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New criteria for the monotonicity of the ratio of two Abelian integrals

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ABSTRACT

New criteria to determine the monotonicity of the ratio of two Abelian integrals are given. When two Abelian integrals have the forms $\int_{\Gamma_h} f_1(x)y\mathrm{d}x$ and $\int_{\Gamma_h} f_2(x)y\mathrm{d}x$ or the forms $\int_{\Gamma_h} \frac{f_1(x)}{y}\mathrm{d}x$ and $\int_{\Gamma_h} \frac{f_2(x)}{y}\mathrm{d}x$ and Γ_h are ovals belonging to the level set $\{(x,y)|H(x,y)=h\}$, where $H(x,y)$ has the form $y^2/2+\Psi(x)$ or $\phi(x)y^2/2+\Psi(x)$, we give new criteria, which are defined directly by the functions which appear in the above Abelian integrals, and prove that the monotonicity of the criteria implies the monotonicity of the ratios of the Abelian integrals. The new criteria are applicable in a large class of problems, some of which simplify the existing proofs and some of which generalize known results.

1. Introduction

This paper concerns the weakened 16th Hilbert problem on the number of limit cycles of plane differential systems, proposed by V. I. Arnold. The problem states as follows.

Consider a polynomial perturbation of a Hamiltonian system

$$\frac{dx}{dt}=\frac{\partial H(x,y)}{\partial y}+\varepsilon P(x,y),\quad \frac{dy}{dt}=-\frac{\partial H(x,y)}{\partial x}+\varepsilon Q(x,y),$$

where $H(x,y)$, $P(x,y)$ and $Q(x,y)$ are real polynomials. Then the Abelian integral associated to the above system is

$$I(h)=\int\limits_{\Gamma_h} P(x,y)\mathrm{d}x-Q(x,y)\mathrm{d}y, \tag{1}$$

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along a closed level set $\Gamma_h \subset \{(x,y)|H(x,y)=h,\ h_1<h<h_2\}$, where Γ_h forms a continuous family of ovals as h varies in the open interval (h_1,h_2). The question one asks is: how large can the number of isolated zeros of the function $I(h)$ be in the above open interval when P,Q and H are polynomials whose degrees are known? This problem is related to the estimation of the number of limit cycles of the perturbed Hamiltonian system. On this theme there have been many excellent works, for example, Binyamini et al. in [4] obtained a double exponential upper bound in n on the number of zeros of Abelian integral where $\deg H=n+1$ and $\deg P,Q=n$. For more works, we recommend the readers the review papers [14,15] or the book [6].

In this paper we consider the case where

$$H(x,y)=\frac{y^2}{2}+\Psi(x), \tag{2}$$

with $\Psi(x)\in C^2(\mu,\nu)$, $\mu,\nu\in\mathbb{R}$. In this case, the Abelian integral (1) can be written as

$$\sum_{k=1}^{m}\alpha_k I_k(h),$$

where $\alpha_1,\alpha_2,\cdots,\alpha_m$ are real constants, and the I_k are in the form

$$I_k(h)=\int\limits_{\Gamma_h} f_k(x)y\mathrm{d}x, \qquad k=1,2,\cdots,m,$$

where $f_k(x)$ are functions of class C^1.

Assume that one of the first two integrals $I_1(h)$ and $I_2(h)$ is non-vanishing, without loss of generality, that is $I_1(h)$. We let

$$u(h)=\frac{I_2(h)}{I_1(h)}.$$

Then the monotonicity of the ratio $u(h)$ shows that the Abelian integral (1) has at most one zero if $m=2$. If $m\geq 2$, as a first step, the monotonicity of the ratio $u(h)$ also play an important role in determining the number of zeros of the Abelian integral (1). For more details, see [16].

There have been many methods to obtain the monotonicity of the ratio $u(h)$, for example:

(i) using Picard Fuchs equations, see [19,7];
(ii) using Green formula, see [5];
(iii) direct estimation of the integral $I_2(h)I_1'(h)-I_1(h)I_2'(h)$ by using some technical tools, see [18,21,17].

Each of the above methods can be only used to some special cases, and one has to repeat the whole procedure of calculations for each individual problem. In [16], Li and Zhang develop a direct method. Concretely, they give a criterion function $\xi(x)$ depending only on $f_1(x),f_2(x),\Psi(x)$ and prove that the monotonicity of $\xi(x)$ implies the monotonicity of $u(h)$. Let us revisit their result.

Consider the Hamiltonian system

$$\dot{x}=y, \qquad \dot{y}=-\Psi'(x), \tag{3}$$

which has the Hamiltonian function $H(x,y)$ in form (2).

Assume that there exists an number $a\in(\mu,\nu)$ such that the following hypothesis is satisfied:

$$(H1) \qquad\qquad \Psi'(x)(x-a)>0, \quad \text{for all } x\in(\mu,\nu)\backslash\{a\}.$$

Obviously, $(a, 0)$ is a center of system (3). Without any loss of generality, we can assume $a = \Psi(a) = 0$ so that center is located at the origin and $\Psi(x) > 0$ for $x \in (\mu, \nu)\backslash\{0\}$.

Let $\Delta = (0, h_s) = \{h | H(x, y) = h$ contains an oval$\}$ and denote by

$$\Gamma_h = \{(x, y) \in \mathbb{R}^2 \mid H(x, y) = h, 0 < h < h_s\}$$

the compact component of the level curve.

For any $h \in (0, h_s)$, Γ_h is a closed orbit. It is easy to see that there exists an involution σ defined in (μ, ν) such that $\Psi(x) = \Psi(\sigma(x))$. Recall that a mapping $\sigma : I \to I$ is an involution if $\sigma^2 = Id$ and $\sigma \neq Id$. Note that here $\sigma(0) = 0$ and $x\sigma(x) < 0$ for other $x \in (\mu, \nu)$.

Define two Abelian integrals

$$I_k(h) = \int\limits_{\Gamma_h} f_k(x) y \mathrm{d}x,$$

where $f_k(x) \in C^1(\mu, \nu)$ for $k = 1, 2$.

Assume that the following hypothesis is also satisfied:

$$(H2) \qquad f_1(x) f_1(\sigma(x)) > 0, \quad \text{for all } x \in (0, \nu).$$

Now $I_1(h)$ is non-vanishing, then one can consider the ratio of the Abelian integral $I_2(h)$ to the Abelian integral $I_1(h)$.

By using double integral, in [16], Li and Zhang prove the following theorem:

Theorem 1.1. *Suppose that the hypotheses* $(H1)$ *and* $(H2)$ *are satisfied. Let*

$$u(h) = \frac{I_2(h)}{I_1(h)} \quad \text{and} \quad \xi(x) = \frac{f_2(x)\Psi'(\sigma(x)) - f_2(\sigma(x))\Psi'(x)}{f_1(x)\Psi'(\sigma(x)) - f_1(\sigma(x))\Psi'(x)}. \tag{4}$$

Then $\xi'(x) < 0$ *(resp.* > 0*) in* (a, ν) *implies* $u'(h) < 0$ *(resp.* > 0*).*

Remark 1.2. From the proof in [16], it is easy to check that the hypothesis $(H2)$ can be replaced by a weaker form

$$(H2') \qquad \frac{f_1(x)}{\Psi'(x)} - \frac{f_1(\sigma(x))}{\Psi'(\sigma(x))} > 0, \quad \text{for all } x \in (0, \nu),$$

and Theorem 1.1 still holds.

Remark 1.3. In fact, in [16], the authors obtained the monotonicity of two Abelian integrals under more general setting: the Abelian integrals have the form $I_k(h) = \int_{\Gamma_h} f_k(x) g(y) \mathrm{d}x$, and $H(x, y)$ has the form $H(x, y) = \Psi(x) + \Phi(y)$. In other words, Theorem 1.1 is only a special case, that is $g(y) = y$ and $\Phi(y) = \frac{y^2}{2}$, of Theorem 1 in [16], but in some sense it is the most important case, all the examples in [16] can be transformed to this case.

This method is quite convenient to use, thus it has been used for many times, for example in [9–11], to deal with zeros of Abelian integrals. On page 360 of V. I. Arnold's book [2], the criterion $\xi(x)$ in Theorem 1.1 is quoted as a useful tool that "despite its seemingly artificial form, it proves to be working in many independently arising particular cases". But on the other hand, it is only a sufficient condition, thus it cannot solve all the cases and sometimes many calculations are needed.

It is interesting to give a better criterion, which is the aim of the present paper. We shall state and prove our new criteria in Section 2. In Section 3, we shall give several applications of our criteria. These applications either generalize known results or simplify the proofs of existing results.

2. Main results and proofs

Let notations be as above. We use some idea of [22] and have the following result that gives a new criterion for the monotonicity of the ratio of two Abelian integrals.

Theorem 2.1. *Suppose that the hypotheses* $(H1)$ *and* $(H2')$ *are satisfied. Let*

$$u(h)=\frac{I_2(h)}{I_1(h)} \quad \textit{and} \quad \bar{\xi}(x)=\frac{\int_{\sigma(x)}^{x} f_2(t)\mathrm{d}t}{\int_{\sigma(x)}^{x} f_1(t)\mathrm{d}t}. \tag{5}$$

Then $\bar{\xi}'(x)>0$ *(resp.* <0*) in* $(0,\nu)$ *implies* $u'(h)<0$ *(resp.* >0*).*

The next proposition shows that for this particular case, our criterion in Theorem 2.1 is more powerful than the criterion in Theorem 1.1 (cf. [16]).

Proposition 2.2. *Suppose that the hypotheses* $(H1)$ *and* $(H2)$ *are satisfied. Then* $\xi'(x)>0$ *(resp.* <0*) in* $(0,\nu)$ *implies* $\bar{\xi}'(x)>0$ *(resp.* <0*) in* $(0,\nu)$*, where* $\xi(x)$ *and* $\bar{\xi}(x)$ *have been defined in Theorem 1.1 and Theorem 2.1 respectively.*

Then, theoretically speaking, for this particular case our method can solve all the cases that Li and Zhang's method can solve, and usually our calculation will be much simpler (see Section 3 for examples).

Moreover Theorem 2.1 in [17] is a special case of our Theorem 2.1, since for $f_1(x)=1, f_2(x)=x$, one has $\bar{\xi}(x)=\frac{1}{2}(x+\sigma(x))$. In other words, Theorem 2.1 also generalizes the results in [17].

For the proof of Theorem 2.1, we introduce two new integrals

$$J_k(h)=\int\limits_{\Gamma_h}\frac{g_k(x)}{y}\mathrm{d}x, \qquad k=1,2,$$

where $g_k(x)\in C^1(\mu,\nu)$. For $x\in(0,\nu)$, we define two functions

$$G_k(x)=\mathscr{B}_\sigma\left(\frac{g_k}{\Psi'}\right)(x)=\frac{g_k(x)}{\Psi'(x)}-\frac{g_k(\sigma(x))}{\Psi'(\sigma(x))}.$$

Here for a given function $\kappa(x)$, $\mathscr{B}_\sigma(\kappa)(x)=\kappa(x)-\kappa(\sigma(x))$ is called as its balance with respect to σ in [13].

We have the following result.

Theorem 2.3. *Suppose the hypothesis* $(H1)$ *and the following hypothesis are satisfied:*

$$(H3) \qquad G_1(x)>0, \quad G_1'(x)>0, \qquad x\in(0,\nu).$$

Let

$$v(h)=\frac{J_2(h)}{J_1(h)} \quad \textit{and} \quad \tau(x)=\frac{G_2(x)}{G_1(x)}.$$

Then $\tau'(x)>0$ *(resp.* <0*) in* $(0,\nu)$ *implies* $v'(h)<0$ *(resp.* >0*).*

Remark 2.4. When $g_1(0) = 0$, $\frac{g_1(x)}{\Psi'(x)}$ is well defined at $x = 0$, thus $G_1(0) = 0$. Then we only need to verify the condition $G_1'(x) > 0$, which will implies that $G_(x) > 0$.

To state another criterion, we need some new notations. In some problems, the Hamiltonian function has the form

$$\tilde{H}(x, y) = \frac{\Phi(x)y^2}{2} + \Psi(x), \tag{6}$$

where $\Psi(x), \Phi(x) \in C^2$ and $\Phi(x) > 0$. We then consider the following ratio of two integrals

$$w(h) = \frac{\int_{\tilde{\Gamma}_h} f_2(x)y\mathrm{d}x}{\int_{\tilde{\Gamma}_h} f_1(x)y\mathrm{d}x},$$

where $\tilde{\Gamma}_h$ is the compact component $\{\tilde{H}(x, y) = h\}$, and $f_1(x), f_2(x)$ are functions of class C^1.

By using the transformation $\tilde{x} = x, \tilde{y} = \sqrt{\Phi(x)}y$, and the results above, we can easily obtain the following.

Theorem 2.5. *Let $\tilde{H}(x, y)$ be as in (6). Suppose that $\Psi(x)$ and $f_1(x)$ verify the hypotheses $(H1)$ and $(H2)$. Let*

$$\zeta(x) = \frac{\int_{\sigma(x)}^{x} \frac{f_2(t)}{\sqrt{\Phi(t)}}\mathrm{d}t}{\int_{\sigma(x)}^{x} \frac{f_1(t)}{\sqrt{\Phi(t)}}\mathrm{d}t}. \tag{7}$$

Then $\zeta'(x) > 0$ (resp. < 0) in $(0, \nu)$ implies $w'(h) < 0$ (resp. > 0).

We now give the proofs of our results. Let us first prove Theorem 2.3.

Proof of Theorem 2.3. When $x \in (0, \nu)$, $\sigma(x) \in (\mu, 0)$ and verifies $\Psi(x) = \Psi(\sigma(x))$. Hence $\frac{\mathrm{d}\sigma(x)}{\mathrm{d}x} = \frac{\Psi'(x)}{\Psi'(\sigma(x))}$.

Let $\mu(h), \nu(h)$ be the intersection of the curve Γ_h with the x-axis, then $\mu < \mu(h) \le \nu(h) < \nu$. Then for $k = 1, 2$,

$$\begin{aligned} J_k(h) &= 2\int_{\mu(h)}^{\nu(h)} \frac{g_k(x)}{y(x; h)}\mathrm{d}x \\ &= 2\int_{0}^{\nu(h)} \frac{g_k(x) - g_k(\sigma(x))\frac{\Psi'(x)}{\Psi'(\sigma(x))}}{y(x; h)}\mathrm{d}x \\ &= 2\int_{0}^{\nu(h)} \frac{\Psi'(x)G_k(x)}{y(x; h)}\mathrm{d}x. \end{aligned}$$

By the hypothesis $(H3)$, when $x \in (0, \nu)$, $G_1(x) > 0$, thus $J_1(h) > 0$. To show the monotonicity of the ratio of J_2 to J_1, we only need to prove that for any constant c, the integral

$$K(h) \triangleq J_2(h) - cJ_1(h) = 2\int_{0}^{\nu(h)} \frac{\Psi'(x)(G_2(x) - cG_1(x))}{y(x; h)}\mathrm{d}x$$

has at most one zero.

Without loss of generality, we suppose that $\tau'(x) > 0$, that is, $\frac{G_2(x)}{G_1(x)}$ is monotone increasing, thus $G_2(x) - cG_1(x)$ has at most one zero in $(0, \nu)$.

If $G_2(x) - cG_1(x)$ has no zero, then $K(h)$ has no zero, the proof is finished. Hence we can suppose that $G_2(x) - cG_1(x)$ has exact one zero in $(0, \nu)$. Denote this zero by c^*, then we have that $G_2(x) - cG_1(x) < 0$ for $x \in (0, c^*)$ and $G_2(x) - cG_1(x) > 0$ for $x \in (c^*, \nu)$.

Denote by $h^* = \Psi(c^*)$, then when $h \in (0, h^*]$, $\nu(h) \le c^*$, and

$$K(h) = 2\int_0^{\nu(h)} \frac{\Psi'(x)(G_2(x) - cG_1(x))}{y(x;h)}\mathrm{d}x < 0,$$

that is, $K(h)$ has no zero in $(0, h^*]$.

When $h > h^*$, $\nu(h) > c^*$,

$$\begin{aligned} K(h) &= 2\int_0^{c^*} \frac{\Psi'(x)(G_2(x) - cG_1(x))}{y(x;h)}\mathrm{d}x + 2\int_{c^*}^{\nu(h)} \frac{\Psi'(x)(G_2(x) - cG_1(x))}{y(x;h)}\mathrm{d}x \\ &= 2\int_0^{c^*} \frac{\Psi'(x)(G_2(x) - cG_1(x))}{y(x;h)}\mathrm{d}x + 2\int_0^{y(c^*)} (G_2(x(y;h)) - cG_1(x(y;h)))\mathrm{d}y, \end{aligned}$$

where $y(c^*) > 0$ satisfies that $\frac{y^2(c^*)}{2} + \Psi(c^*) = h$.

From now on, to simplify the denotation, we will denote x and y instead of $x(y;h)$ respectively. Then

$$\begin{aligned} K'(h) &= 2\int_0^{c^*} -\frac{\Psi'(x)(G_2(x) - cG_1(x))}{y^2}\frac{\partial y}{\partial h}\mathrm{d}x + 2(G_2(c^*) - cG_1(c^*))\frac{\mathrm{d}y(c^*)}{\mathrm{d}h} \\ &\quad +2\int_0^{y(c^*)} (G_2'(x) - cG_1'(x))\frac{\partial x}{\partial h}\mathrm{d}y \\ &= 2\int_0^{c^*} -\frac{\Psi'(x)(G_2(x) - cG_1(x))}{y^3}\mathrm{d}x + 2\int_0^{y(c^*)} \frac{G_2'(x) - cG_1'(x)}{\Psi'(x)}\mathrm{d}y \\ &> 2\int_0^{y(c^*)} \frac{G_2'(x) - cG_1'(x)}{\Psi'(x)}\mathrm{d}y \\ &= 2\int_{c^*}^{\nu(h)} \frac{G_2'(x) - cG_1'(x)}{y}\mathrm{d}x. \end{aligned}$$

When $x > c^*$, $G_2(x) > cG_1(x)$. On the other hand, $\tau'(x) = (\frac{G_2}{G_1})' > 0$, which implies that

$$G_2'(x) - cG_1'(x) > \frac{G_2(x)G_1'(x)}{G_1(x)} - cG_1'(x) > cG_1'(x) - cG_1'(x) = 0,$$

$$K'(h) > 2\int_{c^*}^{\nu(h)} \frac{G_2'(x) - cG_1'(x)}{y}\mathrm{d}x > 0.$$

Immediately, $K(h)$ has at most one zero in (h^*, h_s).

So $K(h)$ has at most one zero in $(0, h_s)$. □

To prove Theorem 2.1, we need the following lemma proved in [13].

Lemma 2.6. *Let Γ_h be an oval inside the level curve $\{(x,y) : \Psi(x) + \frac{y^2}{2} = h\}$. If a function $g(x)$ such that $\frac{g(x)}{\Psi'(x)}$ is analytic at $x = 0$, then*

$$\int\limits_{\Gamma_h} \frac{g(x)}{y} \mathrm{d}x = \int\limits_{\Gamma_h} f(x) y \mathrm{d}x,$$

where $f(x) = (\frac{g(x)}{\Psi'(x)})'$, or equivalently $g(x) = \Psi'(x) \int_0^x f(t)\mathrm{d}t$.

Proof of Theorem 2.1. In Lemma 2.6, for $k = 1, 2$, set $f(x) = f_k(x)$, then we have

$$I_k(h) = \int\limits_{\Gamma_h} \frac{g_k(x)}{y} \mathrm{d}x,$$

where $g_k(x) = \Psi'(x) \int\limits_0^x f_k(t)\mathrm{d}t$. Then

$$G_k(x) = \frac{g_k(x)}{\Psi'(x)} - \frac{g_k(\sigma(x))}{\Psi'(\sigma(x))} = \int\limits_0^x f_k(t)\mathrm{d}t - \int\limits_0^{\sigma(x)} f_k(t)\mathrm{d}t = \int\limits_{\sigma(x)}^x f_k(t)\mathrm{d}t.$$

By hypothesis $(H2')$, $\frac{f_1(x)}{\Psi'(x)} - \frac{f_1(\sigma(x))}{\Psi'(\sigma(x))} > 0$, thus

$$G_1'(x) = f_1(x) - f_1(\sigma(x)) \frac{\Psi'(x)}{\Psi'(\sigma(x))} > 0.$$

Together with that $G_1(0) = 0$, we have that $G_1(x) > 0$ for $x \in (0, \nu)$. The hypothesis $(H3)$ in Theorem 2.3 is satisfied.

At last, by Theorem 2.3, if $\bar{\xi}(x) = \frac{G_2(x)}{G_1(x)}$ is monotone, then the ratio of $I_2(h)$ to $I_1(h)$ is monotone. □

Proof of Proposition 2.2. Denote by $F_k(x) = f_k(x) - f_k(\sigma(x)) \frac{\Psi'(x)}{\Psi'(\sigma(x))}$ for $k = 1, 2$. Then ξ and $\bar{\xi}$, defined in (4) and (5) respectively, can be written as

$$\xi(x) = \frac{F_2(x)}{F_1(x)}, \quad \bar{\xi}(x) = \frac{\int_{\sigma(x)}^x f_2(t)\mathrm{d}t}{\int_{\sigma(x)}^x f_1(t)\mathrm{d}t} = \frac{\int_0^x F_2(t)\mathrm{d}t}{\int_0^x F_1(t)\mathrm{d}t}.$$

By the mean value theorem for integrals, there exists $\delta \in (a, x)$ so that $\bar{\xi}(x) = \frac{F_2(\delta)}{F_1(\delta)} = \xi(\delta)$, thus

$$\bar{\xi}'(x) = \frac{F_1(x)}{\int_0^x F_1(t)\mathrm{d}t}(\xi(x) - \bar{\xi}(x)) = \frac{F_1(x)}{\int_0^x F_1(t)\mathrm{d}t}(\xi(x) - \xi(\delta)).$$

Since the hypotheses $(H1)$ and $(H2)$ are satisfied, $F_1(x)$ and $\int_a^x F_1(t)\mathrm{d}t$ are both positive, obviously, $\xi'(x) > 0$ (resp. < 0) implies $\bar{\xi}'(x) > 0$ (resp. < 0). □

At the end of this section, we introduce two lemmas about the monotonicity of $S(x) \triangleq x + \sigma(x)$ and $T(x) \triangleq (x - a)(\sigma(x) - a)$, which will be used in the next section. For convenience of application, we do not suppose that $a = 0$ any more, since the center is not always at $(0, 0)$. The first lemma has been proved in [17], but for convenience of readers, we still prove it here, since the denotations in the two papers are different.

Lemma 2.7. *Let notations as above. Assume that $H(x, y)$ has the form (2) and the hypothesis (H1) is satisfied. We further assume that the function $\Psi(x)$ has the following asymptotic relation*

$$(H4) \qquad \Psi'(x) \sim (x-a)^{2k-1}, \quad as \quad x \to a,$$

where k is a natural number. Then $S'(x) > 0$ (resp. < 0) for $x \in (a, \nu)$ if $\eta(x) > 0$ (resp. < 0) for all $x \in (\mu, \nu) \setminus \{a\}$, where

$$\eta(x) = (x-a)\Big((2k-1)\Big(\Psi'(x)\Big)^{\frac{2k-2}{2k-1}} - \Psi''(x)\Big). \tag{8}$$

Proof. Define a new function

$$\psi(x) \triangleq 2k\Psi(x) - \Big(\Psi'(x)\Big)^{\frac{2k}{2k-1}}.$$

By direct calculation, we have

$$\psi'(x) = \frac{2k}{2k-1}\Big(\Psi'(x)\Big)^{\frac{1}{2k-1}}\Big((2k-1)\Psi'(x)^{\frac{2k-2}{2k-1}} - \Psi''(x)\Big).$$

By the hypothesis (H1), $\Psi'(x)$ has the same sign with $(x-a)$. Thus, $\psi'(x)$ has the same sign with $(x - a)\Big((2k-1)\Psi'(x)^{\frac{2k-2}{2k-1}} - \Psi''(x)\Big)$, which is $\eta(x)$. Without loss of generality, we assume that $\eta(x) > 0$ for $x \in (\mu, \nu) \setminus \{a\}$. We next verify $S'(x) > 0$ for $x \in (a, \nu)$.

When $\eta(x) > 0$ for $x \in (\mu, \nu) \setminus \{a\}$, we have that $\psi'(x) > 0$ as $x \in (\mu, \nu) \setminus \{a\}$. Since $x > a > \sigma(x)$, we obtain that $\psi(x) > \psi(\sigma(x))$, that is,

$$2k\Psi(x) - \Psi'(x)^{\frac{2k}{2k-1}} > 2k\Psi(\sigma(x)) - \Psi'(\sigma(x))^{\frac{2k}{2k-1}}.$$

Note that $\Psi(x) = \Psi(\sigma(x))$. Hence, $\Psi'(x)^{\frac{2k}{2k-1}} < \Psi'(\sigma(x))^{\frac{2k}{2k-1}}$. This implies that

$$|\Psi'(x)| < |\Psi'(\sigma(x))| \quad \text{and} \quad S'(x) = 1 + \tfrac{\Psi'(x)}{\Psi'(\sigma(x))} > 0. \qquad \square$$

Lemma 2.8. *Let notations be as above. Assume that $H(x, y)$ has the form (2) and the hypothesis (H1) is satisfied, then $T(x)$ is monotone decreasing in (a, ν).*

Proof. By the hypothesis (H1), $\Psi'(x)$ has the same sign with $(x-a)$, then a direct calculation shows

$$\frac{\mathrm{d}T(x)}{\mathrm{d}x} = (\sigma(x) - a) + (x-a)\frac{\Psi'(x)}{\Psi'(\sigma(x))} < 0,$$

which accomplishes the proof. $\square$

3. Applications

We now give several applications of our results of the above section.

Example 1. We first give an application of Theorem 2.5. Consider

$$\tilde{H}(x,y) = xy^2 - \frac{x}{4} + \frac{x^3}{3} \quad \text{and} \quad w(h) = \frac{\int_{\tilde{\Gamma}_h} xy\mathrm{d}x}{\int_{\tilde{\Gamma}_h} y\mathrm{d}x}.$$

The monotonicity of $w(h)$ was firstly established in [8], then reproved in [16] by verifying their criterion, now we will give a simpler proof by using Theorem 2.5.

This system has two centers $(\pm\frac{1}{2}, 0)$. By symmetry, we only need to deal with the center $(\frac{1}{2}, 0)$. In this case

$$a = \frac{1}{2},\ \Phi(x) = 2x,\ \Psi(x) = -\frac{x}{4} + \frac{x^3}{3},\ f_1(x) = 1,\ f_2(x) = x.$$

The hypotheses $(H1)$ and $(H2)$ are both satisfied, and the involution σ satisfies the condition $(x-\frac{1}{2})(\sigma(x)-\frac{1}{2}) < 0$. The criterion function in Theorem 2.5 is

$$\begin{aligned}\zeta(x) &= \frac{\int_{\sigma(x)}^{x} \frac{t}{\sqrt{2t}}\mathrm{d}t}{\int_{\sigma(x)}^{x} \frac{1}{\sqrt{2t}}\mathrm{d}t} = \frac{x\sqrt{x} - \sigma(x)\sqrt{\sigma(x)}}{3(\sqrt{x} - \sqrt{\sigma(x)})} \\ &= \frac{x + \sigma(x) + \sqrt{x\sigma(x)}}{3} = \frac{S(x) + \sqrt{T(x)}}{3}.\end{aligned}$$

Recall that $S(x) = x + \sigma(x), T(x) = x\sigma(x)$.

From $\Psi(x) = \Psi(\sigma(x))$, we obtain

$$\sigma^2(x) + x\sigma(x) + x^2 = \frac{3}{4}, \quad T(x) = S^2(x) - \frac{3}{4}.$$

Thus

$$\zeta(x) = S(x) + \sqrt{S^2(x) - \frac{3}{4}},$$

$$\zeta'(x) = \Big(1 + \frac{S(x)}{\sqrt{S^2(x) - \frac{3}{4}}}\Big)S'(x) = \Big(1 + \frac{S(x)}{\sqrt{S^2(x) - \frac{3}{4}}}\Big)\frac{x^2 + \sigma^2(x) - \frac{1}{2}}{\sigma^2(x) - \frac{1}{4}}.$$

For $x > \frac{1}{2}$, one has $0 < \sigma(x) < \frac{1}{2}$. Hence

$$\frac{3}{2}(x^2 + \sigma^2(x)) > x^2 + \sigma^2(x) + x\sigma(x) = \frac{3}{4},$$

$$x^2 + \sigma^2(x) - \frac{1}{2} > 0,$$

which implies that $\zeta'(x) < 0$ and the function $w(h)$ is monotone.

In [16], it is needed to verify the monotonicity of their criterion function

$$\frac{x\sqrt{\sigma(x)}(\sigma^2(x) - \frac{1}{4}) - \sigma(x)\sqrt{x}(x^2 - \frac{1}{4})}{\sqrt{\sigma(x)}(\sigma^2(x) - \frac{1}{4}) - \sqrt{x}(x^2 - \frac{1}{4})},$$

which is much more complicate than the criterion function $\zeta(x)$ here.

Example 2. We now come back to the general case where $H(x,y) = \frac{y^2}{2} + \Psi(x)$, and we consider the monotonicity of the ratio of the integral $\int\limits_{\Gamma_h} x^n y\mathrm{d}x$ to the integral $\int\limits_{\Gamma_h} y\mathrm{d}x$, where n is a natural number.

Proposition 3.1. *Let $a = 0$ and suppose that the hypotheses $(H1)$ is satisfied. We consider the period annuls around $(0,0)$.*

(i) *If $S(x) = x + \sigma(x)$ is monotone increasing in $(0,\nu)$, then for each natural number n, $\dfrac{\int_{\Gamma_h} x^n y\mathrm{d}x}{\int_{\Gamma_h} y\mathrm{d}x}$ is monotone increasing.*

(ii) *If $S(x) = x + \sigma(x)$ is monotone decreasing in $(0,\nu)$, then for each odd n, $\dfrac{\int_{\Gamma_h} x^n y\mathrm{d}x}{\int_{\Gamma_h} y\mathrm{d}x}$ is monotone decreasing, while for each even n, $\dfrac{\int_{\Gamma_h} x^n y\mathrm{d}x}{\int_{\Gamma_h} y\mathrm{d}x}$ is monotone increasing.*

Proof. Without loss of generality, we assume that $S(x) = x + \sigma(x)$ is monotone increasing, thus $S(x) > 0$ for $x \in (0,\nu)$.

Let $I_1(h) = \int_{\Gamma_h} y\mathrm{d}x$ and $I_2(h) = \int_{\Gamma_h} x^n y\mathrm{d}x$, then $f_k(x)$ defined in Theorem 2.1 are

$$f_1(x) = 1, \qquad f_2(x) = x^n.$$

Obviously, the hypothesis $(H1)$ and $(H2)$ are satisfied. Denote by

$$\xi_n(x) = \frac{(n+1)\int_{\sigma(x)}^{x} f_2(t)\mathrm{d}t}{\int_{\sigma(x)}^{x} f_1(t)\mathrm{d}t} = x^n + x^{n-1}\sigma(x) + \cdots + \sigma^n(x).$$

By Theorem 2.1, we only need to show that $\xi_n'(x) > 0$ for $x \in (0,\nu)$.

If $n = 1$, then $\xi_1(x) = S(x)$, it follows that $\xi_1'(x) > 0$.

If $n = 2$, then

$$\xi_2(x) = x^2 + \sigma^2(x) = S^2(x) - 2T(x),$$

where $T(x) = x\sigma(x) < 0$ is decreasing, thus

$$\xi_2'(x) = 2S(x)S'(x) - 2T'(x) > 0.$$

Suppose that when $n = k, k+1$, $\xi_n'(x) > 0$. With the fact that $\xi_n(0) = 0$, we have $\xi_n(x) > 0$. Then when $n = k+2$,

$$\xi_{k+2}(x) = S(x)\xi_{k+1}(x) - T(x)\xi_k(x),$$
$$\xi_{k+2}' = S'(x)\xi_{k+1}(x) + S(x)\xi_{k+1}'(x) - T'(x)\xi_k(x) - T(x)\xi_k'(x) > 0.$$

By induction, for each n, $\xi_n'(x) > 0$. The proof is finished. □

For the examples satisfying that $S(x)$ is monotone, we recommend the readers the paper [17]. Up to our knowledge, the above result cannot be obtained by existing method.

Example 3. We now consider the monotonicity of the ratios of the integrals $\int_{\Gamma_h} \frac{x}{y}\mathrm{d}x$ to $\int_{\Gamma_h} \frac{1}{y}\mathrm{d}x$. It concerns the seventh Arnold's problem proposed in [1]: for complete hyperelliptic integrals of the first kind

$$J(h)=\int_{\Gamma_h}\frac{\alpha_0+\alpha_1x+\cdots+\alpha_{g-1}x^{g-1}}{y}\mathrm{d}x,\quad H(x,y)=y^2+\Psi(x),$$

where $\deg\Psi=2g+1$ or $2g+2$ with $g\geq 2$, α_i are real constants and $i=0,1,\ldots,g-1$, is this g-dimensional family of $J(h)$ a Chebyshev family in the open interval? Here Chebyshev family means that the number of the isolated zeros of $J(h)$ is no more than $g-1$.

In [12], Gavrilov and Iliev systematically studied this problem and obtained that for any $g>1$, this real vector space of $J(h)$ is not Chebyshev *in general*. They also showed that when $g=2$, exceptional families of ovals exist, such that the corresponding vector space is Chebyshev. In fact, they proved that when $g=2$ and $\deg\Psi=5$, there exist exceptional families of ovals Γ_h such that every Abelian integral of the form $J(h)=\int_{\Gamma_h}\frac{\alpha_0+\alpha_1x}{y}\mathrm{d}x$ is Chebyshev.

Then in [20], Wang and others considered the cases

$$\Psi'(x)=x^3(x-1),\ x(x-1)^3,\ x(x-\frac{2}{5})(x-1)^2,$$

which are all not exceptional families, and prove that the associated integrals $J(h)=\int_{\Gamma_h}\frac{\alpha_0+\alpha_1x}{y}\mathrm{d}x$ are all Chebyshev. By using similar method, in [3], Asheghi and Bakhshalizadeh obtain the Chebyshev property for $\Psi'(x)=x^3(x-1)^3,\ x^5(x-1)$. Here we will generalize the results in [20] and our proofs are much simpler.

Consider the system

$$\dot{x}=-y,\quad \dot{y}=x^{b-1}(x-1),\tag{9}$$

where $b>1$. Its first integral has the form

$$H(x,y)=\frac{y^2}{2}+\Psi(x),\quad \Psi(x)=\frac{x^{b+1}}{b+1}-\frac{x^b}{b}+\frac{1}{b(b+1)}.$$

The integral that we consider is

$$J(h)=\int_{\Gamma_h}\frac{\alpha_0+\alpha_1x}{y}\mathrm{d}x.$$

System (9) has a unique center $(1,0)$ and the period annulus around $(1,0)$ is bounded by the homoclinic loop $\{(x,y)\mid H(x,y)=\frac{1}{b(b+1)}\}$, which intersects x-axis at $(0,0)$ and $(\frac{1+b}{b},0)$, that is, $\mu=0,\nu=\frac{1+b}{b}$. The cases $b=4,6$ have been solved in [20,3] respectively, here we have a more general conclusion.

Proposition 3.2. *The integral $J(h)$ has at most one zero in $(0,\frac{1}{b(b+1)})$.*

Proof. Now $a=1$, it is easy to check that the hypothesis $(H1)$ and $(H4)$ are satisfied if we set $k=1$. Thus $\eta(x)$ defined in (8) has the form

$$\eta(x)=(x-1)(1-bx^{b-1}+(b-1)x^{b-2}).$$

Let

$$l(x)=1-bx^{b-1}+(b-1)x^{b-2}.$$

Obviously $l'(x)$ has at most one zero in $(0,+\infty)$, so $l(x)$ has at most two zeros in $(0,+\infty)$.

Since $b > 1$, $l(x) \sim -bx^{b-1}$ when $x \to +\infty$. So for sufficiently large x, $l(x) < 0$. On the other hand, when $x \to 0^+$, $l(x) > -bx^{b-1} + (b-1)x^{b-2} \sim (b-1)x^{b-2}$. Thus, $l(x) > 0$ for $0 < x \ll 1$. $l(x)$ has the opposite signs at the two endpoints of the interval $(0, +\infty)$. Hence, the number of zeros of $l(x)$ must be odd in the interval $(0, +\infty)$. But $l(x)$ has at most two zeros in $(0, +\infty)$. This implies that $l(x)$ has exactly one zero in $(0, +\infty)$.

Notice that $l(1) = 0$. Consequently, $x = 1$ is the unique zero of $l(x)$ in $(0, +\infty)$. And, $\eta(x) = (x-1)l(x) < 0$ for $x \in (0, +\infty) \setminus \{1\}$. By Lemma 2.7, $S(x) = x + \sigma(x)$ is monotone decreasing in $(1, \frac{1+b}{b})$, which implies that

$$\frac{1+b}{b} < S(x) < 2.$$

For $k = 1, 2$, denote by $J_k(h) = \int_{\Gamma_h} \frac{g_k(x)}{y} \mathrm{d}x$, where $g_1(x) = 1 - x$ and $g_2(x) = 1$. Then $J(h) = (\alpha_0 + \alpha_1)J_2(h) - \alpha_1 J_1(h)$. To finish the proof, we only need to show that the ratio of J_2 to J_1 is monotone. $G_k(x)$ and $\tau(x)$ defined in Theorem 2.3 have the forms

$$\begin{aligned} G_1(x) &= \frac{1}{\sigma^{b-1}(x)} - \frac{1}{x^{b-1}}, \\ G_2(x) &= \frac{1}{x^{b-1}(x-1)} - \frac{1}{\sigma^{b-1}(x)(\sigma(x)-1)}, \\ \tau(x) &= \frac{G_2(x)}{G_1(x)}. \end{aligned}$$

Recall that x and $\sigma(x)$ in the above three formulas satisfy that

$$0 < \sigma(x) < 1 < x < \frac{1+b}{b}$$

and $\Psi(x) = \Psi(\sigma(x))$, thus

$$\frac{\mathrm{d}\sigma(x)}{\mathrm{d}x} = -\frac{\Psi'(x)}{\Psi'(\sigma(x))} = -\frac{\Psi'(x)(\Psi(\sigma(x)) - \frac{1}{b(b+1)})}{\Psi'(\sigma(x))(\Psi(x) - \frac{1}{b(b+1)})} = \frac{(x-1)(b\sigma(x) - b - 1)\sigma(x)}{x(\sigma(x)-1)(bx-b-1)}.$$

Since $b > 1$, obviously $G_1(x) > 0$ and

$$G_1'(x) = \frac{\partial G_1}{\partial \sigma(x)} \frac{\mathrm{d}\sigma(x)}{\mathrm{d}x} + \frac{\partial G_1}{\partial x} = -\frac{(b-1)(x-1)(b\sigma(x)-b-1)}{(\sigma(x)-1)(bx-b-1)x\sigma^{b-3}(x)} + \frac{b-1}{x^{b-2}} > 0.$$

The hypothesis $(H3)$ is satisfied.

Rewrite $\tau(x)$ as $\tau(x) = \tau_1(x)\tau_2(x)$ where

$$\tau_1(x) = -\frac{1}{(x-1)(\sigma(x)-1)}, \qquad \tau_2(x) = \frac{b(x-1)(\sigma(x)-1)+1}{b(x+\sigma(x)) - (b+1)}.$$

By Lemma 2.8, $T(x) = (x-1)(\sigma(x)-1)$ is decreasing. On the other hand, when $x \to 1$, $\sigma(x) \to 1$, $T(x) \to 0$; and $x \to \frac{b+1}{b}$, $\sigma(x) \to 0$, $T(x) \to -\frac{1}{b}$, hence

$$-\frac{1}{b} < T(x) < 0.$$

Immediately, we have that $\tau_1(x), \tau_2(x) > 0$ and $\tau_1'(x) < 0$. Furthermore,

$$\tau_2'(x) = \frac{\partial \tau_2}{\partial \sigma(x)} \frac{\mathrm{d}\sigma(x)}{\mathrm{d}x} + \frac{\partial \tau_2}{\partial x} = \frac{b\sigma(x)(b\sigma(x)-b-1)(x+\sigma(x)-2)}{(\sigma(x)-1)(b(x+\sigma(x))-(b+1))^2} < 0.$$

Hence

$$\tau'(x) = \tau_1'(x)\tau_2(x) + \tau_1(x)\tau_2'(x) < 0,$$

which implies that the ratio of J_2 to J_1 is monotone. $\square$

Example 4. Consider the system

$$\dot{x} = -y, \quad \dot{y} = x^{2m-1}(1-x), \tag{10}$$

where m is a natural number. Its first integral has the form

$$H(x,y) = \frac{y^2}{2} + \Psi(x), \quad \Psi(x) = \frac{x^{2m}}{2m} - \frac{x^{2m+1}}{2m+1}.$$

The Abelian integral that we consider is

$$J(h) = \int_{\Gamma_h} \frac{\alpha_0 + \alpha_1 x}{y} \mathrm{d}x.$$

System (10) has a unique degenerate center $(0,0)$ and the period annulus around $(0,0)$ is bounded by the homoclinic loop $\{(x,y)|H(x,y) = \frac{1}{2m(2m+1)}\}$, which intersects x-axis at $(1,0)$ and $(\mu,0)$, where $\mu < 0$ is the only negative root of $\Psi(x) = \frac{1}{2m(2m+1)}$. The cases $m = 2,3$ have been solved in [20,3]. Here we generalize their results by applying our method in the above section.

Proposition 3.3. *The integral $J(h)$ has at most one zero in $(0, \frac{1}{2m(2m+1)})$.*

Proof. Let $I(h) = hJ(h)$ and we will show that $I(h)$ has at most one zero in $(0, \frac{1}{2m(2m+1)})$.

For $k = 1,2$, denote by $I_k(h) = h\int_{\Gamma_h} \frac{x^{k-1}}{y}\mathrm{d}x$. By using Lemma 2.6,

$$I_1(h) = \int_{\Gamma_h} \frac{\frac{y^2}{2} + \Psi(x)}{y}\mathrm{d}x = \int_{\Gamma_h} \Big(\frac{1}{2} + (\frac{\Psi(x)}{\Psi'(x)})'\Big) y\mathrm{d}x = \int_{\Gamma_h} f_1(x) y\mathrm{d}x,$$

$$I_2(h) = \int_{\Gamma_h} \frac{x(\frac{y^2}{2} + \Psi(x))}{y}\mathrm{d}x = \int_{\Gamma_h} (\frac{x}{2} + (\frac{x\Psi(x)}{\Psi'(x)})') y\mathrm{d}x = \int_{\Gamma_h} f_2(x) y\mathrm{d}x,$$

where

$$f_1(x) = \frac{m(2m+3)x^2 - 2m(2m+3)x + (m+1)(2m+1)}{2((x-1)^2(2m+1)m)},$$

$$f_2(x) = \frac{x(m(2m+5)x^2 + (-4m^2-10m-1)x + (m+2)(2m+1))}{2((x-1)^2(2m+1)m)}.$$

Since the discriminant of $m(2m+3)x^2 - 2m(2m+3)x + (m+1)(2m+1)$ is negative, $m(2m+3)x^2 - 2m(2m+3)x + (m+1)(2m+1)$ is positive for $x \in \mathbb{R}$, $f_1(x) > 0$, for all $x \in (\mu, 1)$. The hypothesis $(H2)$ is satisfied. $\bar{\xi}(x)$, defined in Theorem 2.1, is

$$\bar{\xi}(x) = \frac{S(x)}{2}(1 + \frac{1 + 2mU(x)}{1 + U(x)(2m^2+3m)}) - \frac{T(x)}{1 + U(x)(2m^2+3m)},$$

where

$$S(x) = x + \sigma(x), \quad T(x) = x\sigma(x), \quad U(x) = (x-1)(\sigma(x)-1).$$

Similar to Example 3, one can show that $S'(x) > 0$, $T'(x) < 0$ for $x \in (0,1)$. On the other hand, when $x = 0$, $\sigma(x) = 0$, $S(0) = T(0) = 0$, hence $S(x) > 0, T(x) < 0$ for $x \in (0,1)$. Furthermore, $U(x) > 0$ and $U'(x) = T'(x) - S'(x) < 0$.

At last,

$$\Big(\frac{1+2mU(x)}{1+U(x)(2m^2+3m)}\Big)' = -\frac{m(2m+1)U'(x)}{(1+U(x)(2m^2+3m))^2} > 0,$$
$$\Big(\frac{T(x)}{1+U(x)(2m^2+3m)}\Big)' = \frac{T'(x)}{1+U(x)(2m^2+3m)} - \frac{(2m^2+3m)T(x)U'(x)}{(1+U(x)(2m^2+3m))^2} < 0,$$

so $\bar{\xi}'(x) > 0$, the ratio of the integrals of I_2 to I_1 is monotone, $I(h) = hJ(h)$ has at most one zero in $(0, \frac{1}{2m(2m+1)})$. $\square$

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