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Existence and concentration of ground states for a Choquard equation with competing potentials ☆

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ABSTRACT

In this paper, we are concerned with the following Choquard equation in $\mathbb{R}^3$ that

$$-\epsilon^2\Delta u + V(x)u$$
$$= \epsilon^{\mu-3}\Big[\Big(\int\limits_{\mathbb{R}^3}\frac{P(y)|u(y)|^p}{|x-y|^\mu}\Big)P(x)|u|^{p-2}u + \Big(\int\limits_{\mathbb{R}^3}\frac{Q(y)|u(y)|^q}{|x-y|^\mu}\Big)Q(x)|u|^{q-2}u\Big],$$

where $\epsilon > 0$ is a parameter, $0 < \mu < 3$, $\frac{6-\mu}{3} < q < p < 6-\mu$, the functions V and P are positive and Q may be sign-changing. Via variational methods, we establish the existence of ground states for small ϵ, and investigate the concentration behavior of ground states and show that they concentrate at a global minimum point of the least energy function as $\epsilon \to 0$.

1. Introduction and main results

The Choquard equation

$$-\Delta u + u = \Big(\int\limits_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|}\Big)u,\ \ u \in H^1(\mathbb{R}^3), \tag{1.1}$$

was proposed by Choquard in 1976, and can be described as an approximation to Hartree–Fock theory of a one-component plasma, see [13]. It was also proposed by Penrose in [21] as a model for the self-gravitational collapse of a quantum mechanical wave function. In [13], Lieb proved the existence and uniqueness of a

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minimizer to problem (1.1) by using symmetric decreasing rearrangement inequalities. Later, in [15], Lions showed the existence of infinitely many radially symmetric solutions of (1.1). In [16], Ma and Zhao considered the generalized Choquard equation

$$-\Delta u + u = \Big(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu}\Big)|u|^{q-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{1.2}$$

for $q \geq 2$ and $N \geq 3$. Under some assumptions on N, μ, and q, they proved that every positive solution of (1.2) is radially symmetric and monotone decreasing about some point. Cingolani et al. [8] obtained some existence and multiplicity results in the electromagnetic case, and established the regularity and some decay at infinity of ground states for (1.2). Moroz and Van Schaftingen [17] investigated the qualitative properties of solutions of (1.2) and showed the regularity, positivity and radial symmetry of ground states. The authors [19,20] established the existence of ground states for (1.2) with general nonlinearity in the spirit of Berestycki and Lions and studied the existence of solutions for (1.2) with lower critical exponent due to the Hardy–Littlewood–Sobolev inequality.

On the other hand, some people focused on the semiclassical problem

$$-\epsilon^2\Delta u + u = \epsilon^{\mu-N}\Big(\int\limits_{\mathbb{R}^N} \frac{G(u(y))}{|x-y|^\mu}\Big)g(u), \quad u \in H^1(\mathbb{R}^N), \tag{1.3}$$

where G is the primitive function of g, and there are many results about the existence, multiplicity and concentration of solutions for (1.3) and similar problems as ϵ small. For $N = 3$, $\mu = 1$ and $G(s) = s^2$, Cingolani et al. [10] applied the penalization arguments and showed that there exists a family of solutions having multiple concentration regions which are located around the minimum points of the potential. For $N \geq 3$ and $G(u) = |u|^q$ with $q \in [2, \frac{2N-\mu}{N-2})$, Moroz and Van Schaftingen in [18] developed a nonlocal penalization technique and showed that equation (1.3) has a family of solutions concentrating around the local minimum of V which satisfies a certain decay at infinity. Alves and Yang [5] dealt with the equation (1.3) with general function G, and obtained the multiplicity and concentration of solutions for the equation (1.3) by assuming that V has a global minimum, $g \in C^1(\mathbb{R})$ is of subcritical growth and satisfies Ambrosetti–Rabinowitz type condition. Later, Alves and Yang [6,7] studied the existence and concentration of solutions for (1.3) with both linear and nonlinear potentials which have a global minimum or maximum, and also proved the existence, multiplicity and concentration of solutions for (1.3) with linear potential which has a local minimum. Very recently, Alves et al. [2] considered (1.3) with critical growth that

$$-\epsilon^2\Delta u + V(x)u = \epsilon^{\mu-3}\Big(\int\limits_{\mathbb{R}^3} \frac{|u(y)|^{6-\mu}+f(u(y))}{|x-y|^\mu}\Big)\big(|u|^{4-\mu}u + f(u)\big), \text{ in } \mathbb{R}^3,$$

where $6-\mu$ is the upper critical exponent due to the Hardy–Littlewood–Sobolev inequality, and they investigated the existence, multiplicity and concentration behavior of solutions by variational methods. For the reader's convenience, we recall the definitions of critical exponent for the problem (1.3) with $N = 3$. Firstly we give an important inequality:

Proposition 1.1 *(Hardy–Littlewood–Sobolev inequality). (See [14].) Let $s, r > 1$ and $0 < \mu < 3$ with $\frac{1}{s} + \frac{\mu}{3} + \frac{1}{r} = 2$, $f \in L^s(\mathbb{R}^3)$ and $h \in L^r(\mathbb{R}^3)$. There exists a sharp constant $C(s,\mu,r)$, independent of f, h, such that*

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{f(x)h(y)}{|x-y|^\mu} \leq C(s,\mu,r)|f|_s|h|_r.$$

Remark 1.1. By Proposition 1.1, the term

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^r |u(y)|^r}{|x-y|^\mu}$$

is well defined if $u^r \in L^s(\mathbb{R}^3)$ satisfies $\frac{2}{s}+\frac{\mu}{3}=2$. Therefore, for $u \in H^1(\mathbb{R}^3)$ we will require that $s\cdot r \in [2,6]$. Then $\frac{6-\mu}{3} \le r \le 6-\mu$. Thus $\frac{6-\mu}{3}$ is called the lower critical exponent and $6-\mu$ is called the upper critical exponent in the sense of Hardy–Littlewood–Sobolev inequality. For details, see [2].

For a Schrödinger equation of the form

$$-\epsilon^2\Delta u + V(x)u = K(x)|u|^{p-2}u + Q(x)|u|^{q-2}u,\ \ x \in \mathbb{R}^N,$$

Wang and Zeng [24] proved that the concentration points are located on the middle ground of the competing potential functions and in some cases are given explicitly in terms of these functions. Cingolani and Lazzo [9] obtained multiple solutions and related the number of solutions with the topology of the global minima set of a suitable least energy function. We also mention that, some mathematicians studied other problems with competing potentials, for instance, see [4,23].

By motivation of the above works, we consider a Choquard equation with competing potentials. More precisely, we are devoted to studying the existence and concentration of ground states for the Choquard equation in $\mathbb{R}^3$ that

$$-\epsilon^2\Delta u + V(x)u = \epsilon^{\mu-3}\Big[\Big(\int_{\mathbb{R}^3}\frac{P(y)|u(y)|^p}{|x-y|^\mu}\Big)P(x)|u|^{p-2}u + \Big(\int_{\mathbb{R}^3}\frac{Q(y)|u(y)|^q}{|x-y|^\mu}\Big)Q(x)|u|^{q-2}u\Big], \tag{1.4}$$

where $0<\mu<3$, $\frac{6-\mu}{3}<q<p<6-\mu$, V and P are continuous and positive functions, Q is a continuous function and may be sign-changing. A simple model of (1.4) is the case that $Q=0$, V has a global minimum and P has a global maximum, there is possibly a competition between V and P, and V would attract ground states to its minimum point but P would attract ground states to its maximum point. The competition will be more complex when $Q \neq 0$, and this causes finding the concentration points become more difficult. As we know, there is no work concerning this case.

Throughout the paper, we always assume that:

(V) $V(x) \in C(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ and $V_0 := \inf_{x\in\mathbb{R}^3} V(x) > 0$.
(P) $P(x) \in C(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ is a positive function.
(Q) $Q(x) \in C(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$.

To state the main result, we need two auxiliary problems. For each $\xi \in \mathbb{R}^3$, consider the following problem in $\mathbb{R}^3$ that

$$-\Delta u + V(\xi)u = \Big(\int_{\mathbb{R}^3}\frac{P^2(\xi)|u(y)|^p}{|x-y|^\mu}\Big)|u|^{p-2}u + \Big(\int_{\mathbb{R}^3}\frac{Q^2(\xi)|u(y)|^q}{|x-y|^\mu}\Big)|u|^{q-2}u. \tag{$(P)_\xi$}$$

Denote the corresponding functional of $(\mathrm{P})_\xi$ by I_ξ and the least energy by $C(\xi) := c(V(\xi), P(\xi), Q(\xi))$. In addition, we consider the limit problem in $\mathbb{R}^3$ that

$$-\Delta u + V_\infty u = \Big(\int_{\mathbb{R}^3}\frac{P_\infty^2|u(y)|^p}{|x-y|^\mu}\Big)|u|^{p-2}u + \Big(\int_{\mathbb{R}^3}\frac{Q_\infty^2|u(y)|^q}{|x-y|^\mu}\Big)|u|^{q-2}u, \tag{$(P)_\infty$}$$

and denote the corresponding functional of $(\mathrm{P})_\infty$ by I_∞ and least energy by $c_\infty := c(V_\infty, P_\infty, Q_\infty)$, where

$$V_\infty := \liminf_{|x|\to\infty} V(x),\ \ P_\infty := \limsup_{|x|\to\infty} P(x),\ \ Q_\infty := \limsup_{|x|\to\infty} Q(x).$$

Now we state the main results.

Theorem 1.1. *Let (V), (P) and (Q) hold and suppose that* $0 < \mu < 2$ *and* $2 < q < p < 6 - 2\mu$*. If*

$$\inf_{\xi\in\mathbb{R}^3} C(\xi) < c_\infty, \tag{1.5}$$

then for any $\epsilon > 0$ *small,*

(1) the equation (1.4) has a positive ground state w_ϵ *in* $H^1(\mathbb{R}^3)$*,*

(2) if additionally V*,* P *and* Q *are uniformly continuous, then* w_ϵ *satisfies:*

(i) w_ϵ *possesses a global maximum point* $x_\epsilon \in \mathbb{R}^3$ *such that*

$$\lim_{\epsilon\to 0} C(x_\epsilon) = \inf_{\xi\in\mathbb{R}^3} C(\xi).$$

Setting $v_\epsilon(x) = w_\epsilon(\epsilon x + x_\epsilon)$*, for any sequence* $x_\epsilon \to y_0$*,* $\epsilon \to 0$*,* v_ϵ *converges in* $H^1(\mathbb{R}^3)$ *to a ground state* v *of the equation in* $\mathbb{R}^3$ *that*

$$-\Delta u + V(y_0)u = \Big(\int\limits_{\mathbb{R}^3} \frac{P^2(y_0)|u(y)|^p}{|x-y|^\mu}\Big)|u|^{p-2}u + \Big(\int\limits_{\mathbb{R}^3} \frac{Q^2(y_0)|u(y)|^q}{|x-y|^\mu}\Big)|u|^{q-2}u.$$

(ii) There exists $C_1, C_2 > 0$ *such that*

$$w_\epsilon(x) \le C_1 e^{-\frac{C_2}{\epsilon}|x-x_\epsilon|}.$$

Remark 1.2. We would like to point out that, (1.5) will hold in some cases. Here we give two examples.

(1) There exists a point $s_0 \in \mathbb{R}^3$ such that

$$V_\infty \ge V(s_0),\ \ \ P_\infty \le P(s_0),\ \text{and}\ \ Q_\infty \le Q(s_0),$$

with one of the above inequalities being strict.

(2) There exists a point $s_0 \in \mathbb{R}^3$ such that

$$V_\infty^{-\frac{1}{2}+\frac{5-\mu}{2(p-1)}} P_\infty^{-\frac{2}{p-1}} \ge V^{-\frac{1}{2}+\frac{5-\mu}{2(p-1)}}(s_0) P^{-\frac{2}{p-1}}(s_0),$$
$$V_\infty^{\frac{5-\mu}{2(p-1)}(q-p)} Q_\infty^2 P_\infty^{\frac{2(1-q)}{p-1}} \le V^{\frac{5-\mu}{2(p-1)}(q-p)}(s_0) Q^2(s_0) P^{\frac{2(1-q)}{p-1}}(s_0),$$

with one of the above inequalities being strict. For more details, see Corollary 3.1.

Remark 1.3. As in [6,7], we make some comments on the restrictions that $0 < \mu < 2$ and $2 < q < p < 6-2\mu$ in Theorem 1.1. To obtain the existence results for the autonomous equation $(\mathrm{P})_\xi$, we can eliminate the above restrictions and the parameters μ, p, q just are satisfied that $0 < \mu < 3$ and $\frac{6-\mu}{3} < q < p < 6-\mu$. However, to obtain the existence and concentration results about the equation (1.4), we need the restrictions to ensure that the nonlocal part $\frac{1}{|x|^\mu} * |u|^q$ and $\frac{1}{|x|^\mu} * |u|^p$ are bounded terms.

The proof is based on variational methods. Comparing with the previous existence and concentration results about Choquard equation, for example [5–7], we do not assume that the potentials has a minimum or maximum, and so the previous results cannot be applied to (1.4) even when $Q = 0$. However, the Nehari manifold is still well defined even $Q(x)$ is sign-changing. Hence, we first use the method of Nehari manifold to find ground states as ϵ small and then we find these ground states concentrate at a global minimum point of the least energy function $C(\xi)$ as $\epsilon \to 0$ by virtue of concentration-compactness lemma.

The paper is organized as follows. In Section 2 we introduce the variational framework. In Section 3 we study the autonomous problem. In Section 4 we give a compactness lemma. In Section 5, we are devoted to proving Theorem 1.1.

2. Variational setting

In this paper, we use the following notations. For $1 \le p \le \infty$, the norm in $L^p(\mathbb{R}^3)$ is denoted by $|\cdot|_p$. $\int_{\mathbb{R}^3} f(x)dx$ will be represented by $\int_{\mathbb{R}^3} f(x)$. For any $r > 0$ and $x \in \mathbb{R}^3$, $B_r(x)$ denotes the ball centered at x with the radius r.

For the proof of our theorem, we shall consider an equivalent equation to (1.4). By making the change of variable $x \to \epsilon x$, the problem (1.4) turns out to be

$$-\Delta u + V(\epsilon x)u = \Big(\int_{\mathbb{R}^3} \frac{P(\epsilon y)|u(y)|^p}{|x-y|^\mu}\Big)P(\epsilon x)|u|^{p-2}u + \Big(\int_{\mathbb{R}^3} \frac{Q(\epsilon y)|u(y)|^q}{|x-y|^\mu}\Big)Q(\epsilon x)|u|^{q-2}u. \tag{P*}$$

$H^1(\mathbb{R}^3)$ is the Sobolev space with norm $\|u\| = \big(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2\big)^{\frac{1}{2}}$. By (V), the norms

$$\|u\|_\epsilon^2 = \int_{\mathbb{R}^3} \big(|\nabla u|^2 + V(\epsilon x)u^2\big), \quad \|u\|_\xi^2 = \int_{\mathbb{R}^3} \big(|\nabla u|^2 + V(\xi)u^2\big),$$

are equivalent norms in $H^1(\mathbb{R}^3)$. S_ϵ and S_ξ are the unit sphere of $H^1(\mathbb{R}^3)$ under norms $\|\cdot\|_\epsilon$ and $\|\cdot\|_\xi$ respectively. The functional associated with the equation (P)* is

$$I_\epsilon(u) = \frac{1}{2}\|u\|_\epsilon^2 - \frac{1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P(\epsilon y)|u(y)|^p}{|x-y|^\mu}P(\epsilon x)|u(x)|^p - \frac{1}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q(\epsilon y)|u(y)|^q}{|x-y|^\mu}Q(\epsilon x)|u(x)|^q. \tag{2.1}$$

Now we state the variational setting. We shall apply the method of Nehari manifold developed by Szulkin and Weth [22] to Choquard equation. The Nehari manifold corresponding to (P)* is

$$M_\epsilon = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \langle I_\epsilon'(u), u\rangle = 0\},$$

and the least energy on M_ϵ is defined by $c_\epsilon := \inf_{M_\epsilon} I_\epsilon$. Since the argument of the following results are similar to [25], we omit most of the proof.

Lemma 2.1. *There exists $\rho > 0$ such that $\inf_{\|u\|_\epsilon=\rho} I_\epsilon > 0$.*

Proof. The Hardy–Littlewood–Sobolev inequality implies that

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(y)|^s}{|x-y|^\mu}|u(x)|^s\Big| \le C|u|^{2s}_{s\cdot\frac{6}{6-\mu}}, \tag{2.2}$$

where $s = p$ or q. Then

$$I_\epsilon(u) \ge \frac{1}{2}\|u\|_\epsilon^2 - C(|u|^{2q}_{q\cdot\frac{6}{6-\mu}} + |u|^{2p}_{p\cdot\frac{6}{6-\mu}}),$$

from which the conclusion yields since $2 < q\cdot\frac{6}{6-\mu} < p\cdot\frac{6}{6-\mu} < 6$ and $q, p > 1$. $\square$

Similar to [25, Lemma 3.1], we have that:

Lemma 2.2. *(i) For all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_\epsilon := t_\epsilon(u) > 0$ such that $t_\epsilon u \in M_\epsilon$ and $I_\epsilon(t_\epsilon u) = \max_{t>0} I_\epsilon(tu)$.*

(ii) M_ϵ is bounded away from 0, and there is $\varrho > 0$ such that $t_\epsilon \geq \varrho$ for each $u \in S_\epsilon$.

(iii) For each compact subset $W \subset S_\epsilon$, there exists $C_W > 0$ such that $t_\epsilon \leq C_W$, for all $u \in W$.

By Lemmas 2.1 and 2.2 (i), one easily get that:

Lemma 2.3. $c_\epsilon = \inf_{M_\epsilon} I_\epsilon \geq \inf_{\|u\|_\epsilon = \rho} I_\epsilon > 0$.

Define the mapping $m_\epsilon : S_\epsilon \to M_\epsilon$ by $m_\epsilon(w) := t_\epsilon w$, where t_ϵ is as in Lemma 2.2 (i). Similar to [22, Proposition 3.1], we have:

Lemma 2.4. *The mapping m_ϵ is a homeomorphism between S_ϵ and M_ϵ.*

By Lemma 2.4, the least energy c_ϵ has the following minimax characterization:

$$c_\epsilon := \inf_{u \in M_\epsilon} I_\epsilon(u) = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t \geq 0} I_\epsilon(tu). \tag{2.3}$$

Considering the functional $\Psi_\epsilon : S_\epsilon \to \mathbb{R}$ given by $\Psi_\epsilon(w) := I_\epsilon(m_\epsilon(w))$, as [22, Corollary 3.3] we deduce that:

Lemma 2.5. *(1) If $\{w_n\}$ is a PS sequence for Ψ_ϵ, then $\{m_\epsilon(w_n)\}$ is a PS sequence for I_ϵ. If $\{u_n\} \subset M_\epsilon$ is a bounded PS sequence for I_ϵ, then $\{m_\epsilon^{-1}(u_n)\}$ is a PS sequence for Ψ_ϵ.*

(2) w is a critical point of Ψ_ϵ if and only if $m_\epsilon(w)$ is a nontrivial critical point of I_ϵ. Moreover, $\inf_{M_\epsilon} I_\epsilon = \inf_{S_\epsilon} \Psi_\epsilon$.

Remark 2.1. By Lemma 2.5, we will use the differential structure of S_ϵ to find the PS sequence of I_ϵ. We would like to point out that, M_ϵ is indeed a C^1 regular manifold, and one can make use of the differential structure of M_ϵ to find the PS sequence of I_ϵ.

Lemma 2.6. *Assume that $\{u_n\}$ is a PS sequence of I_ϵ, that is*

$$I_\epsilon(u_n) \to c < \infty, \;\; I'_\epsilon(u_n) \to 0,$$

as $n \to \infty$. Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Proof. Observe that

$$\begin{aligned} & c + o_n(1) + o_n(1)\|u_n\|_\epsilon \\ &= I_\epsilon(u_n) - \frac{1}{2q}\langle I'_\epsilon(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2q}\right)\|u_n\|_\epsilon^2 + \left(\frac{1}{2q} - \frac{1}{2p}\right)\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{P(\epsilon y)|u_n(y)|^p}{|x-y|^\mu} P(\epsilon x)|u_n(x)|^p, \end{aligned} \tag{2.4}$$

from which one easily has that $\|u_n\|_\epsilon$ is bounded. $\square$

3. The autonomous problem

In this section we are concerned with the autonomous equation $(\mathrm{P})_\xi$, giving in Section 1. The functional I_ξ of $(\mathrm{P})_\xi$ is

$$I_\xi(u)=\frac{1}{2}\|u\|_\xi^2-\frac{1}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{P^2(\xi)|u(y)|^p}{|x-y|^\mu}|u(x)|^p-\frac{1}{2q}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{Q^2(\xi)|u(y)|^q}{|x-y|^\mu}|u(x)|^q.$$

The Nehari manifold corresponding to $(\mathrm{P})_\xi$ is defined by

$$M_\xi=\{u\in H^1(\mathbb{R}^3)\backslash\{0\}:\langle I'_\xi(u),u\rangle=0\},$$

and the least energy $C(\xi)$ is defined by $C(\xi)=\inf_{M_\xi} I_\xi$. As Lemma 2.3, $C(\xi)>0$. Similar to (2.3), we get

$$C(\xi)=\inf_{u\in M_\xi} I_\xi(u)=\inf_{w\in H^1(\mathbb{R}^3)\backslash\{0\}}\max_{t>0} I_\xi(tw). \tag{3.1}$$

Defined the mapping $m_\xi:S_\xi\to M_\xi$ by $m_\xi(w):=t(w)w$, and $\Psi_\xi(w)=I_\xi(m_\xi(w))$. Replaced I_ϵ, Ψ_ϵ, m_ϵ and S_ϵ, by I_ξ, Ψ_ξ, m_ξ and S_ξ, Lemma 2.5 still hold.

Lemma 3.1. *For any $\xi\in\mathbb{R}^3$, the problem $(P)_\xi$ has a positive ground state u with $I_\xi(u)=C(\xi)$.*

Proof. Assume that $\{w_n\}\subset S_\xi$ is a minimizing sequence satisfying $\Psi_\xi(w_n)\to\inf_{S_\xi}\Psi_\xi$. By the Ekeland variational principle, we suppose $\Psi'_\xi(w_n)\to 0$. Then, from Lemma 2.5 it follows that $I'_\xi(u_n)\to 0$ and $I_\xi(u_n)\to c_\xi$, where $u_n=m_\xi(w_n)\in M_\xi$. Similar to Lemma 2.6, we obtain that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Up to a subsequence, we assume that $u_n\rightharpoonup\tilde u$ in $H^1(\mathbb{R}^3)$, $u_n\to\tilde u$ in $L^2_{loc}(\mathbb{R}^3)$ and $u_n\to\tilde u$ a.e. on $\mathbb{R}^3$. Then $I'_\xi(\tilde u)=0$. Suppose $\{u_n\}$ is vanishing, then P. L. Lions compactness lemma implies that $u_n\to 0$ in $L^{p\cdot\frac{6}{6-\mu}}(\mathbb{R}^3)$ and $L^{q\cdot\frac{6}{6-\mu}}(\mathbb{R}^3)$. By (2.2), we easily have that $C(\xi)=0$. This is impossible. Hence $\{u_n\}$ is non-vanishing. Then there exists $x_n\in\mathbb{R}^3$ and $\delta_0>0$ such that

$$\int_{B_1(x_n)} u_n^2(x)>\delta_0.$$

Without loss of generality, we assume that $x_n\in\mathbb{Z}^3$. Since the invariant of the functional I_ξ and M_ξ under the translation of the form $v(\cdot)\to v(\cdot-x_n)$, we can assume that $\{x_n\}$ is bounded. Then $\tilde u\neq 0$. So $\tilde u\in M_\xi$ and $I_\xi(\tilde u)\ge C(\xi)$. As (2.4), from Fatou lemma we easily infer that

$$I_\xi(\tilde u)\le\liminf_{n\to\infty}\big(I_\xi(u_n)-\frac{1}{2q}\langle I'_\xi(u_n),u_n\rangle\big)=C(\xi). \tag{3.2}$$

Therefore, $I_\xi(\tilde u)=C(\xi)$. By standard arguments, we can assume that $\tilde u>0$. This ends the proof. □

Next we study the continuity of $C(\xi)$.

Lemma 3.2. *$\xi\to C(\xi)$ is continuous.*

Proof. We shall borrow the idea in [3, Lemma 2.2] to give the proof. For $\xi\in\mathbb{R}^3$, let $\{\xi_n\}$, $\{\lambda_n\}$ be sequences in $\mathbb{R}^3$ such that

(a) $\xi_n\to\xi$, and for any n, $C(\xi_n)\ge C(\xi)$;
(b) $\lambda_n\to\xi$, and for any n, $C(\lambda_n)\le C(\xi)$.

It suffices to show that $C(\xi_n), C(\lambda_n) \to C(\xi)$. First, we show that $C(\xi_n) \to C(\xi)$. By Lemma 3.1, there exists $w \in H^1(\mathbb{R}^3)$ such that $I_\xi(w) = C(\xi)$ and $I'_\xi(w) = 0$. Let $t_n > 0$ be such that $t_n w \in M_{\xi_n}$. Since $w \in M_\xi$ and $t_n w \in M_{\xi_n}$, we get $t_n \to 1$. Then $I_{\xi_n}(t_n w) \to I_\xi(w)$. So

$$\limsup_{n\to\infty} C(\xi_n) \le \limsup_{n\to\infty} I_{\xi_n}(t_n w) = I_\xi(w) = C(\xi).$$

On the other hand, since $C(\xi_n) \ge C(\xi)$, we have $\liminf_{n\to\infty} C(\xi_n) \ge C(\xi)$. Hence, $C(\xi_n) \to C(\xi)$. Below we show that $C(\lambda_n) \to C(\xi)$. Let $\{w_n\} \in M_{\lambda_n}$ be such that $I_{\lambda_n}(w_n) = C(\lambda_n)$ and $I'_{\lambda_n}(w_n) = 0$. As (2.4), one has that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and we may assume that $w_n \rightharpoonup w_0$ in $H^1(\mathbb{R}^3)$. Since $\lambda_n \to \xi$, the Hardy–Littlewood–Sobolev inequality implies that $I'_\xi(w_n) \rightharpoonup 0$. Then $I'_\xi(w_0) = 0$. Since M_{λ_n} is bounded away from 0, it is easy to see that w_n is non-vanishing. Moreover, as in the proof of Lemma 3.1, using the invariant of I_ξ and M_ξ, we can assume that $w_0 \neq 0$. As (2.4), Fatou lemma implies that

$$\liminf_{n\to\infty} I_{\lambda_n}(w_n) \ge I_\xi(w_0).$$

Then $C(\xi) \le \liminf C(\lambda_n)$. On the other hand, using (b), $\limsup C(\lambda_n) \le C(\xi)$. Thus $C(\lambda_n) \to C(\xi)$. This ends the proof. $\square$

We now present an expression for $C(\xi)$ which, in particular, enables us to express $C(\xi)$ explicitly in terms of $V(\xi)$, $P(\xi)$ and $Q(\xi)$.

Let v is a ground state of the equation $(\mathrm{P})_\xi$. Then $I'_\xi(v) = 0$ and $I_\xi(v) = C(\xi)$. Substituting $v = tw(sx)$ with $s^2 = V(\xi)$ and $t^2 = V^{\frac{5-\mu}{2(p-1)}}(\xi)P^{-\frac{2}{p-1}}(\xi)$, and

$$\alpha(\xi) = V^{\frac{5-\mu}{2(p-1)}(q-p)}(\xi)Q^2(\xi)P^{\frac{2(1-q)}{p-1}}(\xi),$$

into the equation $(\mathrm{P})_\xi$, we have

$$-\Delta w + w = \Big(\int\limits_{\mathbb{R}^3} \frac{|w(y)|^p}{|x-y|^\mu}\Big)|w|^{p-2}w + \alpha(\xi)\Big(\int\limits_{\mathbb{R}^3} \frac{|w(y)|^q}{|x-y|^\mu}\Big)|w|^{q-2}w,\ x \in \mathbb{R}^3, \tag{3.3}$$

whose functional we denoted by $I_{\alpha(\xi)}$. Then $\langle I'_{\alpha(\xi)}(w), w\rangle = 0$. Moreover, it is easy to see that $I_\xi(v) = V^{-\frac{1}{2}+\frac{5-\mu}{2(p-1)}}(\xi)P^{-\frac{2}{p-1}}(\xi)I_{\alpha(\xi)}(w)$. Let $c_{\alpha(\xi)} := c(1, 1, \alpha(\xi))$, i.e. the least energy associated with (3.3). Then

$$V^{-\frac{1}{2}+\frac{5-\mu}{2(p-1)}}(\xi)P^{-\frac{2}{p-1}}(\xi)c_{\alpha(\xi)} \le C(\xi).$$

The reverse inequality is obtained in the same way. Then

$$C(\xi) = V^{-\frac{1}{2}+\frac{5-\mu}{2(p-1)}}(\xi)P^{-\frac{2}{p-1}}(\xi)c_{\alpha(\xi)}.$$

Thus we have the following lemma.

Lemma 3.3. *$c_{\alpha(\xi)}$ is a decreasing function of $\alpha(\xi)$ and*

$$C(\xi) = V^{-\frac{1}{2}+\frac{5-\mu}{2(p-1)}}(\xi)P^{-\frac{2}{p-1}}(\xi)c_{\alpha(\xi)}.$$

As a byproduct of the last lemma, we have the following corollary.

Corollary 3.1. *By Lemma 3.3, if there exists a point $s_0 \in \mathbb{R}^3$ such that*

$$V_\infty^{-\frac{1}{2}+\frac{5-\mu}{2(p-1)}}P_\infty^{-\frac{2}{p-1}} \ge V^{-\frac{1}{2}+\frac{5-\mu}{2(p-1)}}(s_0)P^{-\frac{2}{p-1}}(s_0),$$

$$V_\infty^{\frac{5-\mu}{2(p-1)}(q-p)} Q_\infty^2 P_\infty^{\frac{2(1-q)}{p-1}} \leq V^{\frac{5-\mu}{2(p-1)}(q-p)}(s_0) Q^2(s_0) P^{\frac{2(1-q)}{p-1}}(s_0),$$

with one of the above inequalities being strict, then (1.5) holds.

4. A compactness lemma

In this section we shall prove some compactness results for the functional I_ϵ. Firstly, by a nonlocal version of Brezis–Lieb lemma [1, Lemma 3.2], we have the following lemma.

Lemma 4.1. *Fixed $\epsilon > 0$, let $\{u_n\}$ be a bounded $(PS)_c$ sequence for I_ϵ with $c > 0$, then replacing u_n, if necessary, with a subsequence, there exists $u \in H^1(\mathbb{R}^3)$ with $I'_\epsilon(u) = 0$, such that*

(1) $I_\epsilon(u_n - u) \to c - I_\epsilon(u)$;

(2) $I'_\epsilon(u_n - u) \to 0$.

Lemma 4.2. *Fixed $\epsilon > 0$, let $0 < \mu < 2$ and $2 < q < p < 6 - 2\mu$. Then, for any $(PS)_c$ sequence u_n for I_ϵ with $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, either $u_n \to u$ in $H^1(\mathbb{R}^3)$ along a subsequence or*

$$c - I_\epsilon(u) \geq c_\infty.$$

Proof. Define $v_n = u_n - u$ and suppose that $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^3)$. Let $t_n > 0$ be such that $t_n v_n \in M_\infty$, where

$$M_\infty = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'_\infty(u), u\rangle = 0\}.$$

We claim that $\limsup_{n\to\infty} t_n \leq 1$. Otherwise, there exist $\delta > 0$ and a subsequence still denoted by t_n, such that $t_n \geq 1+\delta$ for all $n \in \mathbb{N}$. Since $0 < \mu < 2$ and $2 < q < p < 6-2\mu$, as in [6], there exists $C > 0$ such that

$$\Big|\int_{\mathbb{R}^3} \frac{|v_n(y)|^p}{|x-y|^\mu}\Big|_\infty \leq C, \quad \Big|\int_{\mathbb{R}^3} \frac{|v_n(y)|^q}{|x-y|^\mu}\Big|_\infty \leq C. \tag{4.1}$$

Then it is easy to see that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P(\epsilon y)|v_n(y)|^p}{|x-y|^\mu} P(\epsilon x)|v_n(x)|^p = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P_\infty^2|v_n(y)|^p}{|x-y|^\mu}|v_n(x)|^p + o_n(1),$$
$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q(\epsilon y)|v_n(y)|^p}{|x-y|^\mu} Q(\epsilon x)|v_n(x)|^p = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q_\infty^2|v_n(y)|^p}{|x-y|^\mu}|v_n(x)|^p + o_n(1).$$

By Lemma 4.1, $\langle I'_\epsilon(v_n), v_n\rangle = o_n(1)$. Then

$$\int_{\mathbb{R}^3} (|\nabla v_n|^2 + V_\infty|v_n|^2) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P_\infty^2|v_n(y)|^p}{|x-y|^\mu}|v_n(x)|^p + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q_\infty^2|v_n(y)|^q}{|x-y|^\mu}|v_n(x)|^q + o_n(1).$$

Combining with $t_n v_n \in M_\infty$, we get

$$(t_n^{2p-2q} - 1)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P_\infty^2|v_n(y)|^p}{|x-y|^\mu}|v_n(x)|^p + (1 - t_n^{2-2q})\int_{\mathbb{R}^3} (|\nabla v_n|^2 + V_\infty|v_n|^2) = o_n(1). \tag{4.2}$$

If $\{v_n\}$ is vanishing, then P.L. Lions compactness lemma implies $v_n \to 0$ in $L^{q\cdot\frac{6}{6-\mu}}(\mathbb{R}^3)$ and $L^{p\cdot\frac{6}{6-\mu}}(\mathbb{R}^3)$. Note that $\langle I'_\epsilon(v_n), v_n\rangle = o_n(1)$, one easily gives $v_n \to 0$ in $H^1(\mathbb{R}^3)$, which contradicts with the above assumption. Then there exist $y_n \in \mathbb{R}^3$ and $\delta > 0$ such that

$$\int_{B_1(y_n)} |v_n|^2 \geq \delta. \tag{4.3}$$

Set $\tilde{v}_n = v_n(x + y_n)$, we may suppose that, up to a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H^1(\mathbb{R}^3)$ and $\tilde{v}_n \to \tilde{v}$ a.e. in $\mathbb{R}^3$. By (4.3) we get $\tilde{v} \neq 0$. Then there exists a subset $\Omega \subset \mathbb{R}^3$ with positive measure such that $|\tilde{v}(x)| > 0$ for all $x \in \Omega$. By $t_n \geq 1 + \delta$ and (4.2), letting $n \to \infty$ we infer

$$0 < \int_\Omega \int_\Omega \frac{|\tilde{v}(y)|^p}{|x-y|^\mu} |\tilde{v}(x)|^p = 0,$$

which is absurd. We next distinguish the following two cases:

Case 1: $\limsup_{n\to\infty} t_n = 1$. In this case, there exists a subsequence, still denoted by t_n, such that $t_n \to 1$. From (1) of Lemma 4.1, we have

$$c - I_\epsilon(u) + o_n(1) = I_\epsilon(v_n) \geq I_\epsilon(v_n) + c_\infty - I_\infty(t_n v_n) = o_n(1) + c_\infty.$$

Consequently, $c - I_\epsilon(u) \geq c_\infty$.

Case 2. $\limsup_{n\to\infty} t_n = t_0 < 1$. In this case, without loss of generality, we suppose that $t_n \to t_0 < 1$. Then as (2.4) we obtain

$$\begin{aligned}
c_\infty &\leq (\frac{1}{2} - \frac{1}{2q}) \int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + V_\infty v_n^2\right) + (\frac{1}{2q} - \frac{1}{2p}) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P_\infty^2 |v_n(y)|^p}{|x-y|^\mu} |v_n(x)|^p \\
&= (\frac{1}{2} - \frac{1}{2q}) \|v_n\|_\epsilon^2 + (\frac{1}{2q} - \frac{1}{2p}) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P(\epsilon y)|v_n(y)|^p}{|x-y|^\mu} P(\epsilon x)|v_n(x)|^p + o_n(1) \\
&= I_\epsilon(v_n) - \frac{1}{2q}\langle I'_\epsilon(v_n), v_n\rangle + o_n(1) = c - I_\epsilon(u) + o_n(1).
\end{aligned}$$

Hence, $c - I_\epsilon(u) \geq c_\infty$. $\square$

By Lemmas 2.6 and 4.2, we get:

Lemma 4.3. *I_ϵ satisfies the $(PS)_c$ condition with $c < c_\infty$.*

5. Proof of Theorem 1.1

Lemma 5.1. $\limsup_{\epsilon\to 0} c_\epsilon \leq \inf_{\xi\in\mathbb{R}^3} C(\xi)$. *Moreover,* $\limsup_{\epsilon\to 0} c_\epsilon < c_\infty$.

Proof. Since $(\mathrm{P})_\xi$ has a positive ground state for each $\xi \in \mathbb{R}^3$, we can take $u \in M_\xi$ such that $I_\xi(u) = C(\xi)$ and $I'_\xi(u) = 0$. Then for small $\epsilon > 0$, there holds

$$\begin{aligned}
|\nabla u|_2^2 + \int_{\mathbb{R}^3} V(\epsilon x + \xi)|u|^2 &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P(\epsilon y + \xi)|u(y)|^p}{|x-y|^\mu} P(\epsilon x + \xi)|u(x)|^p \\
&\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q(\epsilon y + \xi)|u(y)|^q}{|x-y|^\mu} Q(\epsilon x + \xi)|u(x)|^q + o_\epsilon(1).
\end{aligned} \tag{5.1}$$

Set $w_\epsilon(x) = u(x - \frac{\xi}{\epsilon})$. By (5.1), we have that

$$\langle I'_\epsilon(w_\epsilon), w_\epsilon\rangle = o_\epsilon(1). \tag{5.2}$$

There exists $t_\epsilon > 0$ such that $t_\epsilon w_\epsilon \in M_\epsilon$. Then $\langle I'_\epsilon(t_\epsilon w_\epsilon), t_\epsilon w_\epsilon\rangle = 0$. By (5.2), it is easy to see that $t_\epsilon \to 1$ as $\epsilon \to 0$. Then

$$I_\epsilon(t_\epsilon w_\epsilon) = I_\xi(w) + o_\epsilon(1) = C(\xi) + o_\epsilon(1).$$

Note that $t_\epsilon w_\epsilon \in M_\epsilon$, one has that $\limsup_{\epsilon\to 0} c_\epsilon \le C(\xi)$. Since ξ is arbitrary, we get

$$\limsup_{\epsilon\to 0} c_\epsilon \le \inf_{\xi\in\mathbb{R}^3} C(\xi).$$

Using (1.5), $\limsup_{\epsilon\to 0} c_\epsilon < c_\infty$. This ends the proof. □

By Lemma 5.1, we may assume that

$$c_\epsilon < c_\infty, \tag{5.3}$$

as ϵ small enough.

Lemma 5.2. *The minimax value c_ϵ is achieved if ϵ is small enough. Hence, problem $(P)^*$ has a positive ground state if ϵ is small enough.*

Proof. Assume that $w_n \in S_\epsilon$ satisfies that $\Psi_\epsilon(w_n) \to \inf_{S_\epsilon} \Psi_\epsilon$. By the Ekeland variational principle, we may suppose that $\Psi'_\epsilon(w_n) \to 0$. Then from Lemma 2.5 it follows that $I'_\epsilon(u_n) \to 0$ and $I_\epsilon(u_n) \to c_\epsilon$, where $u_n = m_\epsilon(w_n) \in M_\epsilon$. By (5.3), Lemma 4.3 implies that there exists $\tilde{u}_\epsilon$ such that $u_n \to \tilde{u}_\epsilon$ in $H^1(\mathbb{R}^3)$. Then $I'_\epsilon(\tilde{u}_\epsilon) = 0$ and $I_\epsilon(\tilde{u}_\epsilon) = c_\epsilon$. By standard arguments, we can further assume that $\tilde{u}_\epsilon > 0$. This ends the proof. □

Lemma 5.3. *Suppose that $0 < \mu < 2$, $2 < q < p < 6-2\mu$ and V, P Q are uniformly continuous. Let u_ϵ be the positive ground state obtained in Lemma 5.2. Then there is $y_\epsilon \in \mathbb{R}^3$ such that $\lim_{\epsilon\to 0} C(\epsilon y_\epsilon) = \inf_{\xi\in\mathbb{R}^3} C(\xi)$, and for each sequence $\epsilon y_\epsilon \to y_0$, $v_\epsilon(x) := u_\epsilon(x + y_\epsilon)$ converges in $H^1(\mathbb{R}^3)$ to a ground state v of*

$$-\Delta u + V(y_0)u = \Big(\int\limits_{\mathbb{R}^3} \frac{P^2(y_0)|u(y)|^p}{|x-y|^\mu}\Big)|u|^{p-2}u + \Big(\int\limits_{\mathbb{R}^3} \frac{Q^2(y_0)|u(y)|^q}{|x-y|^\mu}\Big)|u|^{q-2}u. \tag{5.4}$$

Proof. Let u_n be the positive ground states of problem $(\mathrm{P})^*$ with parameter $\epsilon_n \to 0$. Since $\limsup_{n\to\infty} c_{\epsilon_n} < c_\infty$, as Lemma 2.6 we infer that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Moreover, since $c_{\epsilon_n} \ge c(V_0, |P|_\infty, |Q|_\infty)$, it is easy to see that $\{u_n\}$ is non-vanishing. Then there exists $\delta > 0$ such that

$$\int\limits_{B_1(y_n)} |u_n(x)|^2 \ge \delta. \tag{5.5}$$

Setting $v_n(x) = u_n(x+y_n)$, $\tilde{V}_{\epsilon_n}(x) = V(\epsilon_n(x+y_n))$ and $\tilde{P}_{\epsilon_n}(x) = P(\epsilon_n(x+y_n))$, and $\tilde{Q}_{\epsilon_n}(x) = Q(\epsilon_n(x+y_n))$, we see that v_n solves the below problem

$$-\Delta u + \tilde{V}_{\epsilon_n}(x)u = \Big(\int\limits_{\mathbb{R}^3} \frac{\tilde{P}_{\epsilon_n}(y)|u(y)|^p}{|x-y|^\mu}\Big)\tilde{P}_{\epsilon_n}(x)|u|^{p-2}u + \Big(\int\limits_{\mathbb{R}^3} \frac{\tilde{Q}_{\epsilon_n}(y)|u(y)|^q}{|x-y|^\mu}\Big)\tilde{Q}_{\epsilon_n}(x)|u|^{q-2}u.$$

Since $\{v_n\}$ is also bounded in $H^1(\mathbb{R}^3)$, from (5.5), we may assume that $v_n \rightharpoonup v \ne 0$ in $H^1(\mathbb{R}^3)$.

Claim 1. The sequence $\{\epsilon_n y_n\}$ must be bounded.

Otherwise if $\epsilon_n y_n \to \infty$, then $V(\epsilon_n y_n) \to V_\infty$, $P(\epsilon_n y_n) \to P_\infty$ and $Q(\epsilon_n y_n) \to Q_\infty$. Since V, P and Q are uniformly continuous functions, it follows that for $R > 0$ and $|x| \leq R$,

$$|\tilde{V}_{\epsilon_n}(x) - V_\infty| \leq |V(\epsilon_n(x + y_n)) - V(\epsilon_n y_n)| + |V(\epsilon_n y_n) - V_\infty| \to 0.$$

Similarly,

$$|\tilde{P}_{\epsilon_n}(x) - P_\infty| \to 0, \quad |\tilde{Q}_{\epsilon_n}(x) - Q_\infty| \to 0, \ \forall |x| \leq R.$$

Then for each $\eta \in C_0^\infty(\mathbb{R}^3)$, we claim that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \tilde{V}_{\epsilon_n}(x) v_n \eta \to \int_{\mathbb{R}^3} V_\infty v \eta, \\
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\tilde{P}_{\epsilon_n}(y)|v_n(y)|^p}{|x-y|^\mu} \tilde{P}_{\epsilon_n}(x)|v_n|^{p-2} v_n \eta \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P_\infty^2 |v(y)|^p}{|x-y|^\mu} |v|^{p-2} v \eta, \\
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\tilde{Q}_{\epsilon_n}(y)|v_n(y)|^p}{|x-y|^\mu} \tilde{Q}_{\epsilon_n}(x)|v_n|^{p-2} v_n \eta \to \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q_\infty^2 |v(y)|^p}{|x-y|^\mu} |v|^{p-2} v \eta\ .
\end{aligned} \tag{5.6}$$

Below we only prove the second limit of (5.6) since the others can be similarly obtained. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\tilde{P}_{\epsilon_n}(y)|v_n(y)|^p}{|x-y|^\mu} \tilde{P}_{\epsilon_n}(x)|v_n|^{p-2} v_n \eta - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P_\infty^2 |v(y)|^p}{|x-y|^\mu} |v|^{p-2} v \eta \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\tilde{P}_{\epsilon_n}(y)|v_n(y)|^p - P_\infty |v(y)|^p}{|x-y|^\mu} \tilde{P}_{\epsilon_n}(x)|v_n|^{p-2} v_n \eta \\
&\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{P_\infty |v(y)|^p}{|x-y|^\mu} \Big(\tilde{P}_{\epsilon_n}(x)|v_n|^{p-2} v_n \eta - P_\infty |v|^{p-2} v \eta\Big) \\
&:= I_1 + I_2.
\end{aligned}$$

For I_1, since $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^3$, then for any $x \in \mathbb{R}^3$, $\tilde{P}_{\epsilon_n}(x)|v_n|^p \to P_\infty |v|^p$, therefore $\tilde{P}_{\epsilon_n}(x)|v_n|^p$ converges weakly to $P_\infty |v|^p$ in $L^{\frac{6}{6-\mu}}(\mathbb{R}^3)$. Using the Hardy–Littlewood–Sobolev inequality we know the convolution term

$$\frac{1}{|x|^\mu} * w(x) \in L^{\frac{6}{\mu}}(\mathbb{R}^3)$$

for all $w \in L^{\frac{6}{6-\mu}}(\mathbb{R}^3)$. Then

$$\int_{\mathbb{R}^3} \frac{\tilde{P}_{\epsilon_n}(y)|v_n(y)|^p}{|x-y|^\mu} \rightharpoonup \int_{\mathbb{R}^3} \frac{P_\infty |v(y)|^p}{|x-y|^\mu} \ \text{ in } \ L^{\frac{6}{\mu}}(\mathbb{R}^3).$$

Since $|v|^{p-2} v \eta \in L^{\frac{6}{6-\mu}}(\mathbb{R}^3)$, we infer that $I_1 \to 0$. Observe that $\tilde{P}_{\epsilon_n}(x)|v_n|^{p-2} v_n \to P_\infty |v|^{p-2} v$ in $L_{loc}^{\frac{6}{6-\mu}\cdot\frac{p}{p-1}}(\mathbb{R}^3)$. Then

$$I_2 \leq C |v_n|_{p\cdot\frac{6}{6-\mu}} \Big|\tilde{P}_{\epsilon_n}(x)|v_n|^{p-2} v_n - P_\infty |v|^{p-2} v\Big|_{\frac{6}{6-\mu}\cdot\frac{p}{p-1},\Omega} |\eta|_{p\cdot\frac{6}{6-\mu}} \to 0,$$

where $\Omega = supp\eta$. Thus, from (5.6) v solves

$$-\Delta u + V_\infty u = \Big(\int\limits_{\mathbb{R}^3} \frac{P_\infty^2 |u(y)|^p}{|x-y|^\mu}\Big)|u|^{p-2}u + \Big(\int\limits_{\mathbb{R}^3} \frac{Q_\infty^2 |u(y)|^q}{|x-y|^\mu}\Big)|u|^{q-2}u.$$

Therefore,

$$\begin{aligned}
c_\infty &\le I_\infty(v) - \frac{1}{2q}\langle I'_\infty(v), v\rangle \\
&= \Big(\frac{1}{2}-\frac{1}{2q}\Big)\Big(|\nabla v|_2^2 + \int\limits_{\mathbb{R}^3} V_\infty v^2\Big) + \Big(\frac{1}{2q}-\frac{1}{2p}\Big)\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{P_\infty^2 |v(y)|^q}{|x-y|^\mu}|v|^q \\
&\le \liminf_{n\to\infty}\Big(\frac{1}{2}-\frac{1}{2q}\Big)\Big(|\nabla v_n|_2^2 + \int\limits_{\mathbb{R}^3} \tilde V_{\epsilon_n}(x) v_n^2\Big) + \Big(\frac{1}{2q}-\frac{1}{2p}\Big)\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{\tilde P_{\epsilon_n}(y)|v_n(y)|^q}{|x-y|^\mu}\tilde P_{\epsilon_n}(x)|v_n|^q \qquad (5.7)\\
&= \liminf_{n\to\infty}[\tilde I_{\epsilon_n}(v_n) - \frac{1}{2q}\langle \tilde I'_{\epsilon_n}(v_n), v_n\rangle] \\
&= \liminf_{n\to\infty}[I_{\epsilon_n}(u_n) - \frac{1}{2q}\langle I'_{\epsilon_n}(u_n), u_n\rangle] = \liminf_{n\to\infty} c_{\epsilon_n}.
\end{aligned}$$

This contradicts with $\limsup_{n\to\infty} c_{\epsilon_n} < c_\infty$. Hence $\{\epsilon_n y_n\}$ is bounded and we suppose that $\epsilon_n y_n \to y_0$.

Claim 2. $C(y_0) = \inf_{\xi\in\mathbb{R}^3} C(\xi)$, and v_n converges strongly to v in $H^1(\mathbb{R}^3)$.

In fact, following the proof of Claim 1, we know that v is a solution of the equation (5.4). Moreover, as (5.7) we have

$$\begin{aligned}
C(y_0) &\le I_{y_0}(v) - \frac{1}{2q}\langle I'_{y_0}(v), v\rangle \\
&= \Big(\frac{1}{2}-\frac{1}{2q}\Big)\Big(|\nabla v|_2^2 + \int\limits_{\mathbb{R}^3} V(y_0) v^2\Big) + \Big(\frac{1}{2q}-\frac{1}{2p}\Big)\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{P^2(y_0)|v(y)|^q}{|x-y|^\mu}|v|^q \\
&\le \liminf_{n\to\infty}\Big(\frac{1}{2}-\frac{1}{2q}\Big)\Big(|\nabla v_n|_2^2 + \int\limits_{\mathbb{R}^3} \tilde V_{\epsilon_n}(x) v_n^2\Big) + \Big(\frac{1}{2q}-\frac{1}{2p}\Big)\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} \frac{\tilde P_{\epsilon_n}(y)|v_n(y)|^q}{|x-y|^\mu}\tilde P_{\epsilon_n}(x)|v_n|^q \\
&= \liminf_{n\to\infty} c_{\epsilon_n}.
\end{aligned} \qquad (5.8)$$

By Lemma 5.1, we know that $\limsup_{n\to\infty} c_{\epsilon_n} \le \inf_{\xi\in\mathbb{R}^3} C(\xi)$. Then

$$\inf_{\xi\in\mathbb{R}^3} C(\xi) = C(y_0) = \lim_{n\to\infty} c_{\epsilon_n}.$$

Using (5.8) we have that

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^3}\Big(|\nabla v_n|^2 + \tilde V_{\epsilon_n}(x)|v_n|^2\Big) = \int\limits_{\mathbb{R}^3}\Big(|\nabla v|^2 + V(y_0)v^2\Big),$$

which yields that $v_n \to v$ in $H^1(\mathbb{R}^3)$. In addition, form (5.8) we know that $I_{y_0}(v) = C(y_0)$. So v is a ground state of the equation (5.4). □

Lemma 5.4. *Assume that $0 < \mu < 2$, $2 < q < p < 6-2\mu$ and V, P, Q are uniformly continuous. Set $v_n := u_n(x+y_n)$, where u_n is the positive ground state obtained in Lemma 5.2 and y_n is given in (5.5). Then:*

(i) there exist δ' and $M > 0$ such that $\delta' \le |v_n|_\infty \le M$ for all $n \in \mathbb{N}$.

(ii)

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

Moreover, there exist $C_1,\ C_2 > 0$ *such that*

$$v_n(x) \le C_1 e^{-C_2|x|}, \quad \forall x \in \mathbb{R}^3.$$

Proof. As in the proof of Lemma 5.3, we have that v_n is the solution of

$$-\Delta u + \tilde{V}_{\epsilon_n}(x)u = \Big(\int_{\mathbb{R}^3} \frac{\tilde{P}_{\epsilon_n}(y)|u(y)|^p}{|x-y|^\mu}\Big)\tilde{P}_{\epsilon_n}(x)|u|^{p-2}u + \Big(\int_{\mathbb{R}^3} \frac{\tilde{Q}_{\epsilon_n}(y)|u(y)|^q}{|x-y|^\mu}\Big)\tilde{Q}_{\epsilon_n}(x)|u|^{q-2}u,$$

and $v_n \to v \neq 0$ in $H^1(\mathbb{R}^3)$. Then

$$\lim_{R\to\infty} \int_{|x|\ge R} (v_n^2 + v_n^6) = 0, \text{uniformly for } n \in \mathbb{N}. \tag{5.9}$$

Denote

$$\tilde{K}_n(x) := \int_{\mathbb{R}^3} \frac{\tilde{P}_{\epsilon_n}(y)|v_n(y)|^p}{|x-y|^\mu},\ \tilde{\tilde{K}}_n(x) := \int_{\mathbb{R}^3} \frac{\tilde{Q}_{\epsilon_n}(y)|v_n(y)|^q}{|x-y|^\mu}.$$

As (4.1), there exists $C > 0$ such that $|\tilde{K}_n|_\infty, |\tilde{\tilde{K}}_n|_\infty \le C$ for all $n \in \mathbb{N}$. Using [12, Proposition 3.3], we get that $v_n \in L^t(\mathbb{R}^3)$ for all $t \ge 2$. Then for $t = \frac{12}{3-\mu} > 3$, $v_n^{p-1}, v_n^{q-1} \in L^{\frac{t}{2}}(\mathbb{R}^3)$ for all n. Note that

$$-\Delta v_n \le \tilde{K}_n(x)\tilde{P}_{\epsilon_n}(x)|v_n|^{p-2}v_n + \tilde{\tilde{K}}_n(x)\tilde{Q}_{\epsilon_n}(x)|v_n|^{q-2}v_n.$$

Thus by [11, Theorem 8.17], we infer that for all $y \in \mathbb{R}^3$,

$$\sup_{B_1(y)} v_n(x) \le C\big(|v_n|_{L^2(B_2(y))} + |v_n^{p-1}|_{L^{\frac{t}{2}}(B_2(y))} + |v_n^{q-1}|_{L^{\frac{t}{2}}(B_2(y))}\big). \tag{5.10}$$

Hence $|v_n|_\infty$ is uniformly bounded. Recall that by (5.5),

$$\delta \le \int_{B_1(y_n)} |u_n(x)|^2 \le |B_1||v_n|_\infty^2.$$

Then $|v_n| \ge \delta'$, for all n. Moreover, by (5.9) we get

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly for all } n \in \mathbb{N}.$$

Then we can take $\rho_0 > 0$ such that

$$\tilde{K}_n(x)\tilde{P}_{\epsilon_n}(x)|v_n|^{p-2}v_n \le \frac{V_0}{4}v_n,\ \tilde{\tilde{K}}_n(x)\tilde{Q}_{\epsilon_n}(x)|v_n|^{q-2}v_n \le \frac{V_0}{4}v_n,$$

for all $|x| > \rho_0$. Thus,

$$-\Delta v_n + \frac{\tilde{V}_{\epsilon_n}(x)}{2}v_n \le \frac{V_0}{2}v_n - \frac{\tilde{V}_{\epsilon_n}(x)}{2}v_n \le 0,$$

for all $|x| \geq \rho_0$. Let s and T be positive constants such that $s^2 < \frac{V_0}{2}$ and $v_n(x) \leq Te^{-s\rho_0}$, for all $|x| = \rho_0$. Hence, the function $\psi(x) = Te^{-s|x|}$ satisfies

$$-\Delta\psi + \frac{\tilde{V}_{\epsilon_n}(x)}{2}\psi \geq (\frac{V_0}{2} - s^2)\psi > 0,$$

for all $x \neq 0$. Thereby, taking $\eta = \max\{v_n - \psi, 0\} \in H_0^1(|x| > \rho_0)$ as a test function, we have

$$\begin{aligned}
0 &\geq \int_{\mathbb{R}^3} (\nabla v_n \nabla \eta + \frac{\tilde{V}_{\epsilon_n}(x)}{2} v_n \eta) \\
&\geq \int_{\mathbb{R}^3} ((\nabla v_n - \nabla\psi)\nabla\eta + \frac{\tilde{V}_{\epsilon_n}(x)}{2}(v_n - \psi)\eta) \\
&\geq \frac{V_0}{2} \int_{\{x\in\mathbb{R}^3 : v_n > \psi\}} (v_n - \psi)^2 \geq 0,
\end{aligned}$$

for all $|x| > \rho_0$. Therefore, the set $\Omega_n := \{x \in \mathbb{R}^3 : |x| > \rho_0 \text{ and } v_n > \psi(x)\}$ is empty. Then we know that there exists $C_1, C_2 > 0$ such that

$$v_n(x) \leq C_1 e^{-C_2|x|}, \ \forall x \in \mathbb{R}^3.$$

This ends the proof. $\square$

Proof of Theorem 1.1. Going back to the equation (1.4) with the variable substitution: $x \mapsto \frac{x}{\epsilon}$, Lemma 5.2 implies that (1.4) has a positive ground state $w_\epsilon = u_\epsilon(\frac{x}{\epsilon})$ for $\epsilon > 0$ small. Set $\epsilon_n \to 0$ as $n \to \infty$. If b_n denotes a maximum point of v_n, then from Lemma 5.4, it follows that it is bounded. Then we assume that $b_n \in B_R(0)$. Thereby, the global maximum point of u_n is $z_n := b_n + y_n$ and then $x_n := \epsilon_n z_n$ is the maximum point of w_n. From the boundedness of b_n, by Lemma 5.3 we get that $\lim_{n\to\infty} x_n = y_0$, which together with Lemma 3.2 gives

$$\lim_{n\to\infty} C(x_n) = C(y_0) = \inf_{\xi\in\mathbb{R}^3} C(\xi).$$

Then from Lemma 5.3, the proof of the conclusion (2) (i) in Theorem 1.1 is completed. Moreover, from Lemma 5.4, by the boundedness of b_n we get

$$w_n(x) = u_n(\frac{x}{\epsilon_n}) = v_n(\frac{x}{\epsilon_n} - y_n) = v_n(\frac{x}{\epsilon_n} - \frac{x_n}{\epsilon_n} - b_n) \leq C_1 e^{-C_2|\frac{x}{\epsilon_n} - \frac{x_n}{\epsilon_n} - b_n|} \leq C_1 e^{-\frac{C_2}{\epsilon_n}|x - x_n|}.$$

Thus, for small $\epsilon > 0$, we have that

$$w_\epsilon(x) \leq C_1 e^{-\frac{C_2}{\epsilon}|x - x_\epsilon|}.$$

References

[1] N. Ackermann, On a periodic Schrödinger equation with nonlocal superlinear part, Math. Z. 248 (2004) 423–443.

[2] C.O. Alves, F.S. Gao, M. Squassina, M.B. Yang, Singularly perturbed critical Choquard equations, J. Differential Equations 263 (2017) 3943–3988.

[3] C.O. Alves, S.H.M. Soares, Existence and concentration of positive solutions for a class of gradient systems, NoDEA Nonlinear Differential Equations Appl. 12 (2005) 437–457.

[4] C.O. Alves, César E. Torres Ledesma, Existence and concentration of solution for a non-local regional Schrödinger equation with competing potentials, arXiv:1611.02056, 2016.

[5] C.O. Alves, M.B. Yang, Multiplicity and concentration of solutions for a quasilinear Choquard equation, J. Math. Phys. 55 (2014) 061502, 21 pp.

[6] C.O. Alves, M.B. Yang, Existence of semiclassical ground state solutions for a generalized Choquard equation, J. Differential Equations 257 (2014) 4133–4164.

[7] C.O. Alves, M.B. Yang, Investigating the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016) 23–58.
[8] S. Cingolani, M. Clapp, S. Secchi, Multiple solutions to a magnetic nonlinear Choquard equation, Z. Angew. Math. Phys. 63 (2012) 233–248.
[9] S. Cingolani, M. Lazzo, Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions, J. Differential Equations 160 (2000) 118–138.
[10] S. Cingolani, S. Secchi, M. Squassina, Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities, Proc. Roy. Soc. Edinburgh Sect. A 140 (2010) 973–1009.
[11] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983.
[12] X.M. He, Z.W. Zou, Existence and concentration of ground states for Schrödinger–Poisson equations with critical growth, J. Math. Phys. 53 (2012) 023702.
[13] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1976/1977) 93–105.
[14] E.H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, AMS, Providence, Rhode Island, 2001.
[15] P.L. Lions, The Choquard equation and related equations, Nonlinear Anal. 4 (1980) 1063–1073.
[16] L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 195 (2010) 455–467.
[17] V. Moroz, J. Van Schaftingen, Ground states of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2013) 153–184.
[18] V. Moroz, J. Van Schaftingen, Semi-classical states for the Choquard equation, Calc. Var. Partial Differential Equations 52 (2015) 199–235.
[19] V. Moroz, J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, Trans. Amer. Math. Soc. 367 (2015) 6557–6579.
[20] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy–Littlewood–Sobolev critical exponent, Commun. Contemp. Math. 17 (2015) 1550005, 12 pp.
[21] R. Penrose, On gravity's role in quantum state reduction, Gen. Relativity Gravitation 28 (1996) 581–600.
[22] A. Szulkin, T. Weth, The method of Nehari manifold, in: Handbook of Nonconvex Analysis and Applications, Int. Press, Somerville, MA, 2010.
[23] W.B. Wang, X.Y. Yang, F.K. Zhao, Existence and concentration of ground states to a quasilinear problem with competing potentials, Nonlinear Anal. 102 (2014) 120–132.
[24] X. Wang, B. Zeng, On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions, SIAM J. Math. Anal. 28 (1997) 633–655.
[25] H. Zhang, J.X. Xu, F.B. Zhang, Existence and multiplicity of solutions for a generalized Choquard equation, Comput. Math. Appl. 73 (2017) 1803–1814.