}\right)$ and $v_0 \in \mathcal{C}^1\left(\overline{\Omega}\right)$ be given. Taking a solution (u,v) to (KS) in $\Omega \times (0,T_{max})$ found in lemma 2.1, we assume $T_{max} < \infty$ and pick a blow-up point $x_0 \in \Omega$, meaning there are sequences $(x_k)_{k\in\mathbb{N}} \subset \Omega$ and $(t_k)_{k\in\mathbb{N}} \subset (0,T_{max})$ with*

$$x_k \to x_0 \ \text{as} \ k \to \infty,$$

$$t_k \to T_{max} \ \text{as} \ k \to \infty$$

and

$$u(x_k, t_k) \to \infty \text{ as } k \to \infty.$$

Then for any $\alpha < \frac{n}{\mathfrak{p}}$ *with* $\mathfrak{p}$ *as in* ($\mathfrak{p}$) *we cannot find a* $C > 0$ *such that*

$$u(x,t) \leq C\,|x - x_0|^{-\alpha}$$

holds for all $(x,t) \in \Omega \times (0, T_{max})$.

Proof. Assuming this to be wrong, for any $p_0 \in (\mathfrak{p}, \frac{n}{\alpha})$ and for some positive constants r, C_1 and C_2 we have

$$\frac{1}{C_1^{p_0}} \int_\Omega u(\cdot,t)^{p_0} \leq \int_\Omega |x - x_0|^{-\alpha p_0}\,\mathrm{d}x = C_2 + \int_{B_r(0)} |x|^{-\alpha p_0}\,\mathrm{d}x$$

and the right-hand side is bounded because of $\alpha p_0 < n$, leading to a contradiction in view of theorem 2.2. □

We close this section with two remarks concerning the compatibility of this result with previous works. Firstly, if $m \geq \kappa + \frac{n-2}{n}$, we can pick any $p_0 > 1$ and [28] shows that even $p_0 = 1$ is enough for $\|u\|_{L^\infty((0,T_{\max})));L^{p_0}(\Omega)} < \infty$ to guarantee that u is global and bounded. On the other hand we do not require too much of p_0 comparing our result to one in [1]: lemma 3.2 in that work demands $\mathfrak{p} = \max\left\{\frac{n\kappa}{2}, n(\kappa - 1)\right\}$, which is exactly the same as our result for $m = 1$.

3. Extending the maximal existence time

As seen in condition ($\mathfrak{p}$), there are differences when tackling the problem depending on the relation between κ and $m + 1$. We will discuss in detail how one can proceed in the case $\kappa < m + 1$ and a second chapter will deal with the differences that arise when considering the inverse case.

3.1. Part I: $\kappa < m + 1$

In this first part we demand $1 \leq p_0 \in \left(\frac{n}{2}\left(1 + \kappa - m\right), n\right) \neq \emptyset$ without loss of generality. This in the case $m > \kappa + \frac{n-2}{n}$ admits the choice $p_0 = 1 + \eta$ for arbitrarily small $\eta > 0$ which can even be relaxed to $p_0 = 1$, thereby using nothing more than a property inherent to the first equation. For this we refer to [28].

3.1.1. Choosing parameters

For the sake of clarity in later parts of this work we will now determine as many of the subsequently arising parameters as precisely as possible. Our main goal is to allow for arbitrarily large p in $\|u\|_{L^\infty((0,T);L^p(\Omega))}$ and having fixed such a p we need to ensure the existence of every other quantity needed to achieve this.

Lemma 3.1. *Let* $m \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}$ *with* $\kappa < m + 1$ *as well as some* $p_0 \in \left(\frac{n}{2}(1 + \kappa - m), n\right)$ *with* $p_0 \geq 1$. *There are positive numbers* $p > n + 2$, $\theta > 1$ *and* $\mu > 1$, *their conjugate exponents* $\theta' > 1$ *and* $\mu' > 1$, *as well as a nonempty interval* $(q\_, q_+) \subset (1, \infty)$ *with* $q_- := \max\left\{n, \frac{np_0}{n-p_0}, \frac{1}{2}\frac{np_0}{n-p_0} + 1\right\}$ *and* $q_+ := \frac{n}{2}\frac{m+p-1}{n-p_0}$ *such that together with* $s_+ := \frac{np_0}{n-p_0} > 1$ *the following requirements are met: For* p *we have*

$$p > p_0 + m + 1 - 2\kappa > m + 1 - 2\kappa, \tag{p1}$$

$$p > 2\frac{n - p_0}{n} + p_0 + 1 - m \geq 3 - m > 1 - m, \tag{p2}$$

$$p > 1 - m + 2(n - p_0), \tag{p3}$$

$$p > 2p_0 + 1 - m > p_0 \frac{n-2}{n} + 1 - m, \tag{p4}$$

$$p > 2\frac{n-2}{n} + 1 - m, \tag{p5}$$

$$p > 3m + 1 - 4\kappa \tag{p6}$$

and

$$p > 2 - \kappa. \tag{p7}$$

With respect to q_+ we have

$$q_+ > n, \tag{q1}$$

$$q_+ > \frac{np_0}{n - p_0} > \frac{p_0(n-2)}{2(n-p_0)} \tag{q2}$$

and

$$q_+ > \frac{1}{2}\frac{np_0}{n - p_0} + 1. \tag{q3}$$

Additionally, we can achieve

$$\theta > \frac{m + p - 1}{2(-m + p - 1 + 2\kappa)} > \frac{p_0}{-m + p - 1 + 2\kappa} \tag{θ1}$$

and

$$\theta < \frac{q}{q-2} \text{ for all } q \in (q\_, q_+) \tag{θ2}$$

as well as

$$\mu > \frac{p_0}{2} \tag{μ1}$$

and

$$\mu > \frac{m + p - 1}{4}. \tag{μ2}$$

Lastly, the inequalities

$$s_+ < 2\theta' \tag{s1}$$

and

$$s_+ < 2(q-1)\mu' \text{ for all } q \in (q\_, q_+) \tag{s2}$$

also hold.

Proof. Since

$$n < q_+$$

is equivalent to

$$1 < \frac{m+p-1}{2(n-p_0)},$$

we see that (p3) leads to (q1) not contradicting $q_\_ < q_+$. Moreover

$$\frac{np_0}{n-p_0} < q_+$$

holds if and only if

$$2p_0 < m+p-1$$

which shows that (q2) is admissible if (p4) holds. Finally we see that we may demand (q3) since

$$\frac{1}{2}\frac{np_0}{n-p_0} + 1 < q_+$$

is guaranteed by

$$1 < \frac{n}{2(n-p_0)}\,(m+p-1-p_0)$$

which in turn is a consequence of (p2). Therefore our conditions for p are sufficient to guarantee the existence of a nonempty interval $(q_\_, q_+)$ with the designated borders.

Due to $\frac{q}{q-2} > 1$ for any $q > 2$ and with

$$\frac{m+p-1}{2(-m+p-1+2\kappa)} < 1$$

as a consequence of (p6), the conditions (θ1) and (θ2) can easily be met. So we fix some $\theta \in (1, \frac{q_\_}{q_\_-2})$ and an arbitrary $\mu > 1$ satisfying (μ1) and (μ2). One swiftly sees that

$$s_+ = \frac{np_0}{n-p_0} \leq 2\theta'$$

holds if and only if

$$\frac{\theta}{\theta-1} \geq \frac{np_0}{2(n-p_0)}$$

is true. Therefore, the condition in question is equivalent to

$$1 - \frac{1}{\theta} \leq \frac{2(n-p_0)}{np_0}$$

and

$$\frac{1}{\theta} \geq 1 - \frac{2(n-p_0)}{np_0}.$$

Herein either the term on the right-hand side is nonpositive or $p_0 > \frac{2n}{n+2}$ which at first seems to lead to a new requirement

$$\theta \leq \frac{1}{1 - \frac{2(n-p_0)}{np_0}}.$$

However, (q2) ensures

$$q\left(1 - \frac{2(n-p_0)}{np_0}\right) < q - 2 \text{ for all } q \in (q_-, q_+)$$

which in turn proves that ($\theta 2$) is sufficient here.

Even more directly we can deal with (s2):

$$s_+ = \frac{np_0}{n-p_0} < 2(q-1)\mu' \text{ for all } q \in (q\_, q_+)$$

is true if and only if

$$1 - \frac{1}{\mu} < \frac{2(q-1)(n-p_0)}{np_0} \text{ for all } q \in (q\_, q_+).$$

This is the same as

$$\frac{1}{\mu} > 1 - \frac{2(q-1)(n-p_0)}{np_0} \text{ for all } q \in (q\_, q_+)$$

and because of the equivalence of

$$q > 1 + \frac{1}{2}\frac{np_0}{n-p_0}$$

and

$$1 < \frac{2(q-1)(n-p_0)}{np_0},$$

here the right-hand side can never be positive. For any $s \in [1, s_+)$ we therefore have $s < 2\theta'$ and $s < 2(q-1)\mu'$. $\square$

We now combine these parameters with the conditions on D and S and study the consequences of our choices with respect to the applicability of upcoming estimates.

Lemma 3.2. *Let $m \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ with $\kappa < m+1$ as well as some $p_0 \in \left(\frac{n}{2}(1+\kappa-m), n\right)$ with $p_0 \geq 1$. With p, $q\_$, q_+, s_+, θ and μ as in lemma 3.1, we can find $q \in (q\_, q_+)$ and $s \in [1, s_+)$ such that*

$$\frac{s}{q} < 2$$

holds and such that

$$\beta_1 := \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - \frac{1}{\theta}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_1 := \frac{\frac{n}{2}(\frac{2}{s} - \frac{1}{\theta'})}{1 - \frac{n}{2} + \frac{nq}{s}}$$

as well as

$$\beta_2 := \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_2 := \frac{\frac{n}{2}(\frac{2(q-1)}{s} - \frac{1}{\mu'})}{1 - \frac{n}{2} + \frac{nq}{s}},$$

are positive and that they furthermore allow for the estimates

$$\beta_1 + \gamma_1 < 1$$

and

$$\beta_2 + \gamma_2 < 1.$$

Proof. The positivity of these parameters is a direct consequence of the ranges chosen for p, q, θ, μ and s in lemma 3.1.

With p, θ and μ already fixed, we consider variable variants $\tilde{\gamma}_1(q,s)$ and $\tilde{\gamma}_2(q,s)$ of two of the central quantities of this lemma depending on still unknown q and s. We furthermore set $f(q,s) := \beta_1 + \tilde{\gamma}_1(q,s)$ and $g(q,s) := \beta_2 + \tilde{\gamma}_2(q,s)$ and we seek to prove smallness of these continuous functions close to q_+ and s_+. We begin with

$$\begin{aligned}
g(q_+, s_+) &= \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}} + \frac{\frac{n}{2}\left(\frac{m+p-1}{p_0} - \frac{2(n-p_0)}{np_0} - \frac{1}{\mu'}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{np_0}} \\
&= \frac{\frac{n}{2}}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}} \left[\frac{2n - 2(n-p_0)}{np_0} - 1 + \frac{m+p-1}{p_0}\right] \\
&= \frac{1}{\frac{2}{n} - 1 + \frac{m+p-1}{p_0}} \left[\frac{2}{n} - 1 + \frac{m+p-1}{p_0}\right] \\
&= 1.
\end{aligned}$$

Using $L := \left(1 - \frac{n}{2} + \frac{q(n-p_0)}{p_0}\right)^2 \cdot \frac{2}{n}$ as an abbreviation, we obtain the following statement concerning the derivative of g:

$$\begin{aligned}
L\frac{\partial g}{\partial q}(q, s_+) &= \frac{2(n-p_0)}{np_0}\left(1 - \frac{n}{2} + \frac{q(n-p_0)}{p_0}\right) - \left(\frac{2(q-1)(n-p_0)}{np_0} - \frac{1}{\mu'}\right)\frac{n-p_0}{p_0} \\
&= \frac{n-p_0}{p_0}\left[\frac{2}{n} - 1 + \frac{2q(n-p_0)}{np_0} - \frac{2(q-1)(n-p_0)}{np_0} + \frac{1}{\mu'}\right] \\
&= \frac{n-p_0}{p_0}\left[\frac{2}{p_0} - \frac{1}{\mu}\right] \\
&> 0.
\end{aligned}$$

On the other hand we see that

$$1 > f(q_+, s_+) = \frac{\frac{n}{2}}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}} \left[\frac{-m+p-1+2\kappa}{p_0} - 1 + 2\frac{n-p_0}{np_0} \right]$$

is equivalent to

$$\frac{2}{n} - 1 + \frac{m+p-1}{p_0} > \frac{-m+p-1+2\kappa}{p_0} - 1 + 2\frac{n-p_0}{np_0}$$

which in turn is precisely our overall requirement

$$p_0 > \frac{n}{2}\left(1+\kappa-m\right).$$

This leads to the following result: If $q \in (q\_, q_+)$ and $s \in [1, s_+)$ are marginally smaller than their respective upper bounds, the requested relations hold. Since $\frac{s_+}{q_+} < 2$, demanding $\frac{s}{q} < 2$ as well now is unproblematic. $\square$

We will use these parameters and their relationship to each other to prove our main result after introducing several helpful inequalities.

3.1.2. Some helpful lemmata

In this section we mainly state simple propositions and collect results of previous works starting with the elementary

Lemma 3.3. *For any positive β and γ whose sum is less than 1, and for each $\eta > 0$, there is a constant $c > 0$ such that*

$$(1 + a^\beta)(1 + b^\gamma) \leq \eta(a + b) + c$$

holds for all positive numbers a and b.

This follows directly after several consecutive employments of Young's inequality. It has already been shown that regularity of v can be derived (see [16]) if u belongs to $L^\infty((0, T_{\max}); L^p(\Omega))$ for some $p \geq 1$; in this case we have

Lemma 3.4. *Let (u, v) solve the second equation in (KS) in $\Omega \times (0, T)$ for some $T > 0$. Say for some $p \geq 1$ and $C > 0$ one knows*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \textit{ for all } t \in (0, T).$$

Then

$$\|v\|_{L^\infty((0,T);W^{1,q}(\Omega))} < \infty$$

follows for any $q \in [1, \frac{np}{(n-p)_+})$ and even $q = \infty$ if $p > n$.

This results in two helpful conclusions: Firstly, we may already use the finiteness of all norms with $q < \frac{np_0}{(n-p_0)_+}$, and furthermore we can concentrate our efforts on norms of u as the rest will follow suit.

In the appendix of [28] we find the means to deduce bounds on the $L^\infty(\Omega)$-norm of u in the very general

Lemma 3.5. *For some $T \in (0,\infty]$ and a bounded domain $\Omega \subset \mathbb{R}^n$, consider a nonegative function $u \in \mathcal{C}^0\left(\overline{\Omega} \times [0,T)\right) \cap \mathcal{C}^{2,1}\left(\overline{\Omega} \times (0,T)\right)$ that solves*

$$\begin{cases} u_t \leq \nabla \cdot (D(x,t,u)\nabla u) + \nabla \cdot f(x,t) + g(x,t), & \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial \nu} \leq 0, & \text{on } \partial\Omega \times (0,T) \end{cases}$$

classically for the following functions and parameters: Firstly, we assume

$$0 \leq D \in \mathcal{C}^1\left(\overline{\Omega} \times [0,T) \times [0,\infty)\right)$$

to satisfy

$$D(\cdot,\cdot,s) \geq \delta s^{m-1}$$

in $\Omega \times (0,T)$ for fixed $m \in \mathbb{R}$, $s_0 \geq 1$, $\delta > 0$ and any $s \geq s_0$. Furthermore, we suppose that

$$f \in \mathcal{C}^0\left((0,T); \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega)\right),$$
$$g \in \mathcal{C}^0\left(\Omega \times (0,T)\right)$$

and

$$f \cdot \nu \leq 0 \text{ on } \partial\Omega \times (0,T),$$

and that numbers $q_1 > n+2$, $q_2 > \frac{n+2}{2}$ and $p_0 \geq 1$ fulfil

$$p_0 > 1 - m\frac{(n+1)q_1 - (n+2)}{q_1 - (n+2)}$$

and

$$p_0 > 1 - \frac{m}{1 - \frac{n}{n+2}\frac{q_2}{q_2-1}}$$

as well as

$$p_0 > \frac{n(1-m)}{2},$$

and that $\|f\|_{L^\infty((0,T);L^{q_1}(\Omega))}$, $\|g\|_{L^\infty((0,T);L^{q_2}(\Omega))}$ and $\|u\|_{L^\infty((0,T);L^{p_0}(\Omega))}$ are finite. Then there is a constant $C > 0$ depending on these three norms as well as $\|u(\cdot,0)\|_{L^\infty(\Omega)}$, Ω, δ and m such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C$$

holds for all $t \in (0,T)$.

Since we derive bounds on $\|v(\cdot,t)\|_{W^{1,q}(\Omega)}$ in lemma 3.4, we can rewrite our problem in a way that allows for the employment of this result.

We shall also use the following pointwise estimate for the normal derivative of $|\nabla v|^2$:

Lemma 3.6. *For bounded domains Ω, there always exists a constant $C > 0$ with the property that for any $w \in \mathcal{C}^2(\Omega)$ satisfying $\frac{\partial w}{\partial \nu}\big|_{\partial\Omega} = 0$, the estimate*

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq C|\nabla w|^2 \text{ on } \partial\Omega$$

holds.

This has been accomplished by authors Mizoguchi and Souplet in [20] and replaced the more coarse estimate $\frac{\partial |\nabla v|^2}{\partial \nu}\Big|_{\partial\Omega} \leq 0$ which demanded Ω to be convex.

We near the conclusion of this section by first recalling the Gagliardo–Nirenberg interpolation inequality (for a reference see [8]) as well as (out of a class of generalisations) a fractional variant needed in one of our proofs and for which we refer to a corollary in Section III of [2].

Lemma 3.7 *(Gagliardo–Nirenberg inequality). Assume $p, q \in [1, \infty]$ and $r \in (0, p)$ with $p < \infty$ for $q = n$ and $p \leq \frac{nq}{n-q}$ in the case $q < n$. Then for $a \in (0, 1]$ given by*

$$-\frac{n}{p} = (1 - \frac{n}{q})a - \frac{n}{r}(1 - a).$$

We can find a constant $C > 0$ such that

$$\|w\|_{L^p(\Omega)} \leq C\|\nabla w\|^a_{L^q(\Omega)} \|w\|^{1-a}_{L^r(\Omega)} + C\|w\|_{L^r(\Omega)}$$

holds for any $w \in \mathcal{C}^1\left(\overline{\Omega}\right)$.

Lemma 3.8 *(Gagliardo–Nirenberg inequality for fractional Sobolev spaces). For fixed $r \in \left(0, \frac{1}{2}\right)$ there are $C > 0$ and $a \in (0, 1)$ such that*

$$\|w\|_{W^{r+\frac{1}{2},2}(\Omega)} \leq C\|\nabla w\|^a_{L^2(\Omega)} \|w\|^{1-a}_{L^2(\Omega)} + C\|w\|_{L^2(\Omega)}$$

holds for any $w \in \mathcal{C}^1\left(\overline{\Omega}\right)$.

In this first instance we apply lemma 3.7 to a power of the gradient of a function:

Corollary 3.9. *Given $s \geq 1$ and $q \geq 1$ satisfying $\frac{s}{q} < 2$, there exists $C > 0$ such that for $a \in (0, 1]$ as in lemma 3.7*

$$\int_\Omega |\nabla w|^{2q} \leq C \left[1 + \left(\int_\Omega \left|\nabla |\nabla w|^q\right|^2 \right)^a \right]$$

holds for any smooth w with $\|\nabla w\|_{L^s(\Omega)} < \infty$.

Proof. By lemma 3.7 we have a $C_1 > 0$ and an $a > 0$ such that together with some additional constant $C_2 > 0$, according to our assumed preliminary bound on ∇w we see that

$$\begin{aligned}\int_\Omega |\nabla w|^{2q} &= \| |\nabla w|^q \|^2_{L^2(\Omega)} \\ &\le C_1 \|\nabla |\nabla w|^q\|^{2a}_{L^2(\Omega)} \| |\nabla w|^q \|^{2(1-a)}_{L^{\frac{s}{q}}(\Omega)} + C_1 \| |\nabla w|^q \|^2_{L^{\frac{s}{q}}(\Omega)} \\ &\le C_2 \left[1 + \left(\int_\Omega |\nabla |\nabla w|^q|^2 \right)^a \right]\end{aligned}$$

holds for any $t \in (0, T)$. □

Lemma 3.10. *For any $q \ge 1$ and $\eta > 0$ there is $C_\eta > 0$ such that*

$$\int_{\partial\Omega} |\nabla w|^{2q} \le \eta \int_\Omega |\nabla |\nabla w|^q|^2 + C_\eta \int_\Omega |\nabla w|^{2q}$$

holds for any $w \in \mathcal{C}^2(\overline{\Omega})$.

Proof. We use lemma 3.8 with some $r \in \left(0, \frac{1}{2}\right)$, and together with the embedding $W^{r+\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ this gives us positive constants $a < 1$, C_1 and C_2 such that

$$\begin{aligned}\int_{\partial\Omega} |\nabla w|^{2q} &= \| |\nabla w|^q \|^2_{L^2(\partial\Omega)} \\ &\le C_1 \| |\nabla w|^q \|^2_{W^{r+\frac{1}{2},2}(\Omega)} \\ &\le C_2 \| |\nabla w|^q \|^{2a}_{L^2(\Omega)} \| |\nabla w|^q \|^{2(1-a)}_{L^2(\Omega)} + C_2 \| |\nabla w|^q \|^2_{L^2(\Omega)}\end{aligned}$$

holds. Young's inequality then transforms this into the desired result. □

As a direct consequence of these results we find

Lemma 3.11. *For $w \in \mathcal{C}^2(\Omega)$ with $\frac{\partial w}{\partial \nu}\big|_{\partial\Omega} = 0$ and $\|\nabla w\|_{L^\infty((0,T);L^s(\Omega))} < \infty$ for some $T > 0$, $s \ge 1$ and $q \ge 1$ satisfying $\frac{s}{q} < 2$ we have*

$$\int_\Omega |\nabla w|^{2q-2} \Delta |\nabla w|^2 \le -\frac{q-1}{q^2} \int_\Omega |\nabla |\nabla w|^q|^2 + C \textit{ for all } t \in (0, T)$$

for some positive C.

Proof. From lemma 3.6 we obtain

$$\int_{\partial\Omega} |\nabla w|^{2q-2} \frac{\partial |\nabla w|^2}{\partial \nu} \le C_1 \int_{\partial\Omega} |\nabla w|^{2q} \text{ for all } t \in (0, T)$$

for some $C_1 > 0$ and lemma 3.10 yields

$$C_1 \int_{\partial\Omega} |\nabla w|^{2q} \le 3 \frac{q-1}{2q^2} \int_\Omega |\nabla |\nabla w|^q|^2 + C_2 \int_\Omega |\nabla w|^{2q} \text{ for all } t \in (0, T)$$

for some positive C_2. Lemma 3.9 shows

$$C_2 \int_\Omega |\nabla w|^{2q} \le C_3 + C_3 \left(\int_\Omega |\nabla |\nabla w|^q|^2 \right)^{\lambda_1} \quad \text{for all } t \in (0,T)$$

with $\lambda_1 = \frac{\frac{nq}{s} - \frac{n}{2}}{1 - \frac{n}{2} + \frac{nq}{s}} < 1$ and some positive constant C_3. Young's inequality then gives us $C_4 > 0$ such that

$$C_3 \left(\int_\Omega |\nabla |\nabla w|^q|^2 \right)^{\lambda_1} \le 3\frac{q-1}{2q^2} \int_\Omega |\nabla |\nabla w|^q|^2 + C_4 \text{ for all } t \in (0,T)$$

and thereby we have proven

$$\int_{\partial\Omega} |\nabla w|^{2q-2} \frac{\partial |\nabla w|^2}{\partial \nu} \le 3\frac{q-1}{q^2} \int_\Omega |\nabla |\nabla w|^q|^2 + C_5 \text{ for all } t \in (0,T)$$

for some positive constant C_5. Integration by parts therefore results in

$$\int_\Omega |\nabla w|^{2q-2} \Delta |\nabla w|^2 \le -\frac{q-1}{q^2} \int_\Omega |\nabla |\nabla w|^q|^2 + C_5 \text{ for all } t \in (0,T),$$

as claimed. □

3.1.3. Analysis of a coupled function

In this section we put some of the results of the previous chapter to use by applying them to our context. We begin by proving the crucial

Lemma 3.12. *Let $D \in \mathcal{C}^2([0,\infty))$ with*

$$D(s) \ge C_D (s+1)^{m-1} \textit{ for all } s \in [0,\infty)$$

for some $m \in \mathbb{R}$ and $C_D > 0$ as well as $S \in \mathcal{C}^2([0,\infty))$ with

$$|S(s)| \le C_S (s+1)^\kappa \textit{ for all } s \in [0,\infty)$$

for some $\kappa \in \mathbb{R}$ with $\kappa < m+1$ and $C_S > 0$. Let $T > 0$ and (u,v) a classical solution to the differential equations in (KS) in $\Omega \times (0,T)$ with $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega \times (0,T)$. Furthermore let $p_0 \in \left(\frac{n}{2}(1+\kappa-m), n\right)$ with $p_0 \ge 1$, $C_{p_0} > 0$ and $\|u\|_{L^\infty((0,T),L^{p_0}(\Omega))} \le C_{p_0}$. Taking p and q from lemmata 3.1 and 3.2 we then have $C > 0$ such that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{p} \int_\Omega (u+1)^p + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \right] + \frac{2(p-1)C_D}{(m+p-1)^2} \int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 + \frac{q-1}{2q^2} \int_\Omega |\nabla |\nabla v|^q|^2 \\ &\le C + C \int_\Omega (u+1)^{-m+p-1+2\kappa} |\nabla v|^2 + C \int_\Omega (u+1)^2 |\nabla v|^{2q-2} \end{aligned}$$

holds for any $t \in (0,T)$.

Proof. We begin by computing

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^p &= \int_\Omega (u+1)^{p-1}u_t \\
&= \int_\Omega (u+1)^{p-1}\nabla\cdot(D(u)\nabla u) - \int_\Omega (u+1)^{p-1}\nabla\cdot(S(u)\nabla v) \\
&= -(p-1)\int_\Omega D(u)(u+1)^{p-2}|\nabla u|^2 \\
&\quad + (p-1)\int_\Omega S(u)(u+1)^{p-2}\nabla u\cdot\nabla v \\
&\le -(p-1)C_D\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 \\
&\quad + (p-1)C_S\int_\Omega (u+1)^{\kappa+p-2}|\nabla u||\nabla v| \ \forall t\in(0,T),
\end{aligned}$$

using the assumed estimates for both D and S as well as integration by parts and continue by estimating the rightmost term. Note the positivity of the exponents according to (p2) and (p7). By Young's inequality, we see that

$$\begin{aligned}
\int_\Omega (u+1)^{\kappa+p-2}|\nabla u||\nabla v| &= \int_\Omega (u+1)^{\frac{m+p-3}{2}}|\nabla u|\ (u+1)^{\frac{-m+p-1}{2}+\kappa}|\nabla v| \\
&\le \frac{C_D}{2C_S}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 \\
&\quad + \frac{C_S}{2C_D}\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2 \text{ for all } t\in(0,T),
\end{aligned}$$

and so we have

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^p + \frac{(p-1)C_D}{2}\int_\Omega (u+1)^{m+p-3}|\nabla(u+1)|^2 \\
&\le \frac{(p-1)C_S^2}{2C_D}\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2 \text{ for all } t\in(0,T).
\end{aligned}$$

Using that

$$\frac{\partial}{\partial t}|\nabla v|^2 = 2\nabla v\cdot\nabla v_t = 2\nabla v\cdot\nabla\Delta v - 2|\nabla v|^2 + 2\nabla u\cdot\nabla v$$

as well as

$$\Delta|\nabla v|^2 = 2\nabla\cdot(D^2v\nabla v) = 2|D^2v|^2 + 2\nabla v\cdot\nabla\Delta v,$$

lemma 3.11, which we may invoke due to lemma 3.4, yields some $C_1 > 0$ such that

$$
\begin{aligned}
\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} &= \int_\Omega |\nabla v|^{2q-2}\frac{\partial}{\partial t}|\nabla v|^2 \\
&= \int_\Omega |\nabla v|^{2q-2}\left(\Delta|\nabla v|^2 - 2|D^2 v|^2 - 2|\nabla v|^2 + 2\nabla u\cdot\nabla v\right) \\
&\le -\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 - 2\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 + 2\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v + C_1
\end{aligned}
$$

holds for all $t \in (0,T)$. We rearrange this to see that for any $t \in (0,T)$ we in point of fact have

$$
\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} + \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + 2\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 \le 2\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v + C_1,
$$

and once again continue with the integral on the right-hand side. We employ Young's inequality twice more to find

$$
\begin{aligned}
2\int_\Omega |\nabla v|^{2q-2}\nabla u\cdot\nabla v &= -2\int_\Omega u\nabla\cdot\left(|\nabla v|^{2q-2}\nabla v\right) \\
&= -2(q-1)\int_\Omega u|\nabla v|^{2q-4}\nabla v\cdot\nabla|\nabla v|^2 - 2\int_\Omega u|\nabla v|^{2q-2}\Delta v \\
&\le \frac{q-1}{8}\int_\Omega |\nabla v|^{2q-4}|\nabla|\nabla v|^2|^2 + 2(q-1)\int_\Omega u^2|\nabla v|^{2q-2} \\
&\quad + \frac{2}{n}\int_\Omega |\nabla v|^{2q-2}|\Delta v|^2 + \frac{n}{2}\int_\Omega u^2|\nabla v|^{2q-2} \\
&\le \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 + \left[2(q-1)+\frac{n}{2}\right]\int_\Omega (u+1)^2|\nabla v|^{2q-2} \\
&\quad + 2\int_\Omega |\nabla v|^{2q-2}|D^2v|^2 \text{ for all } t\in(0,T),
\end{aligned}
$$

wherein for the last step we used the pointwise estimate

$$
|\Delta v|^2 \le n|D^2 v|^2.
$$

After cancellation we see for all $t \in (0,T)$

$$
\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^{2q} + \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 \le \left[2(q-1)+\frac{n}{2}\right]\int_\Omega (u+1)^2|\nabla v|^{2q-2} + C_1
$$

which in total means

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{p}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right] + \frac{(p-1)C_D}{2}\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 + \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 \\
&\le C_2\int_\Omega (u+1)^{-m+p-1+2\kappa}|\nabla v|^2 + C_2\int_\Omega (u+1)^2|\nabla v|^{2q-2} + C_2
\end{aligned}
$$

for suitable $C_2 > 0$ and all $t \in (0,T)$. □

3.1.4. A series of applications of the Gagliardo–Nirenberg inequality

In order to compare the terms in lemma 3.12 we interpolate on the right-hand side via Gagliardo–Nirenberg; in fact we do so four times in this chapter and stress the importance of the respective exponents.

Lemma 3.13. *Under the assumptions from the previous lemma 3.12 and also taking θ and μ from lemma 3.1 as well as β_1, β_2, γ_1 and γ_2 from lemma 3.2, we can find a positive constant C such that*

$$\left(\int_\Omega (u+1)^{(-m+p-1+2\kappa)\theta}\right)^{\frac{1}{\theta}} \le C + C\left(\int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\beta_1},$$

$$\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} \le C + C\left(\int_\Omega \left|\nabla |\nabla v|^q\right|^2\right)^{\gamma_1},$$

$$\left(\int_\Omega (u+1)^{2\mu}\right)^{\frac{1}{\mu}} \le C + C\left(\int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\beta_2}$$

and

$$\left(\int_\Omega |\nabla v|^{2(q-1)\mu'}\right)^{\frac{1}{\mu'}} \le C + C\left(\int_\Omega \left|\nabla |\nabla v|^q\right|^2\right)^{\gamma_2}$$

hold in $(0, T)$.

Proof. Firstly, for any $t \in (0, T)$ the Gagliardo–Nirenberg inequality in lemma 3.7 together with ($\theta 1$) gives us positive constants C_1 and C_2 such that for $k := \frac{2(-m+p-1+2\kappa)}{m+p-1}$

$$\begin{aligned}\left\|(u+1)^{\frac{m+p-1}{2}}\right\|^k_{L^{k\theta}(\Omega)} &\le C_1\left\|\nabla(u+1)^{\frac{m+p-1}{2}}\right\|^{ka}_{L^2(\Omega)}\left\|(u+1)^{\frac{m+p-1}{2}}\right\|^{k(1-a)}_{L^{\frac{2p_0}{m+p-1}}(\Omega)} \\ &\quad + C_1\left\|(u+1)^{\frac{m+p-1}{2}}\right\|^k_{L^{\frac{2p_0}{m+p-1}}(\Omega)} \\ &\le C_2\left[1+\left(\int_\Omega\left|\nabla(u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\beta_1}\right]\end{aligned}$$

holds for all $t \in (0, T)$ with

$$\frac{-m+p-1+2\kappa}{m+p-1}a = \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - \frac{1}{\theta}\right)}{1-\frac{n}{2}+\frac{n(m+p-1)}{2p_0}} = \beta_1.$$

Due to (s1), for some $C_3 > 0$ we can also estimate

$$\begin{aligned}\||\nabla v|^q\|^{\frac{2}{q}}_{L^{\frac{2\theta'}{q}}(\Omega)} &\le C_1\|\nabla|\nabla v|^q\|^{\frac{2}{q}c}_{L^2(\Omega)}\||\nabla v|^q\|^{\frac{2}{q}(1-c)}_{L^{\frac{s}{q}}(\Omega)} + C_1\||\nabla v|^q\|^{\frac{2}{q}}_{L^{\frac{s}{q}}(\Omega)} \\ &\le C_3\left[1+\left(\int_\Omega\left|\nabla|\nabla v|^q\right|^2\right)^{\gamma_1}\right]\end{aligned}$$

for

$$\frac{1}{q}c = \frac{\frac{n}{2}(\frac{2}{s} - \frac{1}{\theta'})}{1 - \frac{n}{2} + \frac{nq}{s}} = \gamma_1$$

and all times $t \in (0, T)$. Since ($\mu 1$) holds, we analogously have a positive constant C_4 such that

$$\begin{aligned}\left\|(u+1)^{\frac{m+p-1}{2}}\right\|_{L^{\frac{4\mu}{m+p-1}}(\Omega)}^{\frac{4}{m+p-1}} &\leq C_1 \left\|\nabla(u+1)^{\frac{m+p-1}{2}}\right\|_{L^2(\Omega)}^{\frac{4}{m+p-1}b} \left\|(u+1)^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{\frac{4}{m+p-1}(1-b)} \\ &\quad + C_1 \left\|(u+1)^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{\frac{4}{m+p-1}} \\ &\leq C_4 \left[1 + \left(\int_\Omega \left|\nabla(u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\beta_2}\right]\end{aligned}$$

for any $t \in (0, T)$ with

$$\frac{2b}{m+p-1} = \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}} = \beta_2.$$

Lastly, this time due to (s2), we find $C_5 > 0$ fulfilling

$$\begin{aligned}\||\nabla v|^q\|_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)}^{\frac{2(q-1)}{q}} &\leq C_1 \|\nabla|\nabla v|^q\|_{L^2(\Omega)}^{\frac{2(q-1)}{q}d} \||\nabla v|^q\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2(q-1)}{q}(1-d)} + C_1 \||\nabla v|^q\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2(q-1)}{q}} \\ &\leq C_5 \left[1 + \left(\int_\Omega |\nabla |\nabla v|^q|^2\right)^{\gamma_2}\right]\end{aligned}$$

for

$$\frac{q-1}{q}d = \frac{\frac{n}{2}(\frac{2(q-1)}{s} - \frac{1}{\mu'})}{1 - \frac{n}{2} + \frac{nq}{s}} = \gamma_2$$

for any $t \in (0, T)$.

We note that the Gagliardo–Nirenberg inequality indeed is applicable due to the following observations. Trivially we see

$$\frac{2(-m+p-1+2\kappa)\theta}{m+p-1} > 1$$

and we quickly find

$$\frac{2\theta'}{q} > 1$$

and

$$\theta < \frac{q}{q-2}$$

as well as

$$\frac{4\mu}{m+p-1} > 1$$

and

$$\mu > \frac{m+p-1}{4}$$

to be equivalent by pairs. Additionally,

$$\frac{2(q-1)\mu'}{q} > 1$$

holds if and only if

$$\frac{1}{\mu} > \frac{2-q}{q}$$

is true. The first three requirements are secured by ($\theta 1$), ($\theta 2$) and ($\mu 2$) respectively and the negative right-hand side in the last line completes this set of computations, which means that our previous inequalities truly hold. □

This step immediately results in

Lemma 3.14. *Keeping the assumptions from lemma 3.13 we have*

$$\int_\Omega (u+1)^{-m+p-1+2\kappa} |\nabla v|^2 \leq C \left[1 + \left(\int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 \right)^{\beta_1} \right] \left[1 + \left(\int_\Omega |\nabla |\nabla v|^q|^2 \right)^{\gamma_1} \right]$$

and

$$\int_\Omega (u+1)^2 |\nabla v|^{2q-2} \leq C \left[1 + \left(\int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 \right)^{\beta_2} \right] \left[1 + \left(\int_\Omega |\nabla |\nabla v|^q|^2 \right)^{\gamma_2} \right]$$

with some $C > 0$ and for all $t \in (0, T)$ respectively.

Proof. The Hölder inequality allows for decomposing the integrals into their respective factors for which we then employ lemma 3.13. This means everything follows from the cited lemma as well as the estimates

$$\int_\Omega (u+1)^{-m+p-1+2\kappa} |\nabla v|^2 \leq \left(\int_\Omega (u+1)^{(-m+p-1+2\kappa)\theta} \right)^{\frac{1}{\theta}} \left(\int_\Omega |\nabla v|^{2\theta'} \right)^{\frac{1}{\theta'}}$$

and

$$\int_\Omega (u+1)^2 |\nabla v|^{2q-2} \leq \left(\int_\Omega (u+1)^{2\mu} \right)^{\frac{1}{\mu}} \left(\int_\Omega |\nabla v|^{2(q-1)\mu'} \right)^{\frac{1}{\mu'}}$$

which hold for all $t \in (0, T)$ respectively. □

3.1.5. An ordinary differential inequality

Using lemma 3.3 and the fact that the numbers β_j and γ_j for $j \in \{1,2\}$ from lemma 3.2 satisfy

$$\beta_j + \gamma_j < 1 \text{ for } j \in \{1,2\},$$

we proceed to derive

Lemma 3.15. *Let $D \in \mathcal{C}^2([0,\infty))$ with*

$$D(s) \geq C_D(s+1)^{m-1} \textit{ for all } s \in [0,\infty)$$

for some $m \in \mathbb{R}$ and $C_D > 0$ as well as $S \in \mathcal{C}^2([0,\infty))$ with

$$|S(s)| \leq C_S(s+1)^{\kappa} \textit{ for all } s \in [0,\infty)$$

for some $\kappa \in \mathbb{R}$ with $\kappa < m+1$ and $C_S > 0$. Let $T > 0$ and (u,v) a classical solution to the differential equations in (KS) in $\Omega \times (0,T)$ with $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega \times (0,T)$. Furthermore let $p_0 \in \left(\frac{n}{2}(1+\kappa-m), n\right)$ with $p_0 \geq 1$, $C_{p_0} > 0$ and $\|u\|_{L^\infty((0,T),L^{p_0}(\Omega))} \leq C_{p_0}$. Taking p, θ and μ from lemma 3.1 as well as q, β_1, β_2, γ_1 and γ_2 from lemma 3.2, we can find $\lambda > 0$ as well as a positive constant C satisfying

$$\dot{y}(t) + \frac{1}{C} y^{\lambda}(t) \leq C \textit{ for all } t \in (0,T)$$

for $y(t) := \frac{1}{p}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}$.

Proof. Due to lemmata 3.3, 3.12 and 3.14 we already know that for the parameters from lemma 3.2 there are $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{p}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right] + \frac{2(p-1)C_D}{(m+p-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2 + \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 \\
&\leq C_1 + C_1\left[1 + \left(\int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\beta_1}\right]\left[1 + \left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_1}\right] \\
&\quad + C_1\left[1 + \left(\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2\right)^{\beta_2}\right]\left[1 + \left(\int_\Omega |\nabla|\nabla v|^q|^2\right)^{\gamma_2}\right] \\
&\leq C_2 + \frac{(p-1)C_D}{(m+p-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2 + \frac{q-1}{4q^2}\int_\Omega |\nabla|\nabla v|^q|^2
\end{aligned}$$

holds for any $t \in (0,T)$. This shows that with some $C_3 > 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{p}\int_\Omega (u+1)^p + \frac{1}{q}\int_\Omega |\nabla v|^{2q}\right] + C_3\int_\Omega \left|\nabla (u+1)^{\frac{m+p-1}{2}}\right|^2 + C_3\int_\Omega \left|\nabla\,|\nabla v|^q\right|^2 \leq C_2$$

for all $t \in (0,T)$. Employing the Gagliardo–Nirenberg interpolation inequality once more, we can compare the integral containing the gradient of u to the first term in y. If we estimate

$$
\begin{aligned}
\int_\Omega (u+1)^p &= \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2p}{m+p-1}}(\Omega)}^{\frac{2p}{m+p-1}} \\
&\le C_4 \left\| \nabla (u+1)^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{\frac{2p}{m+p-1}\lambda_2} \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{\frac{2p}{m+p-1}(1-\lambda_2)} \\
&\quad + C_4 \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{\frac{2p}{m+p-1}} \\
&\le C_5 \left[1 + \left(\int_\Omega \left| \nabla (u+1)^{\frac{m+p-1}{2}} \right|^2 \right)^{\frac{p\lambda_2}{m+p-1}} \right]
\end{aligned}
$$

for some positive constants C_4, C_5 and λ_2, and apply lemma 3.9 to v, we see that indeed

$$
\dot{y} + \frac{1}{C_6} y^\lambda \le C_6 \text{ for all } t \in (0, T),
$$

again for some positive constants C_6 and λ. $\square$

Proof of theorem 2.2 for $\kappa < m + 1$. A comparison argument for ordinary differential equations shows that in $\tilde{C} := \max\left\{ y_0, C^{\frac{2}{\lambda}} \right\}$ we have an upper bound for any solution $y \in \mathcal{C}^1(0, T_{\max})$ to

$$
\begin{cases}
\dot{y} & \le -\frac{1}{C} y^\lambda + C \text{ in } (0, T_{\max}), \\
y(0) & = y_0,
\end{cases}
$$

and thus according to lemma 3.15 for any $p > 1$ we find a constant $C_p > 0$ that admits the inequality

$$
\| u(\cdot, t) \|_{L^p(\Omega)} \le C_p \text{ for all } t \in (0, T_{\max}).
$$

Together with lemmata 3.4 and 3.5 this proves the first portion of theorem 2.2 in light of the extensibility criterion in lemma 2.1. $\square$

3.2. Part II: $\kappa \ge m + 1$

In many ways this section follows the previous, and so we may focus on outlining the main differences.

Let us assume $\|u\|_{L^\infty((0,T), L^{p_0}(\Omega))} \le C_{p_0}$ with some $p_0 > n(\kappa - m) \ge n$ and $C_{p_0} > 0$, which by lemma 3.4 immediately results in

$$
\| v(\cdot, t) \|_{W^{1,q}(\Omega)} \le C
$$

for arbitrary $q \ge 1$, some $C > 0$ and all times $t \in (0, T)$ if (u, v) solves the second equation in (KS) in $\Omega \times (0, T)$. Again some parameters need to be fixed and their existence verified. This is the purpose of

Lemma 3.16. *Given $m \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ with $\kappa \ge m + 1$ as well as $p_0 > n(\kappa - m)$ for each arbitrarily large $p > p_0$ with*

$$
p > m + 1 - 2\kappa + p_0 > m + 1 - 2\kappa, \tag{p1'}
$$

$$
p > 4p_0 + 1 - m \tag{p2'}
$$

and

$$p > 3m + 1 - 4\kappa \tag{p3'}$$

there is some $s \in \mathbb{R}$ *with*

$$s > \frac{1}{\frac{1}{n} - \frac{\kappa - m}{p_0}} > 2 \tag{s1'}$$

such that for

$$q = s\frac{m+p-1}{2p_0} > \frac{m+p-1}{2p_0} > 2 > \frac{2p_0}{2+p_0} \tag{q1'}$$

we have

$$\frac{s}{2(q-1)} < 1. \tag{sq'}$$

Proof. Let us first remark on (s1'): Since the denominator is positive and because of $\kappa \geq m+1$ we have $s > \frac{1}{\frac{1}{n} - \frac{1}{p_0}}$ which is bigger than 2 due to $p_0 > \frac{2n}{n+2}$.

The desired inequality

$$1 > \frac{s}{2(q-1)} = \frac{p_0 q}{(q-1)(m+p-1)}$$

is equivalent to

$$[(m+p-1) - p_0] > \frac{m+p-1}{q}$$

and this in turn follows from

$$q > \frac{1}{1 - \frac{p_0}{m+p-1}}.$$

Herein the right-hand side can be estimated from above by $\frac{4}{3}$ which is a trivial lower bound for q in light of (q1'). □

Again we need some Hölder and Gagliardo–Nirenberg exponents consisting of the previously defined parameters. We have to verify that the above choices are admissible for the intended inequalities and do so in the following

Lemma 3.17. *Given* $m \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}$ *with* $\kappa \geq m+1$ *as well as* $p_0 > n(\kappa - m)$ *for the quantities* p, q *and* s *of lemma 3.16 there are two numbers* $\theta > 1$ *and* $\mu > 1$ *satisfying*

$$\theta > \frac{m+p-1}{2(-m+p-1+2\kappa)} \tag{θ1'}$$

and

$$\theta > \frac{p_0}{-m+p-1+2\kappa} \tag{θ2'}$$

as well as

$$\theta < \frac{s}{s-2} \tag{θ3'}$$

and

$$\theta < \frac{q}{q-2}. \tag{θ4'}$$

Furthermore,

$$\mu > \frac{p_0}{2} \tag{μ1'}$$

and

$$\mu > \frac{m+p-1}{4}, \tag{μ2'}$$

hold and for the conjugate exponents $\theta' = \frac{\theta}{\theta-1}$ *and* $\mu' = \frac{\mu}{\mu-1}$ *we can achieve*

$$\frac{2\theta'}{q} > 1 \tag{θ5'}$$

as well as

$$\frac{2(q-1)\mu'}{q} > 1. \tag{μ3'}$$

Proof. The lower bounds for θ are less than 1 due to (p3') and (p1') respectively while the upper bounds obviously are larger than 1.

While (θ5') is equivalent to

$$\frac{1}{\theta'} < \frac{2}{q}$$

and therefore guaranteed by (θ4'), the fact that $q > 2$ is enough to show

$$1 - \frac{1}{\mu} < \frac{2(q-1)}{q}$$

and thereby (μ3'). □

We close this chapter by proving that these choices allow for the employment of lemma 3.3:

Lemma 3.18. *We assume* $p_0 > n(\kappa - m)$ *for some* $m \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}$ *with* $\kappa \geq m+1$ *and the quantities* p, s, q, θ *and* μ *to be as in lemmata 3.16 and 3.17. We define*

$$\beta_1 := \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - \frac{1}{\theta}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}},$$

$$\gamma_1 := \frac{\frac{n}{2}\left(\frac{2}{s} - \frac{1}{\theta'}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}},$$

$$\beta_2 := \frac{\frac{n}{2}\left(\frac{2}{p_0} - \frac{1}{\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

and

$$\gamma_2 := \frac{\frac{n}{2}\left(\frac{m+p-1}{p_0} - \frac{2}{s} - \frac{1}{\mu'}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}.$$

For the quantities $f := \beta_1 + \gamma_1$ and $g := \beta_2 + \gamma_2$ we once more have that

$$f < 1 \text{ and } g < 1$$

hold.

Remark. Note

$$\frac{nq}{s} = \frac{n(m+p-1)}{2p_0}$$

and

$$\frac{2(q-1)}{s} = \frac{2q}{s} - \frac{2}{s} = \frac{m+p-1}{p_0} - \frac{2}{s}$$

and the resulting similarities between the quantities in this lemma and in the previous section.

Proof of lemma 3.18. Firstly we see that

$$1 > f = \frac{\frac{n}{2}\left(\frac{-m+p-1+2\kappa}{p_0} - 1 + \frac{2}{s}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

is equivalent to

$$\frac{2}{n} - 1 + \frac{m+p-1}{p_0} > \frac{-m+p-1+2\kappa}{p_0} - 1 + \frac{2}{s}$$

and therefore we need to ensure

$$\frac{1}{s} < \frac{1}{n} - \frac{\kappa - m}{p_0}.$$

This however is true since we have demanded $p_0 > n(\kappa - m)$ and $s > \frac{1}{\frac{1}{n} - \frac{\kappa - m}{p_0}}$. On the other hand,

$$1 > g = \frac{\frac{n}{2}\left(\frac{2}{p_0} - 1 + \frac{m+p-1}{p_0} - \frac{2}{s}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2p_0}}$$

holds if and only if

$$\frac{2}{n} - 1 + \frac{m+p-1}{p_0} > \frac{2}{p_0} - 1 + \frac{m+p-1}{p_0} - \frac{2}{s}.$$

This is ensured by $p_0 > n$ and $s > 0$ which in combination yield $\frac{1}{n} > \frac{1}{p_0} - \frac{1}{s}$. $\square$

We then return to the previous chapter and repeat the necessary computations to obtain the same results as before meaning we can complete the proof for our theorem by carefully replacing the parameters used in the case $\kappa < m + 1$ by the ones defined in this section.

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