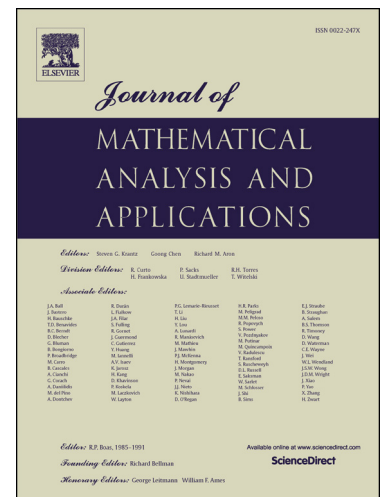


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# Global existence and boundedness in a chemotaxis-haptotaxis system with signal-dependent sensitivity

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**Abstract.** This paper deals with the chemotaxis-haptotaxis system with *signal-dependent sensitivity*

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi(v)u\nabla v) - \xi \nabla \cdot (u\nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ w_t = -vw, & x \in \Omega, t > 0 \end{cases}$$

under homogeneous Neumann boundary conditions and initial conditions, where  $\Omega \subset \mathbb{R}^n$  ( $n \geq 3$ ) is a bounded domain with smooth boundary,  $\xi, \mu > 0$  are constants and  $\chi$  is a function satisfying some conditions. In the case that  $\chi$  is a constant it is known that the above system possesses a global classical solution under some conditions (Cao [4], Tao [10], Tao and Winkler [11]); however, in the case that  $\chi$  is a function, the above system has not been studied. The purpose of this paper is to establish global existence and boundedness in the above system.

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## 1. Introduction

There are many important properties which support our active and comfortable life. One of these important properties is *taxis*. Taxis is the property such that species react some stimuli and move towards, or away from, these stimuli. This property helps us such as, e.g., species move towards their foods, and move away from poison for them. Usually taxis is classified by “which stimuli species react on”. The case that species react on a chemical substance is called *chemotaxis*. Chemotaxis is related to e.g., the movement of sperm, the migration of neurons or lymphocytes: Thus chemotaxis is closely related with not only the beginning but also the maintenance of our life. The system which describes the aggregation of the species by chemotaxis

$$u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \mu u(1 - u), \quad \tau v_t = \Delta v - v + u,$$

where  $\chi, \mu > 0$  and  $\tau = 0, 1$ , is called a *chemotaxis system* and is studied intensively: in the two-dimensional setting for all  $\chi, \mu > 0$  global existence and boundedness were established by Osaki et. al. [9], Tello and Winkler [13]; in the higher-dimensional setting with sufficiently large  $\mu > 0$  Winkler [14, 16] obtained global classical bounded solutions; related works can be found in a survey [1, Section 3] by Bellomo et. al; a model with nonlinear diffusion terms, for instance flux limited diffusion, was studied by Bellomo and Winkler [2, 3]. Recently, Chaplain and Lolas [5] researched a tumor invasion, and proposed the *chemotaxis-haptotaxis* system with constant sensitivity

$$\begin{aligned} u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \\ \tau v_t &= \Delta v - v + u, \\ w_t &= -vw, \end{aligned}$$

where  $\chi, \xi, \mu > 0$  and  $\tau = 0, 1$ . This system mainly consists of the chemotaxis  $\nabla \cdot (u \nabla v)$  and the *haptotaxis*  $\nabla \cdot (u \nabla w)$ . In this system there are some results about global existence and boundedness: In the case that  $\tau = 0$  global existence and boundedness were shown by Tao and Winkler [11]; in the case that  $\tau = 1$  and  $\Omega \subset \mathbb{R}^2$  for all  $\chi, \xi, \mu > 0$  global existence and boundedness were obtained by Tao [10]; in the case that  $\tau = 1$ ,  $\Omega \subset \mathbb{R}^3$  and  $\mu > 0$  is large, global existence of bounded solutions was shown by Cao [4].

In summary, in the case of constant sensitivity some results related to the above chemotaxis-haptotaxis system are known; however, the chemotaxis-haptotaxis system with signal-dependent sensitivity, i.e.,  $\chi \nabla \cdot (u \nabla v)$  is replaced with  $\nabla \cdot (u \chi(v) \nabla v)$ , which determines the chemotactic power depending on the stimuli, has not been studied. In this paper we consider the chemotaxis-haptotaxis system with signal-dependent sensitivity

$$(1.1) \quad \begin{cases} u_t = \Delta u - \nabla \cdot (\chi(v) u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \\ w_t = -vw, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} - \chi(v) u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), & x \in \Omega, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  ( $n \geq 3$ ) with smooth boundary  $\partial \Omega$ ,  $\nu$  is the outward normal vector to  $\partial \Omega$ ,  $\xi, \mu > 0$  are constants and the initial data are regular enough and

satisfy a standard compatibility condition in the sense that

$$(1.2) \quad \begin{cases} u_0 \in C^0(\overline{\Omega}) \text{ with } u_0 \geq 0 \text{ and } u_0 \not\equiv 0, & v_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 \geq 0, \\ w_0 \in C^{2+\alpha}(\overline{\Omega}) \text{ } (\exists \alpha \in (0, 1)) \text{ with } w_0 > 0 \text{ and } \frac{\partial w_0}{\partial \nu} \Big|_{\partial \Omega} = 0. \end{cases}$$

Here  $u$ ,  $v$  and  $w$  denote the density of cells, the concentration of matrix-degrading enzyme (MDE) and the density of extracellular matrix (ECM), respectively. In the case that  $w = 0$  in (1.1), which namely is the chomotaxis system with signal-dependent sensitivity, global existence and boundedness were established under the condition that

$$(1.3) \quad \exists p > n; \quad \chi'(s) + \sqrt{p}|\chi(s)|^2 \leq 0$$

for all  $s > 0$  (c.f. [7]) or under the condition that

$$(1.4) \quad 0 \leq \chi(s) \leq \frac{K}{(a+s)^k} \quad \text{for all } s > 0$$

with some  $a \geq 0$ ,  $k > 1$  and  $K > 0$  satisfying

$$(1.5) \quad K < k(a+\eta)^{k-1} \sqrt{\frac{2}{n}}$$

for some constant  $\eta \geq 0$  ([6, 8]). Here we note that the conditions (1.3) and (1.4)–(1.5) are independent of  $\mu > 0$ ; therefore we expect global existence and boundedness in (1.1) under a condition similar to (1.3) or (1.4)–(1.5) instead of largeness conditions for  $\mu > 0$ . The purpose of this paper is to establish global existence and boundedness in (1.1) under some conditions only for  $\chi$ .

To attain the goal of this paper we will suppose that  $\chi$  satisfies the following conditions:

$$(1.6) \quad \chi \in C^{1+\theta}([0, \infty)) \cap L^1(0, \infty) \quad (0 < \exists \theta < 1), \quad \chi > 0,$$

$$(1.7) \quad \exists C_1 > 0; \quad \chi(s)s \leq C_1 \quad \text{for all } s \geq 0,$$

$$(1.8) \quad \exists p_0 \in (n, n+1) \quad \chi'(s) + \alpha_{p_0}|\chi(s)|^2 \leq 0 \quad \text{for all } s > 0,$$

where  $\alpha_p$  is a positive constant defined as

$$(1.9) \quad \alpha_p := \frac{\eta(p, \varepsilon_1)}{p\varepsilon_1(\varepsilon_1 + 1)}$$

with some  $\varepsilon_1 \in (0, 1)$ , where  $\eta(p, \varepsilon_1)$  is a constant given later (see (3.5)). Now the main result reads as follows.

**Theorem 1.1.** *Let  $n \geq 3$  and let  $\xi > 0$ ,  $\mu > 0$ . Assume that  $\chi$  satisfies (1.6)–(1.8). Then for any  $(u_0, v_0, w_0)$  fulfilling (1.2), (1.1) possesses a unique global classical solution*

$$\begin{aligned} u &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ v &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L_{\text{loc}}^\infty([0, \infty); W^{1,q}(\Omega)), \\ w &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \end{aligned}$$

for some  $q > n$ . Moreover, the solution  $(u, v, w)$  is bounded uniformly-in-time:

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C$$

for all  $t > 0$  with some  $C > 0$ .

**Remark 1.1.** This theorem gives global existence and boundedness in the case that  $n \geq 3$ . On the other hand, in the case that  $n = 1, 2$ , we can prove global existence of bounded solutions to (1.1) under the condition that  $\chi \in C^{1+\theta}([0, \infty)) \cap W^{1,\infty}(0, \infty)$  by using the same strategy as in the proof of [10, Theorem 1.1] (see Remark 3.1).

The strategy for the proof of Theorem 1.1 is to build the  $L^p$ -estimate for  $u$  with some  $p > n$ . In order to establish the desired estimate we use the energy estimate for  $\int_{\Omega} u^p f(v, w)^{-r}$  with some smooth function  $f$  and some constant  $r > 0$ . This strategy is based on ideas in [7] which overcome difficulties of the signal-dependent sensitivity. In the case that  $w = 0$  in (1.1), considering the estimate for  $\int_{\Omega} u^p \varphi(v)^{-r}$  with some smooth function  $\varphi$ , we can see the  $L^p$ -estimate for  $u$  ([7]). However, we cannot give the same energy estimate for  $\int_{\Omega} u^p \varphi(v)^{-r}$  because there is a new unknown function  $w$ . Moreover, in the case that  $\chi(s) \equiv \chi$  we have already obtained a useful estimate for  $\Delta w$  ([4, Lemma 2.2]). Unfortunately (1.1) includes  $\chi(v)$ , and so we cannot use such estimate. Thus we put

$$f(v, w) := \varphi(v) \cdot e^{Cw}$$

with some  $C > 0$  and obtain from an application of the estimate for  $\Delta w$  (see Lemma 2.2) that there is  $C > 0$  such that

$$\int_{\Omega} u^p (f(v, w)^{-r} + 1) \leq C,$$

which entails the  $L^p$ -estimate for  $u$ .

The plan of this paper is as follows. In Section 2 we collect basic facts which will be used later. Section 3 is devoted to proving global existence and boundedness.

## 2. Preliminaries

We first state a result on local existence of classical solutions.

**Lemma 2.1.** *Let  $n \in \mathbb{N}$ ,  $\xi > 0$  and  $\mu > 0$ . Assume that  $\chi$  satisfies (1.6)–(1.8). Then for any  $(u_0, v_0, w_0)$  satisfying (1.2), there is  $T_{\max} \in (0, \infty]$  such that (1.1) admits a unique classical solution*

$$\begin{aligned} u &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ v &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L_{\text{loc}}^{\infty}([0, T_{\max}); W^{1,q}(\Omega)) \quad (q > n), \\ w &\in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \end{aligned}$$

such that

$$u > 0, \quad v \geq 0 \quad \text{and} \quad 0 < w \leq \|w_0\|_{L^{\infty}(\Omega)} \quad \text{for all } t \in (0, T_{\max}).$$

Moreover,

$$(2.1) \quad \text{if } T_{\max} < \infty, \text{ then } \limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^{\infty}(\Omega)} = \infty.$$

*Proof.* The proof of local existence of classical solutions to (1.1) is based on a standard contraction mapping argument which can be found in [12].  $\square$

Let  $s_0 \in (0, T_{\max})$ . According to the regularity of the solution (Lemma 2.1), we note that there exists  $M > 0$  such that

$$(2.2) \quad \|u(\cdot, s_0)\|_{L^\infty(\Omega)} + \|v(\cdot, s_0)\|_{L^\infty(\Omega)} + \|w(\cdot, s_0)\|_{W^{2,\infty}(\Omega)} \leq M.$$

Then an application of the lower estimate for  $\Delta w$  ([4, Lemmas 2.2 and 2.3]) leads to the following lemma.

**Lemma 2.2.** *Let  $\xi > 0$  and assume that (1.2) holds. Then for any  $p > 1$ ,  $s_0 \in (0, T_{\max})$ , some  $\varepsilon_2 \in (0, p)$ ,  $\varepsilon_3 \in (0, p(1 - \varepsilon_1))$ , the solution of (1.1) satisfies*

$$(2.3) \quad p(p-1)\xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w \leq b_1 \int_{\Omega} u^p + \mu \varepsilon_2 \int_{\Omega} u^{p+1} + (p-1)\varepsilon_3 \int_{\Omega} u^{p-2} |\nabla u|^2 \\ + b_2 \int_{\Omega} v^{p+1}$$

for all  $t \in (s_0, T_{\max})$  and some  $b_1 = b_1(p, M, \varepsilon_3, \xi)$ ,  $b_2 = b_2(p, M, \xi, \mu, \varepsilon_2) > 0$ .

*Proof.* Recalling the inequality in [4, Lemmas 2.2 and 2.3], we have already known that

$$p(p-1)\xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w \leq \left(3M\xi + \frac{1}{e}M\xi\right)p \int_{\Omega} u^p + Mp\xi \int_{\Omega} u^p v \\ + 2Mp(p-1)\xi \int_{\Omega} u^{p-1} |\nabla u|,$$

where in accordance with (2.2),  $M$  is an upper bound for  $\|w(\cdot, s_0)\|_{W^{2,\infty}(\Omega)}$ . Hence the Young inequality yields

$$p(p-1)\xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w \leq \left(3M\xi + \frac{1}{e}M\xi\right)p \int_{\Omega} u^p + \mu \varepsilon_2 \int_{\Omega} u^{p+1} + b_2 \int_{\Omega} v^{p+1} \\ + (p-1)\varepsilon_3 \int_{\Omega} u^{p-2} |\nabla u|^2 + b_0 \int_{\Omega} u^p$$

for some  $\varepsilon_2 \in (0, p)$ ,  $\varepsilon_3 \in (0, p(1 - \varepsilon_1))$ ,  $b_0 = b_0(p, M, \varepsilon_3, \xi)$ ,  $b_2 = b_2(p, M, \xi, \mu, \varepsilon_2) > 0$  and all  $t \in (s_0, T_{\max})$ . Thus by putting

$$b_1(p, M, \varepsilon_3, \xi) := \left(3M\xi + \frac{1}{e}M\xi\right)p + b_0(p, M, \varepsilon_3, \xi),$$

we have (2.3).  $\square$

The following lemma holds a key in the proof of Theorem 1.1.

**Lemma 2.3.** *Let  $r \geq 1$  and let  $(u, v, w)$  be a solution to (1.1). Then for all  $\varepsilon > 0$  there exists  $C > 0$  such that*

$$(2.4) \quad \int_{s_0}^t \int_{\Omega} e^{\frac{rs}{2}} |v(x, s)|^r dx ds \leq \varepsilon \int_{s_0}^t \int_{\Omega} e^{\frac{rs}{2}} |u(x, s)|^r dx ds + C e^{\frac{rt}{2}} + \varepsilon$$

for all  $t \in (s_0, T_{\max})$ .

*Proof.* Let  $t \in (s_0, T_{\max})$  and  $r \geq 1$ . Noting the compact embedding  $W^{2,r}(\Omega) \hookrightarrow L^r(\Omega)$  and the continuous embedding  $L^r(\Omega) \hookrightarrow L^1(\Omega)$ , we obtain from the Ehrling lemma that for all  $\tilde{\varepsilon} > 0$  there exists  $C_1 > 0$  such that

$$(2.5) \quad \|e^{\frac{s}{2}} v\|_{L^r(\Omega)} \leq \tilde{\varepsilon} \|e^{\frac{s}{2}} v\|_{W^{2,r}(\Omega)} + C_1 \|e^{\frac{s}{2}} v\|_{L^1(\Omega)} \quad \text{for all } s \in (s_0, t).$$

Now from the standard elliptic regularity argument we can find  $C_2 > 0$  such that

$$(2.6) \quad \|e^{\frac{s}{2}} v\|_{W^{2,r}(\Omega)} \leq C_2 \|e^{\frac{s}{2}} \Delta v\|_{L^r(\Omega)} + C_2 \|e^{\frac{s}{2}} v\|_{L^r(\Omega)} \quad \text{for all } s \in (s_0, t).$$

Then a combination of (2.5) and (2.6) means that for all  $\tilde{\varepsilon} > 0$  there exists  $C_3 > 0$  such that

$$(2.7) \quad \|e^{\frac{s}{2}} v\|_{L^r(\Omega)}^r \leq \tilde{\varepsilon} \|e^{\frac{s}{2}} \Delta v\|_{L^r(\Omega)}^r + C_3 e^{\frac{rs}{2}}.$$

Recalling from an application of the maximal Sobolev regularity ([4, Lemma 2.5]) that there is  $C_4 > 0$  such that

$$\int_{s_0}^t \int_{\Omega} e^{\frac{rs}{2}} |\Delta v(x, s)|^r dx ds \leq C_4 \int_{s_0}^t \int_{\Omega} e^{\frac{rs}{2}} |u(x, s)|^r dx ds + C_4,$$

we can derive from (2.7) that

$$\begin{aligned} \int_{s_0}^t \int_{\Omega} e^{\frac{rs}{2}} |v(x, s)|^r dx ds &\leq \tilde{\varepsilon} \int_{s_0}^t \int_{\Omega} e^{\frac{rs}{2}} |\Delta v(x, s)|^r dx ds + C_3 \int_{s_0}^t e^{\frac{rs}{2}} ds \\ &\leq \tilde{\varepsilon} C_4 \int_{s_0}^t \int_{\Omega} e^{\frac{rs}{2}} |u(x, s)|^r dx ds + \tilde{\varepsilon} C_4 + \frac{2C_3}{r} e^{\frac{rt}{2}} \end{aligned}$$

holds, which implies this lemma.  $\square$

We finally introduce the following lemma which gives us a strategy to show global existence and boundedness.

**Lemma 2.4.** *Let  $p > n$ . If there exists  $m_1 > 0$  such that*

$$(2.8) \quad \|u(\cdot, t)\|_{L^p(\Omega)} \leq m_1 \quad \text{for all } t \in (s_0, T_{\max}),$$

*then there exists  $m_2 > 0$  satisfying*

$$(2.9) \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq m_2 \quad \text{for all } t \in (s_0, T_{\max}).$$

*Proof.* Plugging (2.8) into  $L^p$ - $L^q$  estimates for the Neumann heat semigroup on bounded domains (see [15, Lemma 2.4]) implies that there is  $c_1 > 0$  such that

$$(2.10) \quad \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_1 \quad \text{for all } t \in (s_0, T_{\max}).$$

Hence a combination of (2.10) and an argument similar to that in [4, Lemma 3.5] leads to (2.9).  $\square$

### 3. Global existence and boundedness

In this section we show global existence and boundedness in (1.1) (Theorem 1.1). In the following let  $(u, v, w)$  be the solution of (1.1) on  $[0, T_{\max})$  as in Lemma 2.1. Thanks to Lemma 2.4, it is sufficient to show the  $L^p$ -estimate for  $u$  with some  $p > n$ . We first recall the following elementary estimates for  $u, v$ .

**Lemma 3.1.** *Let  $\xi > 0$  and  $\mu > 0$ . Assume that  $\chi$  satisfies (1.6)–(1.8) and  $(u_0, v_0, w_0)$  satisfies (1.2). Then there exists  $C > 0$  such that*

$$\int_{\Omega} u(\cdot, t) \leq C, \quad \int_{\Omega} v(\cdot, t) \leq C \quad \text{for all } t \in (0, T_{\max}).$$

*Proof.* By simply integrating the first and second equations in (1.1) on  $\Omega$  and using the Hölder inequality  $(\int_{\Omega} u)^2 \leq |\Omega| \int_{\Omega} u^2$  we see the  $L^1$ -boundedness of  $u$ , and then of  $v$ .  $\square$

In order to obtain the desired estimate for  $u$  we introduce the function  $f = f(v, w)$  by

$$f(v, w) = \exp \left\{ \int_0^v \chi(s) ds + \xi w \right\}.$$

Now we shall show the following lemma.

**Lemma 3.2.** *Let  $\xi, \mu, r > 0$  and assume that  $\chi$  satisfies (1.6)–(1.7). Then for all  $p > 1$  and  $\varepsilon_1 > 0$ ,*

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p + \frac{d}{dt} \int_{\Omega} u^p f^{-r} &\leq I_1 + I_2 + I_3 + I_4 + (\mu p + C_1 r) \int_{\Omega} u^p f^{-r} - \mu p \int_{\Omega} u^{p+1} f^{-r} \\ &\quad + r \xi \int_{\Omega} u^p f^{-r} v w - (p-1)(p-p\varepsilon_1-\varepsilon_3) \int_{\Omega} u^{p-2} |\nabla u|^2 \\ &\quad + (b_1 + \mu p) \int_{\Omega} u^p - \mu(p-\varepsilon_2) \int_{\Omega} u^{p+1} + b_2 \int_{\Omega} v^{p+1} \end{aligned}$$

for all  $t \in (s_0, T_{\max})$ , where

$$\begin{aligned} I_1 &:= -p(p-1)\varepsilon_1 \int_{\Omega} u^{p-2} f^{-r} |\nabla u|^2, \quad I_2 := p(p-1) \int_{\Omega} u^{p-1} f^{-r} \chi(v) f^r \nabla u \cdot \nabla v, \\ I_3 &:= p \int_{\Omega} u^{p-1} f^{-r} \nabla \cdot (\nabla u - \chi(v) u \nabla v - \xi u \nabla w), \quad I_4 := -r \int_{\Omega} u^p f^{-r} \chi(v) \Delta v. \end{aligned}$$

*Proof.* Testing the first equation in (1.1) by  $u^{p-1}$  ( $p > 1$ ) and integrating it over  $\Omega$  with positivity of  $u$  imply that for any  $\varepsilon_1 \in (0, 1)$ ,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p &= -p(p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + p(p-1) \int_{\Omega} u^{p-1} \chi(v) \nabla u \cdot \nabla v \\ &\quad + p(p-1) \xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \mu p \int_{\Omega} u^p (1-u-w) \\ &= -p(p-1)\varepsilon_1 \int_{\Omega} u^{p-2} |\nabla u|^2 - p(p-1)(1-\varepsilon_1) \int_{\Omega} u^{p-2} |\nabla u|^2 \\ &\quad + p(p-1) \int_{\Omega} u^{p-1} \chi(v) \nabla u \cdot \nabla v + p(p-1) \xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w \\ &\quad + \mu p \int_{\Omega} u^p (1-u-w). \end{aligned}$$

Noticing from the definition of  $f$  that  $f^{-r} \leq 1$  holds, we derive from the nonnegativity of  $u, w$  that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p &\leq -p(p-1)\varepsilon_1 \int_{\Omega} u^{p-2} f^{-r} |\nabla u|^2 - p(p-1)(1-\varepsilon_1) \int_{\Omega} u^{p-2} |\nabla u|^2 \\ &\quad + p(p-1) \int_{\Omega} u^{p-1} \chi(v) \nabla u \cdot \nabla v + p(p-1)\xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w \\ &\quad + \mu p \int_{\Omega} u^p (1-u). \end{aligned}$$

Denoting by  $I_1$  and  $I_2$  the first and third terms on the right-hand side as

$$\begin{aligned} I_1 &:= -p(p-1)\varepsilon_1 \int_{\Omega} u^{p-2} f^{-r} |\nabla u|^2 \quad \text{and} \\ I_2 &:= p(p-1) \int_{\Omega} u^{p-1} f^{-r} \chi(v) f^r \nabla u \cdot \nabla v, \end{aligned}$$

we can write as

$$\begin{aligned} (3.1) \quad \frac{d}{dt} \int_{\Omega} u^p &\leq I_1 + I_2 - p(p-1)(1-\varepsilon_1) \int_{\Omega} u^{p-2} |\nabla u|^2 \\ &\quad + p(p-1)\xi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \mu p \int_{\Omega} u^p - \mu p \int_{\Omega} u^{p+1}. \end{aligned}$$

Similarly, we have from straightforward calculations that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^p f^{-r} &= p \int_{\Omega} u^{p-1} f^{-r} \nabla \cdot (\nabla u - \chi(v)u \nabla v - \xi u \nabla w) + \mu p \int_{\Omega} u^p f^{-r} (1-u-w) \\ &\quad - r \int_{\Omega} u^p f^{-r} (\chi(v)(\Delta v - v + u) + \xi(-vw)). \end{aligned}$$

Then denoting by  $I_3$  and  $I_4$  as

$$\begin{aligned} I_3 &:= p \int_{\Omega} u^{p-1} f^{-r} \nabla \cdot (\nabla u - \chi(v)u \nabla v - \xi u \nabla w) \quad \text{and} \\ I_4 &:= -r \int_{\Omega} u^p f^{-r} \chi(v) \Delta v, \end{aligned}$$

we obtain from the nonnegativity of the solution that

$$\begin{aligned} (3.2) \quad \frac{d}{dt} \int_{\Omega} u^p f^{-r} &\leq I_3 + I_4 + \mu p \int_{\Omega} u^p f^{-r} (1-u) \\ &\quad - r \int_{\Omega} u^p f^{-r} (-v + u) \chi(v) + r \xi \int_{\Omega} u^p f^{-r} v w. \end{aligned}$$

In light of (1.7), (3.1), (3.2), we have from Lemma 2.2 that

$$\begin{aligned}
 & \frac{d}{dt} \int_{\Omega} u^p + \frac{d}{dt} \int_{\Omega} u^p f^{-r} \\
 & \leq I_1 + I_2 + I_3 + I_4 - p(p-1)(1-\varepsilon_1) \int_{\Omega} u^{p-2} |\nabla u|^2 \\
 & \quad + b_1 \int_{\Omega} u^p + \mu \varepsilon_2 \int_{\Omega} u^{p+1} + (p-1) \varepsilon_3 \int_{\Omega} u^{p-2} |\nabla u|^2 \\
 & \quad + b_2 \int_{\Omega} v^{p+1} + \mu p \int_{\Omega} u^p - \mu p \int_{\Omega} u^{p+1} \\
 & \quad + \mu p \int_{\Omega} u^p f^{-r} - \mu p \int_{\Omega} u^{p+1} f^{-r} + C_1 r \int_{\Omega} u^p f^{-r} + r \xi \int_{\Omega} u^p f^{-r} v w \\
 & \leq I_1 + I_2 + I_3 + I_4 + (\mu p + C_1 r) \int_{\Omega} u^p f^{-r} - \mu p \int_{\Omega} u^{p+1} f^{-r} \\
 & \quad + r \xi \int_{\Omega} u^p f^{-r} v w - (p-1)(p(1-\varepsilon_1) - \varepsilon_3) \int_{\Omega} u^{p-2} |\nabla u|^2 \\
 & \quad + (b_1 + \mu p) \int_{\Omega} u^p - \mu(p - \varepsilon_2) \int_{\Omega} u^{p+1} + b_2 \int_{\Omega} v^{p+1},
 \end{aligned}$$

which implies the end of the proof.  $\square$

We next show the following lemma to obtain a differential inequality which derives the estimate for  $\int_{\Omega} u^p + \int_{\Omega} u^p f^{-r}$ .

**Lemma 3.3.** *Let  $\xi, \mu > 0$  and assume that  $\chi$  satisfies*

$$\chi'(s) + \alpha_p |\chi(s)|^2 \leq 0 \quad \text{for all } s > 0$$

*with some  $p \in (1, n+1)$ , where  $\alpha_p$  is a positive constant defined as (1.9). Then there exists  $r = r(p) > 0$  such that*

$$(3.3) \quad I_1 + I_2 + I_3 + I_4 \leq 0.$$

*Proof.* Due to straightforward calculations, we obtain

$$\begin{aligned}
 I_3 &= -p \int_{\Omega} \nabla(u^{p-1} f^{-r}) \cdot (\nabla u - \chi(v) u \nabla v - \xi u \nabla w) \\
 &= -p(p-1) \int_{\Omega} u^{p-2} f^{-r} \nabla u \cdot (\nabla u - \chi(v) u \nabla v - \xi u \nabla w) \\
 & \quad + pr \int_{\Omega} u^{p-1} f^{-r} (\chi(v) \nabla v + \xi \nabla w) \cdot (\nabla u - \chi(v) u \nabla v - \xi u \nabla w)
 \end{aligned}$$

and

$$\begin{aligned}
 I_4 &= r \int_{\Omega} \nabla(u^p f^{-r} \chi(v)) \cdot \nabla v \\
 &= pr \int_{\Omega} u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla v \\
 & \quad - r^2 \int_{\Omega} u^p f^{-r} \chi(v) (\chi(v) \nabla v + \xi \nabla w) \cdot \nabla v + r \int_{\Omega} u^p f^{-r} \chi'(v) |\nabla v|^2.
 \end{aligned}$$

Therefore it follows that

$$\begin{aligned}
 & I_1 + I_2 + I_3 + I_4 \\
 &= -p(p-1)\varepsilon_1 \int_{\Omega} u^{p-2} f^{-r} |\nabla u|^2 + p(p-1) \int_{\Omega} u^{p-1} f^{-r} \chi(v) f^r \nabla u \cdot \nabla v \\
 &\quad - p(p-1) \int_{\Omega} u^{p-2} f^{-r} |\nabla u|^2 + p(p-1) \int_{\Omega} u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla v \\
 &\quad + p(p-1)\xi \int_{\Omega} u^{p-1} f^{-r} \nabla u \cdot \nabla w + pr \int_{\Omega} u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla v \\
 &\quad + pr\xi \int_{\Omega} u^{p-1} f^{-r} \nabla u \cdot \nabla w - pr \int_{\Omega} u^p f^{-r} |\chi(v)|^2 |\nabla v|^2 - pr\xi \int_{\Omega} u^p f^{-r} \chi(v) \nabla v \cdot \nabla w \\
 &\quad - pr\xi \int_{\Omega} u^p f^{-r} \chi(v) \nabla v \cdot \nabla w - pr\xi^2 \int_{\Omega} u^p f^{-r} |\nabla w|^2 + pr \int_{\Omega} u^{p-1} f^{-r} \chi(v) \nabla u \cdot \nabla v \\
 &\quad - r^2 \int_{\Omega} u^p f^{-r} |\chi(v)|^2 |\nabla v|^2 - r^2 \xi \int_{\Omega} u^p f^{-r} \chi(v) \nabla v \cdot \nabla w + r \int_{\Omega} u^p f^{-r} \chi'(v) |\nabla v|^2.
 \end{aligned}$$

Noting by assumption that

$$\chi'(v) \leq -\alpha_p |\chi(v)|^2,$$

we obtain

$$\begin{aligned}
 & I_1 + I_2 + I_3 + I_4 \\
 &\leq \int_{\Omega} a_1 u^{p-2} f^{-r} |\nabla u|^2 + \int_{\Omega} a_2 u^{p-1} f^{-r} |\nabla u| |\nabla v| + \int_{\Omega} a_3 u^{p-1} f^{-r} |\nabla u| |\nabla w| \\
 &\quad + \int_{\Omega} a_4 u^p f^{-r} |\nabla v|^2 + \int_{\Omega} a_5 u^p f^{-r} |\nabla v| |\nabla w| + \int_{\Omega} a_6 u^p f^{-r} |\nabla w|^2 \\
 &= \int_{\Omega} u^p f^{-r} (a_1 x^2 + a_2 xy + a_3 xz + a_4 y^2 + a_5 yz + a_6 z^2),
 \end{aligned}$$

where  $x, y, z$  and  $a_i$  ( $i = 1, 2, 3, 4, 5, 6$ ) are given as

$$\begin{aligned}
 x &= u^{-1} |\nabla u|, \quad y = |\nabla v|, \quad z = |\nabla w|, \\
 a_1 &:= -p(p-1)(1 + \varepsilon_1), \\
 a_2 &:= p((p-1) + 2r + (p-1)f^r) \chi(v), \\
 a_3 &:= p((p-1) + r) \xi, \\
 a_4 &:= -r(p + r + \alpha_p) |\chi(v)|^2, \\
 a_5 &:= -r(2p + r) \xi \chi(v), \\
 a_6 &:= -pr\xi^2.
 \end{aligned}$$

Thus our goal is to find some  $r > 0$  such that

$$(3.4) \quad a_1 x^2 + a_2 xy + a_3 xz + a_4 y^2 + a_5 yz + a_6 z^2 \leq 0.$$

We shall show (3.4) by using the Sylvester criterion. We first confirm that  $a_1a_6 - \frac{a_3^2}{4} > 0$  holds with some  $r > 0$ . Indeed, we can see that

$$\begin{aligned} a_1a_6 - \frac{a_3^2}{4} &= p(p-1)(1+\varepsilon_1)(pr\xi^2) - p^2 \frac{((p-1)+r)^2}{4} \xi^2 \\ &= p^2 \xi^2 \left( -\frac{r^2}{4} + (p-1)(1+\varepsilon_1)r - \frac{(p-1)}{2}r - \frac{(p-1)^2}{4} \right) \\ &= p^2 \xi^2 \left( -\frac{r^2}{4} + (p-1) \left( \frac{1}{2} + \varepsilon_1 \right) r - \frac{(p-1)^2}{4} \right) \\ &= p^2 \xi^2 \left( -\frac{1}{4} \left( r - 2(p-1) \left( \frac{1}{2} + \varepsilon_1 \right) \right)^2 + (p-1)^2 \varepsilon_1 (\varepsilon_1 + 1) \right). \end{aligned}$$

Thus by choosing  $r = 2(p-1)(\frac{1}{2} + \varepsilon_1)$  we can verify that  $a_1a_6 - \frac{a_3^2}{4} > 0$ . We next confirm that

$$A_3 := \begin{vmatrix} a_1 & \frac{a_3}{2} & \frac{a_2}{2} \\ \frac{a_3}{2} & a_6 & \frac{a_5}{2} \\ \frac{a_2}{2} & \frac{a_5}{2} & a_4 \end{vmatrix} \leq 0.$$

Straightforward calculations and the fact that  $a_1a_6 - \frac{a_3^2}{4} = p^2 \xi^2 (p-1)^2 \varepsilon_1 (\varepsilon_1 + 1)$  when  $r = 2(p-1)(\frac{1}{2} + \varepsilon_1)$  lead to

$$\begin{aligned} A_3 &= \left( a_1a_6 - \frac{a_3^2}{4} \right) a_4 + \frac{a_2a_3a_5}{4} - \frac{a_2^2a_6}{4} - \frac{a_1a_5^2}{4} \\ &= -p^2 \xi^2 (p-1)^3 \varepsilon_1 (\varepsilon_1 + 1) (1 + 2\varepsilon_1) (p + (p-1)(1 + 2\varepsilon_1) + \alpha_p) |\chi(v)|^2 \\ &\quad - \frac{p^2}{4} r ((p-1) + 2r + (p-1)f^r) ((p-1) + r)(2p + r) \xi^2 |\chi(v)|^2 \\ &\quad + \frac{p^2}{4} ((p-1) + 2r + (p-1)f^r)^2 (pr\xi^2) |\chi(v)|^2 \\ &\quad + \frac{p(p-1)(1+\varepsilon_1)}{4} r^2 (2p + r)^2 \xi^2 |\chi(v)|^2. \end{aligned}$$

Now letting  $d := 1 + 2\varepsilon_1$  and noting that  $r = (p-1)(1 + 2\varepsilon_1) = (p-1)d$ , we deduce that

$$\begin{aligned} A_3 &= -p^2 (p-1)^3 \varepsilon_1 (\varepsilon_1 + 1) d (p + (p-1)d + \alpha_p) \xi^2 |\chi(v)|^2 \\ &\quad - \frac{p^2}{4} (p-1)^3 d (1 + 2d + f^{(p-1)d}) (1 + d) (2p + (p-1)d) \xi^2 |\chi(v)|^2 \\ &\quad + \frac{p^3}{4} (p-1)^3 d (1 + 2d + f^{(p-1)d})^2 \xi^2 |\chi(v)|^2 \\ &\quad + \frac{p}{4} (p-1)^3 d^2 (1 + \varepsilon_1) (2p + (p-1)d)^2 \xi^2 |\chi(v)|^2 \\ &= p(p-1)^3 d \xi^2 |\chi(v)|^2 (-p\varepsilon_1 (\varepsilon_1 + 1) \alpha_p + \psi(p, \varepsilon_1)), \end{aligned}$$

where  $\psi(p, \varepsilon_1)$  is given by

$$\begin{aligned}\psi(p, \varepsilon_1) &:= -p\varepsilon_1(\varepsilon_1 + 1)(p + (p - 1)d) \\ &\quad - \frac{p}{4}(1 + 2d + f^{(p-1)d})(1 + d)(2p + (p - 1)d) \\ &\quad + \frac{p^2}{4}(1 + 2d + f^{(p-1)d})^2 \\ &\quad + \frac{d}{4}(1 + \varepsilon_1)(2p + (p - 1)d)^2.\end{aligned}$$

Then since the boundedness of  $f(v, w)$

$$1 \leq f(v(x, t), w(x, t)) \leq k \quad \text{for all } x \in \Omega \text{ and all } t > 0$$

with  $k = \exp\{\int_0^\infty \chi(s)ds + \xi\|w_0\|_{L^\infty}\} > 1$  and the fact  $(p - 1)d \leq nd$  imply

$$\begin{aligned}(3.5) \quad \psi(p, \varepsilon_1) &\leq -p\varepsilon_1(\varepsilon_1 + 1)(p + (p - 1)d) \\ &\quad - \frac{p}{4}(1 + 2d + 1)(1 + d)(2p + (p - 1)d) \\ &\quad + \frac{p^2}{4}(1 + 2d + k^{nd})^2 \\ &\quad + \frac{d}{4}(1 + \varepsilon_1)(2p + (p - 1)d)^2 \\ &= -\varepsilon_1(\varepsilon_1 + 1)((1 + d)p - d)p \\ &\quad - \frac{1}{4}(1 + 2d + 1)(1 + d)((2 + d)p - d)p \\ &\quad + \frac{1}{4}(1 + 2d + k^{nd})^2 p^2 + \frac{d}{4}(1 + \varepsilon_1)((2 + d)p - d)^2 \\ &= -\varepsilon_1(\varepsilon_1 + 1)(1 + d)p^2 + \varepsilon_1(\varepsilon_1 + 1)dp - \frac{d + 1}{2}(1 + d)(2 + d)p^2 \\ &\quad + \frac{d + 1}{2}(1 + d)dp + \frac{1}{4}(1 + 2d + k^{nd})^2 p^2 \\ &\quad + \frac{d}{4}(1 + \varepsilon_1)(2 + d)^2 p^2 - \frac{d}{2}(1 + \varepsilon_1)(2 + d)pd \\ &\quad + \frac{d}{4}(1 + \varepsilon_1)d^2 \\ &=: \eta(p, \varepsilon_1),\end{aligned}$$

we can see from the definition of  $\alpha_p$  (see (1.9)) that

$$A_3 \leq p(p - 1)^3 |\chi(v)|^2 \xi^2 (1 + 2\varepsilon_1)(-p\varepsilon_1(\varepsilon_1 + 1)\alpha_p + \eta(p, \varepsilon_1)) = 0.$$

Thus noticing that  $a_1 = -p(p - 1)(1 + \varepsilon_1) < 0$ , from the Sylvester criterion we have (3.4), which yields (3.3).  $\square$

Now we are in a position to show the desired  $L^p$ -estimate for  $u$  with some  $p > n$ .

**Lemma 3.4.** *Let  $n \geq 3$  and let  $\xi, \mu > 0$ . Assume that  $\chi$  satisfies (1.6)–(1.7) and*

$$\chi'(s) + \alpha_p |\chi(s)|^2 \leq 0 \quad \text{for all } s > 0$$

*with some  $p \in (1, n+1)$ , where  $\alpha_p$  is a positive constant defined as (1.9). Then there exists  $C > 0$  such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (s_0, T_{\max}).$$

*Proof.* Lemmas 3.2 and 3.3 yield

$$\begin{aligned} (3.6) \quad & \frac{d}{dt} \int_{\Omega} u^p + \frac{d}{dt} \int_{\Omega} u^p f^{-r} \\ & \leq (\mu p + C_1 r) \int_{\Omega} u^p f^{-r} - \mu p \int_{\Omega} u^{p+1} f^{-r} + (b_1 + \mu p) \int_{\Omega} u^p \\ & \quad - (\mu p - \mu \varepsilon_2) \int_{\Omega} u^{p+1} - (p-1)(p(1-\varepsilon_1) - \varepsilon_3) \int_{\Omega} u^{p-2} |\nabla u|^2 \\ & \quad + b_2 \int_{\Omega} v^{p+1} + r \xi \int_{\Omega} u^p f^{-r} v w \end{aligned}$$

with  $r = 2(p-1)(\frac{1}{2} + \varepsilon_1)$  and some  $C_1 > 0$ . Noting that

$$w(x, t) \leq \|w_0\|_{L^\infty} \quad \text{and} \quad f^{-r}(v(x, t), w(x, t)) \leq 1 \quad \text{for all } x \in \Omega \text{ and all } t > 0,$$

we derive from the Young inequality that

$$\begin{aligned} r \xi \int_{\Omega} u^p f^{-r} v w &= (p-1)(1+2\varepsilon_1) \xi \int_{\Omega} u^p f^{-r} v w \\ &\leq \varepsilon_4 \int_{\Omega} u^{p+1} f^{-r} + \ell \int_{\Omega} v^{p+1} \end{aligned}$$

with some  $\varepsilon_4 \in (0, \mu p)$  and  $\ell = \ell(\xi, \|w_0\|_{L^\infty}, p, \varepsilon_1, \varepsilon_4) > 0$ . Thus a combination of (3.6) and the definitions of  $\varepsilon_2, \varepsilon_3, \varepsilon_4$  enables us to see that

$$\begin{aligned} (3.7) \quad & \frac{d}{dt} \int_{\Omega} u^p + \frac{d}{dt} \int_{\Omega} u^p f^{-r} \\ & \leq c_1 \int_{\Omega} u^p f^{-r} - c_2 \int_{\Omega} u^{p+1} f^{-r} + c_3 \int_{\Omega} u^p - c_4 \int_{\Omega} u^{p+1} + c_5 \int_{\Omega} v^{p+1} \\ & \leq c_6 \int_{\Omega} u^p (f^{-r} + 1) - c_7 \int_{\Omega} u^{p+1} (f^{-r} + 1) + c_5 \int_{\Omega} v^{p+1} \end{aligned}$$

with some  $c_1, c_2, c_3, c_4, c_5, c_6, c_7 > 0$ . Here we put

$$y_q(t) := \int_{\Omega} |u(\cdot, t)|^q (f^{-r}(v(\cdot, t), w(\cdot, t)) + 1)$$

for  $q \geq 1$  and  $t \in (s_0, T_{\max})$ . Noting from the Hölder inequality, the Young inequality and  $f^{-r}(v, w) \leq 1$  on  $\Omega \times (s_0, T_{\max})$  that there is  $c_8 > 0$  satisfying

$$\left(c_6 + \frac{p+1}{2}\right) y_p \leq \frac{c_7}{2} y_{p+1} + c_8 \quad \text{on } (s_0, T_{\max}),$$

we infer that

$$\begin{aligned}
 (3.8) \quad \frac{d}{dt} y_p(t) &\leq c_6 y_p(t) - c_7 y_{p+1}(t) + c_5 \int_{\Omega} |v(\cdot, t)|^{p+1} \\
 &= -\frac{p+1}{2} y_p(t) + \left( c_6 + \frac{p+1}{2} \right) y_p(t) - c_7 y_{p+1}(t) + c_5 \int_{\Omega} |v(\cdot, t)|^{p+1} \\
 &\leq -\frac{p+1}{2} y_p(t) - \frac{c_7}{2} y_{p+1}(t) + c_8 + c_5 \int_{\Omega} |v(\cdot, t)|^{p+1}
 \end{aligned}$$

for all  $t \in (s_0, T_{\max})$  with some  $c_8 > 0$ . Then thanks to Lemma 2.3 with  $\varepsilon \in (0, \frac{c_7}{2c_5})$  and the relation  $\int_{\Omega} u^{p+1} \leq y_{p+1}$ , for all  $t \in (s_0, T_{\max})$  integrating (3.8) over  $(s_0, t)$  implies that

$$\begin{aligned}
 e^{\frac{p+1}{2}t} y_p(t) &\leq e^{\frac{p+1}{2}s_0} y_p(s_0) - \frac{c_7}{2} \int_{s_0}^t e^{\frac{p+1}{2}s} y_{p+1}(s) ds + c_8 \int_{s_0}^t e^{\frac{p+1}{2}s} ds \\
 &\quad + c_5 \left( \varepsilon \int_{s_0}^t \int_{\Omega} e^{\frac{p+1}{2}s} |u(x, s)|^{p+1} dx ds + c_9 e^{\frac{p+1}{2}t} + \varepsilon \right) \\
 &\leq e^{\frac{p+1}{2}s_0} y_p(s_0) - \left( \frac{c_7}{2} - \varepsilon c_5 \right) \int_{s_0}^t e^{\frac{p+1}{2}s} y_{p+1}(s) ds + \frac{2c_8}{p+1} e^{\frac{p+1}{2}t} + c_5 c_9 e^{\frac{p+1}{2}t} + \varepsilon c_5
 \end{aligned}$$

with some  $c_9 > 0$ . Thus we can verify that

$$\begin{aligned}
 y_p(t) &\leq e^{-\frac{p+1}{2}(t-s_0)} y_p(s_0) + \frac{2c_8}{p+1} + c_5 c_9 + \varepsilon c_5 e^{-\frac{p+1}{2}t} \\
 &\leq c_{10}
 \end{aligned}$$

for all  $t \in (s_0, T_{\max})$  with some  $c_{10} > 0$ , which concludes the proof.  $\square$

**Remark 3.1.** In the case that  $n \geq 3$ , in order to obtain the  $L^p$ -estimate for  $u$ , we use the energy estimate for  $\int_{\Omega} u^p \varphi(v)$  with some function  $\varphi$ . On the other hand, in the case that  $n = 2$ , by considering the energy estimate for  $\int_{\Omega} u \log u + \int_{\Omega} |\nabla v|^2$ , we can attain the  $L^p$ -estimate for  $u$ . Here we give a short proof. Since a combination of the Gagliardo–Nirenberg inequality and the  $L^2$ -estimate for  $\nabla v$  (from [10, (3.11)] with the boundedness of  $\int_t^{t+\tau} \int_{\Omega} u^2$  for all  $t \in (0, T_{\max} - \tau)$  with  $\tau := \min\{1, \frac{1}{2}T_{\max}\}$ ) yields

$$\int_{\Omega} |\nabla v|^4 \leq c_1 \left( \int_{\Omega} |\Delta v|^2 + 1 \right)$$

with some  $c_1 > 0$ , integration by parts and the Young inequality imply that

$$\begin{aligned}
 (3.9) \quad \int_{\Omega} \chi(v) \nabla u \cdot \nabla v &= - \int_{\Omega} \chi'(v) u |\nabla v|^2 - \int_{\Omega} \chi(v) u \Delta v \\
 &\leq c_1 \|\chi'\|_{L^\infty(\Omega)}^2 \int_{\Omega} u^2 + \frac{1}{4c_1} \int_{\Omega} |\nabla v|^4 + \|\chi\|_{L^\infty(\Omega)}^2 \int_{\Omega} u^2 + \frac{1}{4} \int_{\Omega} |\Delta v|^2 \\
 &\leq (c_1 \|\chi'\|_{L^\infty(\Omega)}^2 + \|\chi\|_{L^\infty(\Omega)}^2) \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} |\Delta v|^2 + \frac{1}{4} \quad \text{on } (0, T_{\max}).
 \end{aligned}$$

Thus noting

$$\frac{d}{dt} \int_{\Omega} u \log u + \int_{\Omega} \frac{|\nabla u|^2}{u} = \int_{\Omega} \chi(v) \nabla u \cdot \nabla v + \xi \int_{\Omega} \nabla u \cdot \nabla w + \mu \int_{\Omega} u(1 + \log u)(1 - u - w),$$

and applying (3.9) to the first term on the right-hand-side of the above identity, from arguments similar to those in the proof of [10, Lemma 3.3] we can see the boundedness of the energy function  $\int_{\Omega} u \log u + \int_{\Omega} |\nabla v|^2$ . Then due to a generalized Gagliardo–Nirenberg inequality, the boundedness of the energy function leads to the boundedness of  $\|u\|_{L^{\infty}(0, T_{\max}; L^2(\Omega))}$  and  $\|\nabla v\|_{L^{\infty}(0, T_{\max}; L^4(\Omega))}$ . Then the standard  $L^p$ – $L^q$  estimate for the Neumann heat semigroup implies the boundedness of  $\|\nabla v\|_{L^{\infty}(0, T_{\max}; L^q(\Omega))}$  for all  $q \in [4, \infty)$ . Thanks to this boundedness, from a standard testing argument (see [10, Lemma 3.10]) we can attain the boundedness of  $\|u\|_{L^{\infty}(0, T_{\max}; L^p(\Omega))}$  with some  $p > 2$ . Here we note that these arguments do not work in the case that  $n \geq 3$ ; because of lacking the generalized Gagliardo–Nirenberg inequality, we could not obtain the estimate for  $\|u\|_{L^{\infty}(0, T_{\max}; L^n(\Omega))}$  from the boundedness of the energy function. As we mentioned before, in the case that  $n = 2$  the estimate for the energy function yields the  $L^p$ -estimate for  $u$ . On the other hand, in the case that  $n = 1$ , we do not have to use an energy function; since we have already established the boundedness of  $\|u\|_{L^{\infty}(0, T_{\max}; L^1(\Omega))}$ , arguments similar to those in the case that  $n = 2$  lead to  $\|u\|_{L^{\infty}(0, T_{\max}; L^p(\Omega))}$  for some  $p > 1$ .

*Proof of Theorem 1.1.* Under the condition that  $\chi$  satisfies (1.6)–(1.8), a combination of Lemmas 2.4 and 3.4, along with (2.1) directly leads to Theorem 1.1.  $\square$

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