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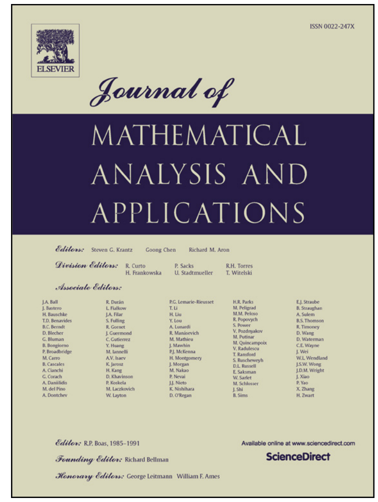

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NEW INTEGRAL REPRESENTATIONS FOR THE FOX-WRIGHT FUNCTIONS AND ITS APPLICATIONS

KHALED MEHREZ

Abstract. Our aim in this paper is to derive several new integral representations for the Fox–Wright functions. In particular, we give new Laplace and Stieltjes transforms for this special function under some restrictions on parameters. From the positivity conditions on the weight in these representations, we found sufficient conditions to be imposed on the parameters of the Fox–Wright functions which allow us to conclude that it is completely monotonic. As applications, we derive a class of functions that are related to the Fox H-functions and are positive definite. Moreover, we extended the Luke's inequalities and we establish new Turán type inequalities for the Fox-Wright function. Finally, by appealing to each of the Luke's inequalities, two sets of two–sided bounding inequalities for the generalized Mathieu's type series are proved.

1. Introduction

In this paper, we use the Fox-Wright generalized hypergeometric function ${}_p\Psi_q[.]$ with p numerator parameters $\alpha_1, ..., \alpha_p$ and q denominator parameters $\beta_1, ..., \beta_q$, which are defined by [39, p. 4, Eq. (2.4)]

$$(1.1)\qquad {}_p\Psi_q\left[^{(\alpha_1,A_1),...,(\alpha_p,A_p)}_{(\beta_1,B_1),...,(\beta_q,B_q)}\Big|z\right] = {}_p\Psi_q\left[^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|z\right] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_l+kA_l)}{\prod_{j=1}^{q}\Gamma(\beta_l+kB_l)}\frac{z^k}{k!},$$

$$\left(\alpha_i, \beta_j \in \mathbb{C}, \text{ and } A_i, B_j \in \mathbb{R}^+ \ (i = 1, ..., p, j = 1, ..., q)\right),$$

where, as usual,

$$\mathbb{N} = \{1, 2, 3, ...\}, \ \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

$\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ stand for the sets of real, positive real and complex numbers, respectively. The convergence conditions and convergence radius of the series at the right-hand side of (1.1) immediately follow from the known asymptotic of the Euler Gamma–function. The defining series in (1.1) converges in the whole complex z-plane when

$$(1.2)\qquad \Delta = \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1.$$

If $\Delta = -1$, then the series in (1.1) converges for $|z| < \rho$, and $|z| = \rho$ under the condition $\Re(\mu) > \frac{1}{2}$, (see [17] for details), where

$$(1.3)\qquad \rho = \left(\prod_{i=1}^{p} A_i^{-A_i}\right)\left(\prod_{j=1}^{q} B_j^{B_j}\right), \quad \mu = \sum_{j=1}^{q}\beta_j - \sum_{k=1}^{p}\alpha_k + \frac{p-q}{2}$$

If, in the definition (1.1), we set

$$A_1 = ... = A_p = 1 \quad \text{and} \quad B_1 = ... = B_q = 1,$$

we get the relatively more familiar generalized hypergeometric function ${}_pF_q[.]$ given by

$$(1.4)\qquad {}_pF_q\left[^{\alpha_1,...,\alpha_p}_{\beta_1,...,\beta_q}\Big|z\right] = \frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\,{}_p\Psi_q\left[^{(\alpha_1,1),...,(\alpha_p,1)}_{(\beta_1,1),...,(\beta_q,1)}\Big|z\right]$$

The Fox-Wright function appeared recently as a fundamental solutions of diffusion-like equations containing fractional derivatives in time of order less than 1. In the physical literature, such equations are in general referred to as fractional sub-diffusion equations, since they are used as model equations

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for the kinetic description of anomalous diffusion processes of slow type, characterized by a sub-linear growth of the variance (the mean squared displacement) with time (see for example [22]).

The H-function was introduced by Fox in [11] as a generalized hypergeometric function defined by an integral representation in terms of the Mellin-Barnes contour integral

$$H_{q,p}^{m,n}\left(z\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right) = H_{q,p}^{m,n}\left(z\Big|_{(A_1,\alpha_1),\dots,(A_p,\alpha_p)}^{(B_1,\beta_1),\dots,(B_q,\beta_q)}\right) = \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(A_j s+\alpha_j)\prod_{j=1}^{n}\Gamma(1-\beta_j-B_j s)}{\prod_{j=n+1}^{q}\Gamma(B_k s+\beta_k)\prod_{j=m+1}^{p}\Gamma(1-\alpha_j-A_j s)} z^{-s}ds. \tag{1.5}$$

Here $\mathcal{L}$ is a suitable contour in $\mathbb{C}$ and $z^{-s}=\exp(-s\log|z|+i\arg(z))$, where $\log|z|$ represents the natural logarithm of $|z|$ and $\arg(z)$ is not necessarily the principal value.

The definition of the H-function is still valid when the A_i's and B_j's are positive rational numbers. Therefore, the H-function contains, as special cases, all of the functions which are expressible in terms of the G–function. More importantly, it contains the Fox-Wright generalized hypergeometric function defined in (1.1), the generalized Mittag-Leffler functions, etc. For example, the function ${}_p\Psi_q[.]$ is one of these special case of H-function. By the definition (1.1) it is easily extended to the complex plane as follows [18, Eq. 1.31],

$${}_p\Psi_q\left[^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| z\right] = H_{p,q+1}^{1,q}\left(-z\Big|^{(A_p,1-\alpha_p)}_{(0,1),(B_q,1-\beta_q)}\right). \tag{1.6}$$

The representation (1.6) holds true only for positive values of the parameters A_i and B_j.

The special case for which the H-function reduces to the Meijer G-function is when $A_1=...=A_p=B_1=...=B_q=A,\ A>0$. In this case,

$$H_{q,p}^{m,n}\left(z\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right) = \frac{1}{A}G_{p,q}^{m,n}\left(z^{1/A}\Big|_{\alpha_p}^{B_q}\right). \tag{1.7}$$

Additionally, when setting $A_i=B_j=1$ in (1) (or $A=1$ in (1.7)), the H- and Fox-Wright functions turn readily into the Meijer G-function.

Each of the following definitions will be used in our investigation.

A real valued function f, defined on an interval I, is called completely monotonic on I, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x)\geq 0,\ n\in\mathbb{N}_0,\ \text{ and } x\in I. \tag{1.8}$$

The celebrated Bernstein Characterization Theorem gives a necessary and sufficient condition that the function f should be completely monotonic for $0<x<\infty$ is that

$$f(x)=\int_0^\infty e^{-xt}d\mu(t), \tag{1.9}$$

where $\mu(t)$ is non-decreasing and the integral converges for $0<x<\infty$.

A function f is said to be absolutely monotonic on an interval I, if f has derivatives of all orders and satisfies

$$f^{(n)}(x)\geq 0,\ x\in I,\ n\in\mathbb{N}_0.$$

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\log f$ satisfies

$$(-1)^n(\log f)^{(n)}(x)\geq 0,\ n\in\mathbb{N},\ \text{ and }\ x\in I. \tag{1.10}$$

In [4, Theorem 1.1] and [13, Theorem 4], it was found and verified once again that a logarithmically completely monotonic function must be completely monotonic, but not conversely.

An infinitely differentiable function $f: I\longrightarrow[0,\infty)$ is called a Bernstein function on an interval I, if f' is completely monotonic on I. The Bernstein functions on $(0,\infty)$ can be characterized by [31, Theorem 3.2] which states that a function $f:(0,\infty)\longrightarrow[0,\infty)$ is a Bernstein function if and only if it admits the representation

$$f(x)=a+bx+\int_0^\infty(1-e^{-xt})d\mu(t), \tag{1.11}$$

where $a,b\geq 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_0^\infty \min\{1,t\}d\mu(t)<\infty$. The formula (6) is called the Lévy–Khintchine representation of f. In [7, pp.161–162, Theorem 3] and [31, Proposition 5.25], it was proved that the reciprocal of a Bernstein function is logarithmically completely monotonic.

In [31, Definition 2.1], it was defined that a Stieltjes transform is a function $f:(0,\infty)\longrightarrow[0,\infty)$ which can be written in the following form:

$$f(x)=\frac{a}{x}+b+\int_0^\infty\frac{d\mu(t)}{t+x}, \tag{1.12}$$

where a,b are non-negative constant and μ is a non-negative measure on $(0,\infty)$ such that the integral $\int_0^\infty \frac{d\mu(t)}{t+1}<\infty$. In [4, Theorem 2.1] it was proved that a positive Stieltjes transform must be a logarithmically completely monotonic function on $(0,\infty)$, but not conversely. We define S to be the class of functions representable by (1.12). Functions representable in one of the forms

$$f(z)=a+\int_0^\infty\frac{d\mu(t)}{(z+t)^\alpha}=\frac{b}{z^\alpha}+\int_0^\infty\frac{d\nu(t)}{(1+zt)^\alpha}, \tag{1.13}$$

are known as generalized Stieltjes functions of order α. Here, $\alpha>0$, μ and ν are non-negative measures supported on $[0,\infty), a,b\geq 0$ are constants and we always choose the principal branch of the power function. The measures μ and ν are assumed to produce convergent integrals (1.13) for each $z\in\mathbb{C}\setminus(-\infty,0]$. We denote by S_α to be the class of functions representable by (1.13).

The present sequel to some of the aforementioned investigations is organized as follows. In Section 2, we derive the Laplace integral representations for the Fox H-function $H_{q,p}^{p,0}$ and for the Fox-Wright function ${}_p\Psi_q$. We give a numbers of consequences, some monotonicity and log-convexity properties for the Fox-Wright function are researched, and an Turán type inequality are proved. In Section 3, we find the generalized Stieltjes transform representation of the Fox-Wright function ${}_{p+1}\Psi_p$. As applications, we present some class of completely monotonic functions related to the Fox-Wright function. In addition, we deduce new Turán type inequalities for this special function. In Section 4, some further applications are proved, firstly, a class of positive definite function related to the Fox H-function are given. As consequences, we find the non-negativity for a class of function involving the Fox H-function. Next, we show that the Fox-Wright function ${}_p\Psi_q[z]$ has no real zeros and all its zeros lie in the open right half plane $\Re(z)>0$. Moreover, two-sided exponential inequalities for the Fox-Wright function are given, in particular, we gave a generalization of Luke's inequalities. Finally, by appealing to each of these two-sided exponential inequalities, two sets of two-sided bounding inequalities for generalized Mathieu's type series are proved.

2. Laplace transform representation and completely monotonic functions for the Fox-Wright functions

In the first main result we will need a particular case of Fox's H-function defined by

$$H_{q,p}^{p,0}\left(z\middle|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)=\frac{1}{2i\pi}\int_{\mathcal{L}}\frac{\prod_{j=1}^p\Gamma(A_js+\alpha_j)}{\prod_{k=1}^q\Gamma(B_ks+\beta_k)}z^{-s}ds, \tag{2.14}$$

where $A_j,B_k>0$ and α_j,β_k are real. The contour $\mathcal{L}$ can be either the left loop $\mathcal{L}_-$ starting at $-\infty+i\alpha$ and ending at $-\infty+i\beta$ for some $\alpha<0<\beta$ such that all poles of the integrand lie inside the loop, or the right loop $\mathcal{L}_+$ starting $\infty+i\alpha$ at and ending $\infty+i\beta$ and leaving all poles on the left, or the vertical line $\mathcal{L}_{ic}$, $\Re(z)=c$, traversed upward and leaving all poles of the integrand on the left. Denote the rightmost pole of the integrand by γ :

$$\gamma=\min_{1\leq j\leq p}(\alpha_j/A_j).$$

Existence conditions of Fox's H-function under each choice of the contour $\mathcal{L}$ have been thoroughly considered in the book [18]. Let $z>0$ and under the conditions:

$$\sum_{j=1}^p A_j=\sum_{k=1}^q B_k,\quad \rho\leq 1,$$

we get that the function $H_{q,p}^{p,0}(z)$ exists by means of [18, Theorem 1.1], if we choose $\mathcal{L}=\mathcal{L}_+$ or $\mathcal{L}=\mathcal{L}_{ic}$ under the additional restriction $\mu>1$. Only the second choice of the contour ensures the existence of the Mellin transform of $H_{q,p}^{p,0}(z)$, see [18, Theorem 2.2]. In [15, Theorem 6], the author extend the condition

$\mu > 1$ to $\mu > 0$ and proved that the function $H^{p,0}_{q,p}(z)$ is a compact support.

In the course of our investigation, one of the main tools is the following result providing the Laplace transform of the Fox's H-function $z^{-1}H^{p,0}_{q,p}(z)$.

Theorem 1. *Suppose that $\mu > 0$, $\gamma \geq 1$, and $\sum_{j=1}^p A_j = \sum_{k=1}^q B_k$. Then, the following integral representation*

$$ {}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| z\right] = \int_0^\rho e^{zt} H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\frac{dt}{t}, \quad (z \in \mathbb{R}), \tag{2.15}$$

hold true. Moreover, the function

$$z \mapsto {}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right]$$

is completely monotonic on $(0,\infty)$, if and only if, the function $H^{p,0}_{q,p}(z)$ is non-negative on $(0,\rho)$.

Proof. Upon setting $k = s$, $k \in \mathbb{N}_0$ in the Mellin transform for the Fox's H-function $H^{p,0}_{q,p}(z)$ [15, Theorem 6]:

$$\frac{\prod_{i=1}^p \Gamma(A_i s + \alpha_i)}{\prod_{k=1}^q \Gamma(B_k s + \beta_k)} = \int_0^\rho H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right) t^{s-1}dt, \; \Re(s) > \gamma, \tag{2.16}$$

we get

$$\begin{aligned}
{}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| z\right] &= \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(A_i k + \alpha_i) z^k}{k! \prod_{j=1}^q \Gamma(B_j k + \beta_j)} \\
&= \sum_{k=0}^\infty \int_0^\rho H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\frac{(zt)^k}{k!}\frac{dt}{t} \\
&= \int_0^\rho H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\left(\sum_{k=0}^\infty \frac{(zt)^k}{k!}\right)\frac{dt}{t} \\
&= \int_0^\rho e^{zt} H^{p,0}_{q,p}\left(t\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right)\frac{dt}{t}.
\end{aligned}$$

For the exchange of the summation and integration, we use the asymptotic relation [18, Theorem 1.2, Eq. 1.94]

$$H^{m,n}_{q,p}(z) = \theta(z^\gamma), \; |z| \longrightarrow 0. \tag{2.17}$$

Now, suppose that the function is completely monotonic on $(0,\infty)$, therefore by means of Bernstein Characterization Theorem and using the fact of the uniqueness of the measure with given Laplace transform (see [38, Theorem 6.3]), we deduce that $H^{p,0}_{q,p}(z)$ is non-negative on $(0,\rho)$, which evidently completes the proof of Theorem 1. □

We next calculate the finite Laplace transform of several special functions. In the following example, we present the finite Laplace transform of the function

$$\frac{t^{a_2-2}(1-t)^{b_1+b_2-1}}{\Gamma(b_1+b_2)} {}_2F_1\left[{}^{a_2+b_2-a_1,b_1}_{b_1+b_2}\Big|1-z\right].$$

Example 1. *Suppose that $a_1, a_2 > 1$ and $b_1, b_2 > 0$. Then the following identity holds:*

$$\begin{aligned}
\int_0^1 e^{zt}\frac{t^{a_2-2}(1-t)^{b_1+b_2-1}}{\Gamma(b_1+b_2)} {}_2F_1\left[{}^{a_2+b_2-a_1,b_1}_{b_1+b_2}\Big|1-t\right]dt &= {}_2\Psi_2\left[{}^{(a_1-1,1),(a_2-1,1)}_{(a_1+b_1-1,1),(a_2+b_2-1,1)}\Big|z\right] \\
&= \frac{\Gamma(a_1-1)\Gamma(a_2-1)}{\Gamma(a_1+b_1-1)\Gamma(a_2+b_2-1)} \\
&\quad \times {}_2F_2\left[{}^{a_1-1,a_2-2}_{a_1+b_1-1,a_2+b_2-1}\right].
\end{aligned} \tag{2.18}$$

In particular, the following formula

$$\begin{aligned}
\int_0^1 e^{zt}t^{a-1} {}_2F_1\left[{}^{2/3,1/3}_{1}\Big|1-t\right]dt &= {}_2\Psi_2\left[{}^{(a,1),(a,1)}_{(a+1/3,1),(a+2/3,1)}\Big|z\right] \\
&= \frac{\Gamma^2(a)}{\Gamma(a+1/3)\Gamma(a+2/3)} {}_2F_2\left[{}^{a,\;a}_{a+\frac{1}{3},a+\frac{2}{3}}\right].
\end{aligned} \tag{2.19}$$

is valid for all $a>0$. *Indeed, we combined the following formula* [18, Eq. (1.142)]

$$H_{2,2}^{2,0}\left(t\Big|_{(a_1-1,1),(a_2-1,1)}^{(a_1+b_1-1,1),(a_2+b_2-1,1)}\right)=\frac{t^{a_2-1}(1-t)^{b_1+b_2-1}}{\Gamma(b_1+b_2)}\,{}_2F_1\left[{}_{b_1+b_2}^{a_2+b_2-a_2,b_1}\Big|1-t\right] \tag{2.20}$$

with (2.15) we get (2.18).

Example 2. *Keeping (2.15) and the formula* [18, Eq. (1.143)] *in mind, we arrive at*

$$\frac{1}{\sqrt{\pi}}\int_0^\infty e^{tz}t^{\alpha-1}(1-t)^{-\frac{1}{2}}dt={}_1\Psi_1\left[{}_{(\alpha+\frac{1}{2},1)}^{(\alpha,1)}\Big|z\right]. \tag{2.21}$$

Example 3. *The four parameters Wright function is defined by the series (in the case it is a convergent one)*

$$\phi\left((\mu,a),(\nu,b);z\right)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(a+k\mu)\Gamma(b+k\nu)},\ \mu,\nu\in\mathbb{R},\ a,b\in\mathbb{C}. \tag{2.22}$$

The series from the right-hand side of (2.22) is absolutely convergent for all $z\in\mathbb{C}$ *if* $\mu+\nu>0$. *If* $\mu+\nu=0$, *the series is absolutely convergent for* $|z|<|\mu|^\mu|\nu|^\nu$ *and* $z|=|\mu|^\mu|\nu|^\nu$ *under the condition* $\Re(a+b)>2$. *Some of the basic properties of the four parameters Wright function was proved in* [20]. *So, by means of Theorem 1 we deduce that the four parameters Wright function* $\phi\left((\mu,a),(\nu,b);z\right)$ *possess the following integral representation:*

$$\phi\left((\mu,a),(\nu,b);z\right)=\int_0^{\mu^\mu\nu^\nu}e^{zt}H_{2,1}^{1,0}\left[t\Big|_{(1,1)}^{(\mu,a),(\nu,b)}\right]\frac{dt}{t}, \tag{2.23}$$

where a,b,μ *and* ν *be a real number such that* $\mu+\nu=1$ *and* $a+b>3/2$.

Corollary 1. *Suppose that the hypotheses of Theorem 1 are satisfied. We define the sequence* $(\psi_{n,m})_{n,m\geq0}$ *by*

$$\psi_{n,m}=\frac{\prod_{i=1}^p\Gamma(\alpha_i+(n+m)A_i)}{\prod_{j=1}^q\Gamma(\beta_j+(n+m)B_j)},\ n,m\in\mathbb{N}_0.$$

If $(H_1^n):\psi_{n,2}<\psi_{n,1}$ *and* $\psi_{n,1}^2<\psi_{n,0}\psi_{n,2}$, *for all* $n\in\mathbb{N}_0$, *then the function*

$$z\mapsto{}_p\Psi_q\left[{}_{(\beta_q,B_q)}^{(\alpha_p,A_p)}\Big|-z\right]$$

is completely monotonic on $(0,\infty)$, *and consequently, the function* $H_{q,p}^{p,0}(z)$ *is non-negative on* $(0,\rho)$.

Proof. In [28, Theorem 4], the authors proved that the function ${}_p\Psi_q\left[{}_{(\beta_q,B_q)}^{(\alpha_p,A_p)}\Big|z\right]$ satisfying the following inequality

$$\psi_{0,0}e^{\psi_{0,1}\psi_{0,0}^{-1}|z|}\leq{}_p\Psi_q\left[{}_{(\beta_q,B_q)}^{(\alpha_p,A_p)}\Big|z\right]\leq\psi_{0,0}-\psi_{0,1}(1-e^{|z|}),\ z\in\mathbb{R}, \tag{2.24}$$

if $\psi_{0,1}>\psi_{0,2}$ and $\psi_{0,1}^2<\psi_{0,0}\psi_{0,2}$. On the other hand, by the left hand side of the above inequalities, we get for $n\geq0$

$$(-1)^n\frac{d^n}{dz^n}{}_p\Psi_q\left[{}_{(\beta_p,B_q)}^{(\alpha_p,A_p)}\Big|-z\right]={}_p\Psi_q\left[{}_{(\beta_q+nB_q,B_q)}^{(\alpha_p+nA_p,A_p)}\Big|-z\right]\geq\psi_{n,0}e^{\psi_{n,1}\psi_{n,0}^{-1}|z|}>0.$$

So, the function $z\mapsto{}_p\Psi_q\left[{}_{(\beta_q,B_q)}^{(\alpha_p,A_p)}\Big|-z\right]$ is completely monotonic on $(0,\infty)$, and consequently, the function $H_{q,p}^{p,0}(z)$ is non-negative on $(0,\rho)$, by means of Theorem 1. □

Remark 1. ***a.*** *Combining (2.15) with (1.6), we obtain*

$$H_{p,q+1}^{1,q}\left(z\Big|_{(0,1),(B_q,1-\beta_q)}^{(A_p,1-\alpha_p)}\right)=\int_0^\rho e^{-zt}H_{q,p}^{p,0}\left(t\Big|_{(A_p,\alpha_p)}^{(B_q,\beta_q)}\right)\frac{dt}{t}. \tag{2.25}$$

b. *In view of (2.15) and (1.7), we get*

$${}_p\Psi_p\left[{}_{(\beta_q,A)}^{(\alpha_p,A)}\Big|z\right]=A^{-1}\int_0^1e^{uz}G_{p,p}^{p,0}\left(u^{1/A}\Big|_{\alpha_p}^{\beta_p}\right)\frac{du}{u},\quad A>0,z\in\mathbb{R}. \tag{2.26}$$

Letting in the above formula, the value $A=1$, we get [16, Corollary 1, Eq. 11]

$$ {}_pF_p\left[{}^{\alpha_1,...,\alpha_p}_{\beta_1,...,\beta_p}\Big| z\right]=\prod_{j=1}^p\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)}\int_0^1 e^{zt}G_{p,p}^{p,0}\left(t\Big|^{\beta_p}_{\alpha_p}\right)\frac{dt}{t}. \tag{2.27}$$

Theorem 2. *Let $\alpha_i,\ \beta_i,\ i=1,...,p$ be a real number such that*

$$(H_2):\ 0<\alpha_1\leq...\leq\alpha_p,\ 0<\beta_1\leq...\leq\beta_p,\ \sum_{j=1}^k\beta_j-\sum_{j=1}^k\alpha_j\geq 0,\ for\ k=1,...,p$$

In addition, assume that $\psi=\sum_{j=1}^p(\beta_j-\alpha_j)>0$. Then, the function

$$z\mapsto {}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_q,A)}\Big| -z\right],$$

is completely monotonic on $(0,\infty)$.

Proof. In [14, Lemma 2], the authors proved that the function $G_{p,p}^{p,0}\left(t\Big|^{\beta_p}_{\alpha_p}\right)$ is non-negative on $(0,1)$, and since the hypotheses of thus Theorem implies the hypotheses of Theorem 1 and so we can used the integral representation (2.26). Therefore, we deduce that all prerequisites of the Bernstein Characterization Theorem for the complete monotone functions are fulfilled, that is, the function ${}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_q,A)}\Big| -z\right]$, is completely monotonic on $(0,\infty)$. It is important to mention here that there is another proof for proved the completely monotonic for the function ${}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_q,A)}\Big| -z\right]$, without using the integral representation (2.26). For this we make use the inequalities (2.24). In our case, we have

$$\psi_{n,m}=\prod_{j=1}^p\frac{\Gamma(\alpha_j+(m+n)A)}{\Gamma(\beta_j+(m+n)A)}.$$

Under the condition (H_2), Alzer [1, Theorem 10] proved that the function

$$\varphi: z\mapsto\prod_{j=1}^p\frac{\Gamma(\alpha_j+z)}{\Gamma(\beta_j+z)},$$

is completely monotonic on $(0,\infty)$ this yields that $\varphi(A)\geq\varphi(2A)$ and consequently $\psi_{0,1}>\psi_{0,2}$. On the other hand, Bustoz and Ismail [6] proved that the function

$$p(z;a,b)=\frac{\Gamma(z)\Gamma(z+a+b)}{\Gamma(z+a)\Gamma(z+b)},\ a,b\geq 0,$$

is completely monotonic on $(0,\infty)$, then the function $z\mapsto p(z;a,b)$ is decreasing on $(0,\infty)$. Now, we choosing $a=b=A$, we obtain $p(\beta_j;A,A)<p(\alpha_j;A,A)$, thus implies that $\psi_{0,1}^2<\psi_{0,0}\psi_{0,2}$. So, from the inequality (2.24), we deduce that the function ${}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_p,A)}\Big| -z\right]$ is non-negative on $(0,\infty)$. On the other hand, for $n\in\mathbb{N}_0$ we have

$$(-1)^n\frac{d^n}{dz^n}{}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_p,A)}\Big| -z\right]={}_p\Psi_p\left[{}^{(\alpha_p+nA,A)}_{(\beta+nA,A)}\Big| -z\right]={}_p\Psi_p\left[{}^{(\delta_p,A)}_{(\lambda_p,A)}\Big| -z\right]\geq 0,$$

where $\delta_p=\alpha_p+nA$ and $\lambda_p=\beta_p+nA$ satisfying the hypothesis (H_2). Thus implies that the function ${}_p\Psi_p\left[{}^{(\alpha_p+nA,A)}_{(\beta+nA,A)}\Big| -z\right]$ is non-negative on $(0,\infty)$, and consequently the function ${}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_p,A)}\Big| -z\right]$ is completely monotonic on $(0,\infty)$. This completes the proof of Theorem 2. □

Remark 2. *We see in the second proof of the above Theorem that the condition $\psi>0$ is not necessary for proved the complete monotonicity property for the function ${}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_p,A)}\Big| -z\right]$, and consequently this property is also true under the conditions hypotheses (H_2) only, and consequently the H-function $H_{p,p}^{p,0}\left[t\big|^{(A,\beta_p)}_{(A,\alpha_p)}\right]$ is non-negative under thus hypotheses.*

Remark 3. *We see that the conditions of hypotheses (H_1^n) are satisfied in the cases when $p=q,\ \alpha_i\leq\beta_i$ and $A_i=B_i$ for $i=1,...,p$. Indeed, under this conditions, we get $\varphi((n+1)A)\geq\varphi((n+2)A),\ n\geq 0$ and consequently $\psi_{n,1}>\psi_{n,2}$. Moreover, we have $p(\beta_j+nA_j;A_j,A_j)\leq p(\alpha_j+nA_j;A_j,A_j),\ n\geq 0$, i.e., $\psi_{n,1}^2<\psi_{n,0}\psi_{n,2}$.*

Conjecture 1. *We suppose that* $p=q,\ \alpha_i\leq\beta_i,\ A_i\leq B_i$ *and* $\beta_i+B_i>x^*$ *for* $i=1,...,p$, *where* x^* *is the abscissa of the minimum of Gamma function. Since the function* $z\mapsto\frac{\Gamma(z+a)}{\Gamma(z)},\ a>0$ *is increasing on* $(0,\infty)$, *we get*

$$\Gamma(z+a)\Gamma(z+b)\leq\Gamma(z)\Gamma(z+a+b),\ a,b>0.$$

We set $z=\alpha_i+nA_i,\ a=A_i$ *and* $b=\beta_i-\alpha_i+(n+1)(B_i-A_i)$ *in the above inequality, we get*

$$\frac{\Gamma(\alpha_i+(n+2)A_i)}{\Gamma(\beta_i+(n+1)B_i+A_i)}<\frac{\Gamma(\alpha_i+(n+1)Ai)}{\Gamma(\alpha_i+(n+1)B_i)}. \tag{2.28}$$

Since the Gamma function $\Gamma(z)$ *is increasing on* (x^*,∞), *we get* $\Gamma(\beta_i+(n+1)B_i+A_i)<\Gamma(\beta_i+(n+2)B_i)$ *and consequently*

$$\frac{\Gamma(\alpha_i+(n+2)A_i)}{\Gamma(\beta_i+(n+2)B_i)}<\frac{\Gamma(\alpha_i+(n+1)Ai}{\Gamma(\alpha_i+(n+1)B_i)}, \tag{2.29}$$

i.e the condition $\psi_{n,2}<\psi_{n,1}$ *in* (H_1^n) *holds true. We were not able to show that the inequality* $\psi_{n,1}^2<\psi_{n,0}\psi_{n,2}$. *For this we state the following conjecture: Proved a sufficient condition imposed on the parameters* α_i,β_j,A_i *and* B_j $(A_i\neq B_j)$ *such that the conditions of hypotheses* (H_1^n) *are satisfied.*

Corollary 2. *Keeping the notation and constraints of hypotheses* (H_2) *of Theorem 2. Then, the function*

$$A\mapsto {}_p\Psi_p\left[\left.{}^{(\alpha_p,A)}_{(\beta_q,A)}\right|z\right],$$

is log-convex on $(0,\infty)$ *for all* $z\in\mathbb{R}$. *Furthermore, then the following Turán type inequality*

$$ {}_p\Psi_p\left[\left.{}^{(\alpha_p,A)}_{(\beta_q,A)}\right|z\right]{}_p\Psi_p\left[\left.{}^{(\alpha_p,A+2)}_{(\beta_q,A+2)}\right|z\right]-\left({}_p\Psi_p\left[\left.{}^{(\alpha_p,A+1)}_{(\beta_q,A+1)}\right|z\right]\right)^2\geq0, \tag{2.30}$$

holds true for all $A\in(0,\infty)$ *and* $z\in\mathbb{R}$.

Proof. Rewriting the integral representation (2.26) in the following form:

$${}_p\Psi_p\left[\left.{}^{(\alpha_p,A)}_{(\beta_q,A)}\right|z\right]=\int_0^1 e^{t^Az}G_{p,p}^{p,0}\left(t\left|{}^{\beta_p}_{\alpha_p}\right.\right)\frac{dt}{t},\ \ A>0,\ z\in\mathbb{R}. \tag{2.31}$$

Let us recall the Hölder inequality [24, p. 54], that is

$$\int_a^b|f(t)g(t)|dt\leq\left[\int_a^b|f(t)|^pdt\right]^{1/p}\left[\int_a^b|g(t)|^pdt\right]^{1/q}, \tag{2.32}$$

where $p\geq1,\ \frac1p+\frac1q=1,\ f$ and g are real functions defined on (a,b) and $|f|^p,\ |g|^q$ are integrable functions on (a,b). From the Hölder's inequality and integral representation (2.31) and using the fact that the function $A\mapsto x^A$ is convex on $(0,\infty)$ when $x>0$. For $A_1,A_2>0$ and $t\in[0,1]$, we thus get

$$\begin{aligned}
{}_p\Psi_p\left[\left.{}^{(\alpha_p,\ tA_1+(1-t)A_2}_{(\beta_p,\ tA_1+(1-t)A_2)}\right|z\right]&=\int_0^1 e^{zu^{tA_1+(1-t)A_2}}G_{p,p}^{p,0}\left(u\left|{}^{\beta_p}_{\alpha_p}\right.\right)\frac{du}{u}\\
&\leq\int_0^1 e^{tzu^{A_1}}e^{(1-t)zu^{A_2}}G_{p,p}^{p,0}\left(u\left|{}^{\beta_p}_{\alpha_p}\right.\right)\frac{du}{u}\\
&=\int_0^1\left[\frac{e^{zu^{A_1}}}{u}G_{p,p}^{p,0}\left(u\left|{}^{\beta_p}_{\alpha_p}\right.\right)\right]^t\left[\frac{e^{zu^{A_2}}}{u}G_{p,p}^{p,0}\left(u\left|{}^{\beta_p}_{\alpha_p}\right.\right)\right]^{1-t}du\\
&\leq\left[\int_0^1 e^{zu^{A_1}}G_{p,p}^{p,0}\left(u\left|{}^{\beta_p}_{\alpha_p}\right.\right)\frac{du}{u}\right]^t\left[\int_0^1 e^{zu^{A_2}}G_{p,p}^{p,0}\left(u\left|{}^{\beta_p}_{\alpha_p}\right.\right)\frac{du}{u}\right]^{1-t}\\
&=\left[{}_p\Psi_p\left[\left.{}^{(\alpha_p,\ A_1)}_{(\beta_p,\ A_1)}\right|z\right]\right]^t\left[{}_p\Psi_p\left[\left.{}^{(\alpha_p,\ A_2)}_{(\beta_p,\ A_2)}\right|z\right]\right]^{1-t},
\end{aligned} \tag{2.33}$$

and hence the required result follows. Now, choosing $A_1=A,A_2=A+2$ and $t=\frac12$ in the above inequality we get the Turán type inequality (2.30). □

Corollary 3. *Let* $\lambda,\ \omega>0$. *Suppose that the hypotheses* (H_2) *of Theorem 2 are satisfied. , we deduce that the function*

$$z\mapsto z^{-\lambda}{}_{p+1}\Psi_p\left[\left.{}^{(\lambda,1),(\alpha_p,A)}_{(\beta_q,A)}\right|-\frac1z\right]$$

is completely monotonic on $(0,\infty)$, *and consequently, the Hypergeometric function*

$$z \mapsto z^{-\lambda}{}_{p+1}F_p\left[{}^{\lambda,\alpha_1,\dots,\alpha_p}_{\beta_1,\dots,\beta_p}\Big| -\frac{1}{z}\right]$$

is completely monotonic on $(0,\infty)$. (*see* [16, Theorem 3].)

Proof. From the integral representation [28, Eq. 7]

$$z^{-\lambda}{}_{p+1}\Psi_p\left[{}^{(\lambda,1),(\alpha_p,A)}_{(\beta_p,A)}\Big| -\frac{\omega}{z}\right] = \int_0^\infty e^{-zt}t^{\lambda-1}{}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_p,A)}\Big| -\omega t\right]dt,$$

and using the fact that the function ${}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_p,A)}\Big| -z\right]dt$, is non-negative on $(0,\infty)$, we deduce that the function $z^{-\lambda}{}_{p+1}\Psi_P\left[{}^{(\lambda,1),(\alpha_p,A)}_{(\beta_p,A)}\Big| -\frac{\omega}{z}\right]$ is completely monotonic on $(0,\infty)$. □

Remark 4. *Let* $\lambda,\ \omega > 0$. *Repeating the same calculations in Corollary 3 and under the assumptions of Theorem 1 such that the conditions of hypotheses* (H_1^n) *(or the function* $H^{p,0}_{q,p}[.]$ *is non-negative) are satisfied, we get that the function*

$$z \mapsto z^{-\lambda}{}_{p+1}\Psi_q\left[{}^{(\lambda,1),(a_p,A_p)}_{(b_q,B_q)}\Big| -\frac{\omega}{z}\right]$$

is completely monotonic on $(0,\infty)$.

In the following example we present some new properties for the $\tau-$Kummer hypergeometric φ^τ defined by [34]

$$\varphi^\tau(b,c,z) = \sum_{k=0}^\infty \frac{\Gamma(b+k\tau)}{\Gamma(c+k\tau)}\frac{z^k}{k!},\ (c>b>0,\ \tau>0,\ |z|<1.)$$

Example 4. *The following assertions are true:*
1. *In view of Theorem 1 and the identity* [23, p.127]

$$z^\alpha(1-z)^\beta = \Gamma(\beta+1)G^{1,0}_{1,1}\left(z\Big|^{\alpha+\beta+1}_{\alpha}\right),\ |z|<1,$$

we deduce that the $\tau-$*Kummer hypergeometric* φ^τ *admits the following integral representation:*

$$\begin{aligned}\varphi^\tau(b,c,z) &= \frac{\Gamma(c)}{\tau\Gamma(b)}\int_0^1 e^{zt}G^{1,0}_{1,1}\left(t^{\frac{1}{\tau}}\Big|^{c}_{b}\right)\frac{dt}{t}\\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 e^{zt^\tau}t^{b-1}(1-t)^{c-b-1}dt.\end{aligned} \tag{2.34}$$

2. *The function* $z\mapsto\varphi^\tau(b,c,-z)$ *is completely monotonic on* $(0,1)$. *Moreover, the* $\tau\mapsto\varphi^\tau(b,c,z)$ *is log-convex on* $(0,\infty)$ *and satisfies the following Turán type inequality*

$$\varphi^{\tau+2}(b,c,z)\varphi^\tau(b,c,z) - \varphi^{\tau+1}(b,c,z)^2 \geq 0.$$

Theorem 3. *The function* $z\mapsto {}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|\frac{1}{z}\right]$ *admits the following Laplace integral representation*

$$ {}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|\frac{1}{z}\right] = \int_0^\infty e^{-zt}\left({}_p\Psi_{q+1}\left[{}^{(\alpha_p+1,A_p)}_{(\beta_q+1,B_q),(2,1)}\Big|t\right] + \frac{\prod_{i=1}^p\Gamma(\alpha_i)}{\prod_{j=1}^q\Gamma(\beta_j)}\delta_0\right)dt, \tag{2.35}$$

where δ_0 *is the Dirac measure with mass* 1 *concentrated at zero. Moreover, the function*

$$z\mapsto {}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big|\frac{1}{z}\right]$$

is completely monotonic on $(0,\infty)$.

Proof. Straightforward calculation would yield

$$\int_0^\infty e^{-zt}\left({}_p\Psi_{q+1}\left[{}^{(\alpha_p+1,A_p)}_{(\beta_q+1,B_q),(2,1)}\Big|t\right] + \frac{\prod_{i=1}^p\Gamma(\alpha_i)}{\prod_{j=1}^q\Gamma(\beta_j)}\delta_0\right)dt =$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+A_i m+1)}{\prod_{j=1}^{q}\Gamma(\beta_i+B_i m+1)\Gamma(m+2)m!}\int_0^{\infty} t^m e^{-zt}dt + \frac{\prod_{i=1}^{p}\Gamma(\alpha_i)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\\
&= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+A_i m+1)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j m+1)(m+1)!z^{m+1}} + \frac{\prod_{i=1}^{p}\Gamma(\alpha_i)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\\
&= \sum_{m=1}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+A_i m)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j m)m!z^{m}} + \frac{\prod_{i=1}^{p}\Gamma(\alpha_i)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\\
&= \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+A_i m)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_j m)m!z^{m}}\\
&= {}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\middle|\frac{1}{z}\right].
\end{aligned}
$$

Therefore, the integral representation (2.35) of the function Fox-Wright function ${}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\middle|\frac{1}{z}\right]$ is fulfilled. Simultaneously, the function ${}_p\Psi_{q+1}\left[{}^{(\alpha_p+1,A_p)}_{(\beta_q+1,B_q),(2,1)}\middle|t\right]$ being positive, all prerequisites of the Bernstein Characterization Theorem for the complete monotone functions are fulfilled, that is, the function ${}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\middle|\frac{1}{z}\right]$ is completely monotone on $(0,\infty)$. It is important to mention here that there is another proof for the completely monotone of the Fox-Wright function ${}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\middle|\frac{1}{z}\right]$. By using the fact that if the function $f(x)$ is absolutely monotonic then the function $f(1/x)$ is completely monotonic [38, p. 151], and since the function ${}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\middle|z\right]$ is absolutely monotonic and consequently the function ${}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\middle|\frac{1}{z}\right]$ is completely monotonic on $(0,\infty)$, which evidently completes the proof of Theorem 3. □

Example 5. *The four parameters Wright function $\phi((\mu,a),(\nu,b);1/z)$ is completely monotonic on $(0,\infty)$, and admits the following integral representation:*

$$\phi((\mu,a),(\nu,b);1/z) = \int_0^{\infty} e^{-zt}\left({}_1\Psi_3\left[{}^{(2,1)}_{(a+1,\mu),(b+1,\nu),(2,1)}\middle|t\right] + \frac{1}{\Gamma(a+1)\Gamma(b+1)}\delta_0\right)dt. \tag{2.36}$$

3. Stieltjes transform representation for the Fox-Wright functions and its consequences

In this section, we show that the Fox-Wright function

$${}_{p+1}\Psi_q\left[{}^{(\sigma,1),(\alpha_p,A_p)}_{(\beta_p,B_p)}\middle|-z\right]$$

is a generalized Stieltjes functions of order σ. As applications, some class of logarithmically completely monotonic functions related to the Fox-Wright function are derived. Moreover, we deduce new Turán type inequalities for thus special function.

Theorem 4. *Let $\sigma>0$ and $z\in\mathbb{C}$ such that $|\arg z|<\pi$ and $|z|<1$. Assume that the hypotheses (H_2) of Theorem 2 are satisfied. Then, the following Stieltjes transform hold true:*

$${}_{p+1}\Psi_p\left[{}^{(\sigma,1),(\alpha_p,A)}_{(\beta_P,A)}\middle|-z\right] = \int_0^1 \frac{d\mu(t)}{(1+tz)^{\sigma}}, \tag{3.37}$$

where

$$d\mu(t) = H^{p,0}_{p,p}\left(t\middle|{}^{(A,\beta_p)}_{(A,\alpha_p)}\right)\frac{dt}{t}. \tag{3.38}$$

Furthermore, the function

$$z\mapsto {}_{p+1}\Psi_p\left[{}^{(\sigma,1),(\alpha_p,A)}_{(\beta_p,A)}\middle|-z\right]$$

is completely monotonic on $(0,1)$.

Proof. Consider the right-hand side of (3.40) with $d\mu(t)$ is given by (3.41). We make use of the formula (2.16) and applying the binomial expansion to

$$(1+z)^{-\sigma} = \sum_{k=0}^{\infty}(\sigma)_k\frac{(-1)^k z^k}{k!},\quad z\in\mathbb{C}\ \text{such that}\ |z|<1,$$

and integrating term by term we obtain the left-hand side of (3.40). Finally, its easy to see that the function $z \mapsto {}_{p+1}\Psi_p\left[\substack{(\sigma,1),(\alpha_p,A)\\(\beta_p,A)}\middle| -z\right]$ is completely monotonic on $(0,1)$. $\square$

Corollary 4. *Let $0<\sigma\leq 1$. Assume that the hypotheses (H_2) of Theorem 2 are satisfied. Then the following assertions are true:*
a. *The function*

$$z \mapsto {}_{p+1}\Psi_p\left[\substack{(\sigma,1),(\alpha_p,A)\\(\beta_p,A)}\middle| -z\right]$$

is logarithmically completely monotonic on $(0,1)$.
b. *The function*

$$z \mapsto 1\Big/ {}_{p+1}\Psi_p\left[\substack{(\sigma,1),(\alpha_p,A)\\(\beta_p,A)}\middle| -z\right] \tag{3.39}$$

is a Bernstein function on $(0,1)$. In particular, the function

$$z \mapsto {}_{p+1}\Psi_p\left[\substack{(\sigma+1,1),(\alpha_p+A,A)\\(\beta_p+A,A)}\middle| -z\right]\Big/ {}_{p+1}\Psi_q\left[\substack{(\sigma,1),(\alpha_p,A)\\(\beta_p,A)}\middle| -z\right]$$

is completely monotonic on $(0,1)$.

Proof. a. By using the fact that $S_\alpha \subseteq S_\beta$ whenever $\alpha\leq\beta$ (see [30]), we deduce

$${}_{p+1}\Psi_p\left[\substack{(\sigma,1),(\alpha_p,A)\\(\beta_p,A)}\middle| -z\right] \in S_1 = S,$$

where $0<\sigma\leq 1$. On the other hand, it was proved in [4, Theorem 1.2] that the set of Stieltjes transforms $\mathcal{S}\setminus\{0\}$ is a proper subset of the class of logarithmically completely monotonic functions.
b. The result follows from Theorem 4 and Proposition 1.3 [5]. $\square$

Example 6. *We consider the $\tau-$Gauss hypergeometric function ${}_2\varphi_1^\tau(a,b,c,z)$ defined by [35]:*

$${}_2\varphi_1^\tau(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(a+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!},\ a,b,c>0, c>b, |z|<1.$$

Then, the following assertions are true:

(1) *The $\tau-$Gauss hypergeometric function possesses the following integral representation:*

$${}_2\varphi_1^\tau(a,b,c,-z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1+zt^\tau)^a}dt.$$

(2) *The $\tau-$Gauss hypergeometric function ${}_2\varphi_1^\tau(a,b,c,-z)$ is logarithmically completely monotonic on $(0,1)$.*

Remark 5. *Assume that the assumption of Theorem 1 and the hypotheses (H_1^n) are satisfied (or the function $H_{q,p}^{p,0}[.]$ is non-negative). Then, the following Stieltjes transform hold true:*

$${}_{p+1}\Psi_q\left[\substack{(\sigma,1),(\alpha_p,A_p)\\(\beta_q,B_q)}\middle| -z\right] = \int_0^\rho \frac{d\mu(t)}{(1+tz)^\sigma}, \tag{3.40}$$

where

$$d\mu(t) = H_{q,p}^{p,0}\left(t\middle|\substack{(B_q,\beta_q)\\(A_p,\alpha_p)}\right)\frac{dt}{t}. \tag{3.41}$$

Moreover, the following assertions are true:

(1) *The function*

$$z \mapsto {}_{p+1}\Psi_q\left[\substack{(\sigma,1),(\alpha_p,A_p)\\(\beta_q,B_q)}\middle| -z\right]$$

is logarithmically completely monotonic on $(0,1)$.

(2) *The function*

$$z \mapsto 1\Big/ {}_{p+1}\Psi_q\left[\substack{(\sigma,1),(\alpha_p,A_p)\\(\beta_q,B_q)}\middle| -z\right] \tag{3.42}$$

is a Bernstein function on $(0,1)$. In particular, the function

$$z \mapsto {}_{p+1}\Psi_q\left[\substack{(\sigma+1,1),(\alpha_p+A_p,A_p)\\(\beta_q+B_q,B_q)}\middle| -z\right]\Big/ {}_{p+1}\Psi_q\left[\substack{(\sigma,1),(\alpha_p,A_p)\\(\beta_q,B_q)}\middle| -z\right]$$

is completely monotonic on $(0,1)$.

Theorem 5. *Under the assumptions (H_2) stated in Theorem 2. The function*

$$\sigma \mapsto \Xi(\sigma) = {}_{p+1}\Psi_p\left[\left.{}^{(\sigma,1),(\alpha_p,A)}_{(\beta_p,A)}\right|z\right]$$

is log-convex on $(0,\infty)$ for each $z \in (0,1)$. Furthermore, the following Turán type inequality

$$(3.43)\qquad {}_{p+1}\Psi_p\left[\left.{}^{(\sigma,1),(\alpha_p,A)}_{(\beta_p,A)}\right|z\right]{}_{p+1}\Psi_p\left[\left.{}^{(\sigma+2,1),(\alpha_p,A)}_{(\beta_p,A)}\right|z\right] - \left({}_{p+1}\Psi_p\left[\left.{}^{(\sigma+1,1),(\alpha_p,A)}_{(\beta_p,A)}\right|z\right]\right)^2 \geq 0,$$

holds true for all $\sigma \in (0,\infty)$ and $z \in (0,1)$.

Proof. Recall the Chebyshev integral inequality [24, p. 40]: if $f, g : [a,b] \longrightarrow \mathbb{R}$ are synchoronous (both increasing or decreasing) integrable functions, and $p : [a,b] \longrightarrow \mathbb{R}$ is a positive integrable function, then

$$(3.44)\qquad \int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt.$$

Note that if f and g are asynchronous (one is decreasing and the other is increasing), then (3.44) is reversed. Let $\sigma_2 > \sigma_1 \geq 0$ and arbitrary $\epsilon > 0$ and we consider the functions $p, f, g : [0,1] \longrightarrow \mathbb{R}$ defined by:

$$p(t) = \frac{t^{-1}H^{p,0}_{p,p}\left(t\left|^{(A,\beta_p)}_{(A,\alpha_p)}\right.\right)}{(1-zt)^{\sigma_1}},\quad f(t) = \frac{1}{(1-zt)^{\sigma_2-\sigma_1}},\quad g(t) = \frac{1}{(1-zt)^{\epsilon}}.$$

Since the function p is non-negative on $(0,1)$ and the functions f and g are increasing on $(0,1)$ if $z \in (0,1)$, we gave

$$\Xi(\sigma_1+\epsilon)\Xi(\sigma_2) \leq \Xi(\sigma_1)\Xi(\sigma_2+\epsilon).$$

The above inequality is equivalent to log-convexity for the function $\sigma \mapsto \Xi(\sigma)$ on $(0,\infty)$ for each $z \in (0,1)$ (see [25, Chapter I.4]). Now, focus on the Turán type inequality (3.45). Since the function $\sigma \mapsto \Xi(\sigma)$ is log-convex on $(0,\infty)$ for each $x \in (0,1)$. it follows that for all $\sigma_1, \sigma_2 > 0,\ t \in [0,1]$ and $x \in (0,1)$, we have

$$\Xi(t\sigma_1 + (1-t)\sigma_2) \leq [\Xi(\sigma_1)]^t[\Xi(\sigma_2)]^{1-t}.$$

Upon setting

$$\sigma_1 = \sigma,\ \sigma_2 = \sigma + 2 \text{ and } t = \frac{1}{2},$$

the above inequality reduces to the Turán type inequality (3.45), which evidently completes the proof of Theorem 5. □

Remark 6. *Under the assumptions stated in Theorem 1 and hypotheses (H_1^n) (or the function $H^{p,0}_{q,p}[.]$ is non-negative) and by repeating the same calculations as above we deduce that The function*

$$\sigma \mapsto \Xi(\sigma) = {}_{p+1}\Psi_q\left[\left.{}^{(\sigma,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\right|z\right]$$

is log-convex on $(0,\infty)$ for each $z \in (0,1)$. Furthermore, the following Turán type inequality

$$(3.45)\qquad {}_{p+1}\Psi_q\left[\left.{}^{(\sigma,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\right|z\right]{}_{p+1}\Psi_q\left[\left.{}^{(\sigma+2,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\right|z\right] - \left({}_{p+1}\Psi_q\left[\left.{}^{(\sigma+1,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\right|z\right]\right)^2 \geq 0,$$

holds true for all $\sigma \in (0,\infty)$ and $z \in (0,1)$.

4. Further Applications

4.1. **A class of positive definite functions related to the Fox H-function.** The purpose of this section is to prove a class of positive definite functions related to the Fox H-function. As an application, we derive a class of function involving the Fox H-function is non-negative. Let us remind the reader that a continuous function $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ is called positive definite function, if for all $N \in \mathbb{N}$, all sets of pairwise distinct centers $X = \{x_1, ..., x_N\} \subseteq \mathbb{R}^d$ and $z = \{\xi_1, ..., \xi_N\} \subset \mathbb{C}^N$, the quadratic form

$$\sum_{j=1}^N \sum_{k=1}^N \xi_j \bar{\xi_k} f(x_j - x_k)$$

is non-negative.

Theorem 6. *Let the parameters $\rho, \nu \in \mathbb{C}$, satisfy the conditions*

$$\Re(\rho) + \Re(\nu) + \min_{1\le j\le p}\left[\frac{\alpha_j}{A}\right] > -1,$$

$$\Re(\nu) > -\frac{1}{2} \;\; and \;\; \Re(\rho)+\Re(\nu) < \frac{3}{2}.$$

Moreover, assume that the hypotheses (H_2) of Theorem 2 are satisfies. Then the function

$$\chi : z \mapsto z^{-(\rho+\nu)} H^{1,p}_{p,p+2}\left[2z\Big|^{(A,1-\alpha_p)}_{(\frac{1}{2},\frac{\rho+\nu}{2}),(A,1-\beta_p),(\frac{1}{2},\frac{\rho-\nu}{2})}\right] \tag{4.46}$$

is positive definite function on $\mathbb{R}$.

Proof. We can write the following formula [18, Eq. (2.45), pp. 57]

$$\int_0^\infty x^{\rho-1} J_\nu(zx) H^{m,n}_{q,p}\left[x\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right]dx = \frac{2^{\rho-1}}{z^\rho} H^{m,n+1}_{q+2,p}\left[\frac{2}{z}\Big|^{(\frac{1}{2},1-\frac{\rho+\nu}{2}),(B_q,\beta_q),(\frac{1}{2},1-\frac{\rho-\nu}{2})}_{(A_p,\alpha_p)}\right] \tag{4.47}$$

in the following form

$$\int_0^\infty x^{\rho+\nu-1} \mathcal{J}_\nu(zx) H^{m,n}_{q,p}\left[x\Big|^{(B_q,\beta_q)}_{(A_p,\alpha_p)}\right]dx = \frac{\Gamma(\nu+1)2^{\rho+\nu-1}}{z^{\rho+\nu}} H^{m,n+1}_{q+2,p}\left[\frac{2}{z}\Big|^{(\frac{1}{2},1-\frac{\rho+\nu}{2}),(B_q,\beta_q),(\frac{1}{2},1-\frac{\rho-\nu}{2})}_{(A_p,\alpha_p)}\right] \tag{4.48}$$

where

$$\mathcal{J}_\nu(x) = 2^\nu\Gamma(\nu+1)\frac{J_\nu(x)}{x^\nu},\; \Re(\nu) > -\frac{1}{2},$$

with $J_\nu(x)$ is the Bessel function of index ν. On the other hand, as the function $\mathcal{J}_\nu(x)$ is positive definite function [9, Proposition 2] and the function $H^{p,0}_{p,p}[t|^{(A,\beta_p)}_{(A,\alpha_p)}]$ is non-negative (Remark 2), we deduce that for any finite list of complex numbers $\xi_1, ..., \xi_N$ and $z_1, ..., z_N \in \mathbb{R}$,

$$\sum_{j=1}^N\sum_{k=1}^N \xi_j\bar{\xi_k}(z_j-z_k)^{-(\rho+\nu)} H^{p,1}_{p+2,p}\left[2(z_j-z_k)^{-1}\Big|^{(\frac{1}{2},1-\frac{\rho+\nu}{2}),(A,\beta_q),(\frac{1}{2},1-\frac{\rho-\nu}{2})}_{(A,\alpha_p)}\right] = \tag{4.49}$$

$$= \frac{1}{\Gamma(\nu+1)2^{\rho+\nu-1}}\int_0^\infty x^{\rho+\nu-1}\left[\sum_{j=1}^N\sum_{k=1}^N \xi_j\bar{\xi_k}\mathcal{J}_\nu(xz_j-xz_k)\right] H^{p,0}_{p,p}\left[x\Big|^{(A,\beta_p)}_{(A,\alpha_p)}\right]dx \ge 0.$$

Thus, implies that the function

$$\chi_1 : z \mapsto z^{-(\rho+\nu)} H^{p,1}_{p+2,p}\left[\frac{2}{z}\Big|^{(\frac{1}{2},1-\frac{\rho+\nu}{2}),(A,\beta_p),(\frac{1}{2},1-\frac{\rho-\nu}{2})}_{(A,\alpha_p)}\right]$$

is positive definite function on $\mathbb{R}$. So, the [18, Property 1.3, p. 11] completes the proof of Theorem 6. □

Theorem 7. *Let the parameters $\rho, \nu \in \mathbb{C}$, satisfy the conditions*

$$\Re(\rho) + \Re(\nu) + \min_{1\le j\le p}\left[\frac{\alpha_j}{A}\right] > 0,\; \Re(\nu) > -\frac{1}{2}, \tag{4.50}$$

$$\Re(1-(\rho+\nu)) + \max_{1\le j\le p}\left(\frac{\alpha_j}{A}\right) < 1,\; and\; \Re(\rho)+\Re(\nu) < \frac{3}{2}. \tag{4.51}$$

Then, the function

$$\mathcal{K}^{\nu,\rho}_{p,q}(z) = z^{\rho+\nu-1} H^{p+1,1}_{p+2,p+2}\left[8z\Big|^{(\frac{1}{2},1-\frac{\rho+\nu}{2}),(A,\beta_p),(\frac{1}{2},1-\frac{\rho-\nu}{2})}_{(\frac{1}{2},\frac{1-(\rho+\nu)}{2}),(A,\alpha_p),(\frac{1}{2},1-\frac{\rho+\nu}{2})}\right] \tag{4.52}$$

is non-negative on $\mathbb{R}$.

Proof. Firstly, we proved that the function $z \longmapsto \chi(z)$ is in $L^1(0,\infty)$. By using the asymptotic expansion [18, Eq. 1.94, pp. 19]

$$H^{m,n}_{p,q}\left[z\Big|^{(A_p,a_p)}_{(B_q,b_q)}\right] = \theta(z^c),\; |z| \longrightarrow 0, \;\; \text{where } c = \min_{1\le j\le m}\left[\frac{\Re(b_j)}{B_j}\right].$$

In our case $m = 1, b_1 = \frac{\rho+\nu}{2}$ and $B_1 = \frac{1}{2}$, and consequently $c = \rho+\nu$. Thus implies that

$$\chi(z) = \theta(1) \;\text{ as }\; z \longrightarrow 0. \tag{4.53}$$

On the other hand, by using the asymptotic [18, Eq. 1.94, pp. 19]

$$H_{p,q}^{m,n}\left[z\Big|_{(B_q,b_q)}^{(A_p,a_p)}\right]=\theta(z^d),\;|z|\longrightarrow\infty,\quad\text{where } d=\min_{1\leq j\leq n}\left[\frac{\Re(a_j)-1}{A_j}\right].$$

In our case $n=p$ and $a_j=1-\alpha_j$ and consequently $d=-(\Re(\rho)+\Re(\nu)+\min_{1\leq j\leq p}\frac{\Re(\alpha_j)}{A})$, thus we get

$$\chi(z)=\theta\left(z^{-(\rho+\nu+M)}\right),\text{ where } M=\min_{1\leq j\leq p}\frac{\alpha_j}{A}. \tag{4.54}$$

Now, combining (4.54) with the hypotheses (4.50) and (4.53), we deduce that the function $z\longmapsto\chi(z)$ is in $L^1(0,\infty)$. In addition, as $z\mapsto\mathcal{J}_\nu(zx)$ is an even function and using the integral representation (4.48), we deduce that $z\longmapsto\chi(z)$ is an even function, and consequently thus function is in $L^1(\mathbb{R})$.
Secondly, we calculate the Fourier transform of the function $z\longmapsto\chi(z)$. Since $z\longmapsto\chi(z)$ is an even function then the the Fourier transform can be written as a Hankel transform (see [26, Lemma 1.1], when $\alpha=-1/2$), more precisely,

$$\mathcal{F}(\chi)(z)=\sqrt{\frac{2}{\pi}}\int_0^\infty\chi(x)\cos(xz)dz. \tag{4.55}$$

We now make use of the following formula [18, Eq. 2.50, p. 58]

$$\int_0^\infty x^{\rho-1}\cos(xz)H_{p,q}^{m,n}\left[x\Big|_{(B_q,b_q)}^{(A_p,a_p)}\right]dx=\frac{2^{\rho-1}\sqrt{\pi}}{z^\rho}H_{p+2,q}^{m,n+1}\left[\frac{2}{z}\Big|_{(B_q,b_q)}^{(\frac12,\frac{2-\rho}{2}),(A_p,a_p),(\frac12,\frac{1-\rho}{2})}\right] \tag{4.56}$$

where $z>0,\;\rho\in\mathbb{C}$ such that

$$\Re(\rho)+\min_{1\leq j\leq m}\Re\left(\frac{b_j}{B_j}\right)>0\;\text{ and }\;\Re(\rho)+\max_{1\leq j\leq n}\Re\left(\frac{a_j-1}{A_j}\right)<1.$$

In our case $\rho\longrightarrow1-(\rho+\nu),\;m=1\;n=p,\;b_1=\frac{\rho+\nu}{2}$ and $B_1=\frac12$, thus

$$\Re(\rho)+\min_{1\leq j\leq m}\left(\frac{b_j}{B_j}\right)=1>0.$$

Therefore,

$$\begin{aligned}\mathcal{F}(\chi)(z)&=2^{1-\rho-\nu}z^{\rho+\nu-1}H_{p+2,p+2}^{1,p+1}\left[\frac{8}{z}\Big|_{(\frac12,\frac{\rho+\nu}{2}),(A,1-\beta_p),(\frac12,\frac{\rho-\nu}{2})}^{(\frac12,\frac{1+\rho+\nu}{2}),(A,1-\alpha_p),(\frac12,\frac{\rho+\nu}{2})}\right]\\&=2^{1-\rho-\nu}z^{\rho+\nu-1}H_{p+2,p+2}^{p+1,1}\left[8z\Big|_{(\frac12,\frac{1-(\rho+\nu)}{2}),(A,\alpha_p),(\frac12,1-\frac{\rho+\nu}{2})}^{(\frac12,1-\frac{\rho+\nu}{2}),(A,\beta_p),(\frac12,1-\frac{\rho-\nu}{2})}\right].\end{aligned} \tag{4.57}$$

Finally, using the fact that the Fourier transform for a function in L^1 and positive definite function is non-negative (see for example [8, theorem 6.6] or [36, Theorem 6.11, p. 74]). So, the proof is completes. □

4.2. **Zeros of the Fox-Wright functions.**

Theorem 8. *Keeping the notation and constraints of hypotheses (H_2) of Theorem 2. Then, all the roots of the Fox-Wright function ${}_p\Psi_p\left[{}^{(\beta_p,A)}_{(\alpha_p,A)}\Big|z\right]$ are in the left-hand half-plane $\Re z\leq0$.*

Proof. By using the following identity [15, Theorem 8]

$$H_{p,p}^{p,0}\left(z\Big|_{(A,\alpha_p)}^{(A,\beta_p)}\right)=\frac{1}{\log(1/z)}\int_z^1H_{p,p}^{p,0}\left(\frac{z}{u}\Big|_{(A,\alpha_p)}^{(A,\beta_p)}\right)\frac{Q(u)}{u}du \tag{4.58}$$

where $Q(u)$ is defined by

$$Q(u)=\sum_{i=1}^p\frac{t^{\alpha_i/A}}{1-t^{1/A}}-\sum_{j=1}^q\frac{t^{\beta_j/A}}{1-t^{1/A}},\;t\in(0,1),$$

we deduce that the function

$$t\mapsto H_{p,p}^{p,0}\left(t\Big|_{(A,\alpha_p)}^{(A,\beta_p)}\right)$$

decreasing on $(0,1)$. On the other hand, by means of the integral representation (2.15) and we make the following change of variables $t=1-u$ we get

$$e^{-z}{}_p\Psi_p\left[{}^{(\alpha_p,A)}_{(\beta_p,A)}\Big|z\right]=\int_0^1e^{-zu}H_{p,p}^{p,0}\left(1-u\Big|_{(A,\alpha_p)}^{(A,\beta_p)}\right)\frac{du}{1-u} \tag{4.59}$$

Taking into account the obvious equation [18, Property 2.5, Eq. 2.1. 5]

$$z^\sigma H^{m,n}_{q,p}\left(z\Big|^{(A_p,\alpha_p)}_{(B_q,\beta_q)}\right)=H^{m,n}_{q,p}\left(z\Big|^{(A_p,\alpha_p+\sigma A_p)}_{(B_q,\beta_q+\sigma B_q)}\right),\ \sigma\in\mathbb{C}, \tag{4.60}$$

and using (4.76) we get

$$e^{-z}{}_p\Psi_p\left[^{(\alpha_p,A)}_{(\beta_p,A)}\Big|z\right]=\int_0^1 e^{-zu}H^{p,0}_{p,p}\left(1-u\Big|^{(A,\beta_p-A)}_{(A,\alpha_p-A)}\right)du. \tag{4.61}$$

Since the function $H^{p,0}_{p,p}\left(1-u\Big|^{(A,\beta_p-A)}_{(A,\alpha_p-A)}\right)$ is non-negative and increasing on $(0,1)$, we deduce that the hypothesis of Theorem 2.1.7 in [29] is fulfilled. □

It is worth mentioning that a complete description of location and asymptotic behavior of zeros of the Wright function for all values of its parameters was presented by Y. Luchko in [19].

Remark 7. *We suppose that the hypotheses of Theorem 1 and the function $H^{p,0}_{q,p}[.]$ is non-negative (or the condition of (H_1^n) are satisfies). Then, by repeating the same calculations as above, we deduce that all the roots of the Fox-Wright function ${}_p\Psi_q\left[^{(\beta_p,B_q)}_{(\alpha_p,A_p)}\Big|z\right]$ are in the left-hand half-plane $\Re z\leq 0$.*

4.3. **Extended Luke's inequalities.** Our aim in the this section is to present two-sided exponential inequalities for the Fox-Wright function. As an application, we gave a generalization of Luke's inequalities.

Theorem 9. *Under the hypotheses (H_2) of Theorem 2, we get*

$$\psi_{0,0}e^{-\psi_{0,1}\psi_{0,0}^{-1}z}\leq {}_p\Psi_p\left[^{(\alpha_p,A)}_{(\beta_q,A)}\Big|-z\right]\leq\psi_{0,0}-\psi_{0,1}(1-e^{-z}),\ z\in\mathbb{R}. \tag{4.62}$$

Proof. We recall the Jensen's integral inequality [25, Chap. I, Eq. (7.15)],

$$\varphi\left(\int_a^b f(s)d\mu(s)\Big/\int_a^b d\mu(s)\right)\leq\int_a^b\varphi(f(s))d\mu(s)\Big/\int_a^b d\mu(s), \tag{4.63}$$

if φ is convex and f is integrable with respect to a probability measure μ. Letting $\varphi_z(s)=e^{-zt},\ f(t)=t,$ and

$$d\mu(t)=H^{p,0}_{p,p}\left(t\Big|^{(A,\beta_p)}_{(A,\alpha_p)}\right)\frac{dt}{t}.$$

Thus,

$$\int_0^1 d\mu(t)=\frac{\prod_{i=1}^p\Gamma(\alpha_i)}{\prod_{j=1}^p\Gamma(\beta_j)},\ \text{and}\ \int_0^1 f(t)d\mu(t)=\frac{\prod_{i=1}^p\Gamma(\alpha_i+A)}{\prod_{j=1}^p\Gamma(\beta_j+A)},$$

and

$$\int_0^1\phi_z(f(t))d\mu(t)={}_p\Psi_p\left[^{(\alpha_p,A)}_{(\beta_q,A)}\Big|z\right].$$

This proves the lower bound asserted by Theorem 9. In order to demonstrate the upper bound, we will apply the converse Jensen inequality, due to Lah and Ribarić, which reads as follows. Set

$$A(f)=\int_m^M f(s)d\sigma(s)\Big/\int_m^M d\sigma(s),$$

where σ is a non-negative measure and f is a continuous function. If $-\infty<m<M<\infty$ and φ is convex on $[m,M]$, then according to [27, Theorem 3.37]

$$(M-m)A(\varphi(f))\leq(M-A(f))\varphi(m)+(A(f)-m)\varphi(M). \tag{4.64}$$

Setting $\varphi_z(t)=e^{-zt},\ d\sigma(t)=d\mu(t),\ f(s)=s$ and $[m,M]=[0,1]$, we complete the proof of the upper bound in (4.62). □

Corollary 5. *Let $\lambda>0$ and under the conditions of hypotheses (H_2) of Theorem 2, then the following two-sided inequality holds true:*

$$\frac{\psi_{0,0}\Gamma(\lambda)}{\left(1+\frac{\psi_{0,1}}{\psi_{0,0}}z\right)^\lambda}\leq{}_{p+1}\Psi_p\left[^{(\lambda,1),(\alpha_p,A)}_{(\beta_q,A)}\Big|-z\right]\leq\Gamma(\lambda)\left[\psi_{0,0}-\psi_{0,1}\left(1-\frac{1}{(1+z)^\lambda}\right)\right],\ z>0. \tag{4.65}$$

Proof. Multiply inequalities (4.62) by $e^{-t}t^{\lambda-1}$, and integrate using the integral representation [28, Eq. (7)]

$$\int_0^\infty e^{-t}t^{\lambda-1}{}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -zt\right]dt = {}_{p+1}\Psi_q\left[{}^{(\lambda,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right] \tag{4.66}$$

and make use of the following known formula

$$\int_0^\infty t^\lambda e^{-\sigma t}dt = \frac{\Gamma(\lambda+1)}{\sigma^{\lambda+1}},$$

where $\lambda > -1$ and $\sigma > 0$. This completes the proof of the two-sided inequalities (4.65) asserted by Corollary 5. □

Remark 8. *Suppose that the hypotheses (H_2) of Theorem 2 are satisfied and taking in (4.65) the value $A = 1$ and using the identities (1.4), we re-obtain the Luke's inequalities for the hypergeometric function ${}_{p+1}F_p$: (see* [21, Theorem 13, Eq. (4.20)]*)*

$$\frac{1}{(1+\theta z)^\sigma} \leq {}_{p+1}F_p\left[{}^{\sigma,\alpha_1,\ldots,\alpha_p}_{\beta_1,\ldots,\beta_p}\Big| -z\right] \leq 1-\theta+\frac{\theta}{(1+x)^\sigma},\ \left(\theta=\prod_{j=1}^p\frac{\alpha_j}{\beta_j},\ z>0.\right) \tag{4.67}$$

Example 7. *The following two–sided inequality*

$$\frac{1}{\left(1+\frac{\Gamma(c)\Gamma(b+\tau)}{\Gamma(b)\Gamma(c+\tau)}z\right)^a} \leq {}_2\varphi_1^\tau(a,b,c,z) \leq \left[1-\frac{\Gamma(c)\Gamma(b+\tau)}{\Gamma(b)\Gamma(c+\tau)}\left(1-\frac{1}{(1+z)^a}\right)\right] \tag{4.68}$$

is valid for all $a, b, c, \tau > 0$, such that $c > b$.

Remark 9. *By repeating the same calculations in the proof of the Theorem 9 and Corollary (5), we present a generalization of the inequalities (4.62) and (4.65), respectively. Suppose that the hypotheses of Theorem 1 such that the function $H^{p,0}_{q,p}[.]$ is non-negative (or the hypotheses (H_1^n) be satisfied), then the following inequalities holds:*

$$\psi_{0,0}e^{-\psi_{0,1}\psi_{0,0}^{-1}z} \leq {}_p\Psi_q\left[{}^{(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right] \leq \psi_{0,0}-\frac{\psi_{0,1}}{\rho}(1-e^{-\rho z}),\ z\in\mathbb{R}, \tag{4.69}$$

and

$$\frac{\psi_{0,0}\Gamma(\lambda)}{\left(1+\frac{\psi_{0,1}}{\psi_{0,0}}z\right)^\lambda} \leq {}_{p+1}\Psi_p\left[{}^{(\lambda,1),(\alpha_p,A_p)}_{(\beta_q,B_q)}\Big| -z\right] \leq \Gamma(\lambda)\left[\psi_{0,0}-\frac{\psi_{0,1}}{\rho}\left(1-\frac{1}{(1+\rho z)^\lambda}\right)\right],\ z>0,\ \lambda>0. \tag{4.70}$$

4.4. **New inequalities for the generalized Mathieu's series.** The generalized Mathieu series is defined by [37]:

$$S_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = S_\mu^{(\alpha,\beta)}(r;\{a_k\}_{k=0}^\infty) = \sum_{k=1}^\infty \frac{2a_k^\beta}{(r^2+a_k^\alpha)^\mu},\ (r,\alpha,\beta,\mu>0), \tag{4.71}$$

where it is tacitly assumed that the positive sequence

$$\mathbf{a} = (a_k)_k,\ \text{such that}\ \lim_{k\longrightarrow\infty} a_k = \infty,$$

is so chosen that the infinite series in the definition (4.71) converges, that is, that the following auxiliary series:

$$\sum_{k=0}^\infty \frac{1}{a_k^{\mu\alpha-\beta}}$$

is convergent.

Theorem 10. *Let $\alpha, \beta, \nu, \mu > 0$ such that $\nu(\mu\alpha-\beta) > 1$ and $\nu\alpha = 1$. Then the following inequalities holds true:*

$$L \leq S_\mu^{(\alpha,\beta)}(r;\{k^\nu\}_{k=1}^\infty) \leq R,\ r>0, \tag{4.72}$$

where

$$L = 2\zeta(\nu(\mu\alpha-\beta), \frac{\mu}{\nu(\mu\alpha-\beta)}r^2+1),$$

and

$$R = 2\left(1-\frac{\mu}{\nu(\mu\alpha-\beta)}\right)\zeta(\nu(\mu\alpha-\beta))+\frac{2\mu}{\nu(\mu\alpha-\beta)}\zeta(\nu(\mu\alpha-\beta),r^2+1),$$

and $\zeta(s,a)$ is the Hurwitz Zeta Function defined by:

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s},\ \Re(s)>1.$$

Proof. We make use the representation integral for the Mathieu's series [33],

$$S_{\mu}^{(\alpha,\beta)}(r;\{k^\nu\}_{k=1}^{\infty})=\frac{2}{\Gamma(\mu)}\int_0^\infty \frac{x^{\nu(\mu\alpha-\beta)-1}}{e^x-1}\,{}_1\Psi_1\left(\left.{(\mu,1)\atop(\nu(\mu\alpha-\beta),\nu\alpha)}\right|-r^2x^{\nu\alpha}\right)dx,$$

with (4.69) and using the following formula [10, Eq. 8, p.313]

$$\int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx=\Gamma(s)\zeta(s,a),\ \Re(s)>1,\ \Re(a)>0,$$

we obtain the inequalities (4.72) asserted by Theorem 10. □

Corollary 6. *Assume that $\alpha,\beta,\nu,\mu>0$ such that $\nu(\mu\alpha-\beta)>2$ and $\nu\alpha=1$. Then*

$$L_1\le S_{\mu}^{(\alpha,\beta)}(r;\{k^\nu\}_{k=1}^{\infty})\le R_1,\ r>0, \tag{4.73}$$

where

$$L_1=\frac{2e^{-(\nu(\mu\alpha-\beta)-1)\psi\left(\frac{\mu r^2}{\nu(\mu\alpha-\beta)}+\frac{3}{2}\right)}}{\nu(\mu\alpha-\beta)-1}$$

and

$$R_1=2\left(1-\frac{\mu}{\nu(\mu\alpha-\beta)}\right)\frac{e^{(\nu(\mu\alpha-\beta)-1)\gamma}}{\nu(\mu\alpha-\beta)-1}+\frac{2\mu}{\nu(\mu\alpha-\beta)}\frac{e^{-(\nu(\mu\alpha-\beta)-1)\psi(r^2+1)}}{\nu(\mu\alpha-\beta)-1}$$

with γ is Euler-Mascheroni constant and ψ is the digamma function.

Proof. The result follows from Theorem 10 combined with [3, Theorem 3.1]. □

4.5. **Some inequalities involving the Riemann zeta function.** In this section, we establish some new various inequalities for the Riemann zeta function.

Theorem 11. *The Riemann zeta function satisfies the bounds*

$$L(\mu)\le\zeta(\mu)\le R(\mu),\quad \mu>1 \tag{4.74}$$

where

$$L(\mu)=\frac{e^{(\mu-1)\gamma}}{\mu-1},\ \ R(\mu)=2^{2(\mu-1)}e^{-2(\mu-1)}R(\mu).$$

Proof. Letting $\alpha=\beta$ and tends r to 0 in (4.73), we get

$$\frac{e^{-(\mu-2)\psi(1)}}{\mu-2}<\zeta(\mu-1)<\left(1-\frac{\mu}{\mu-1}\right)\frac{e^{-(\mu-2)\psi(3/2)}}{\mu-2}+\frac{\mu}{\mu-1}\frac{e^{-(\mu-2)\psi(3/2)}}{\mu-2},\ \ \mu>2, \tag{4.75}$$

and using the fact that $\psi(1)=-\gamma$ and $\psi(3/2)=2-2\log(2)-\gamma$, we obtain the desired result. □

Remark 10. *We note that the right hand side of inequalities (4.74), i.e*

$$\zeta(\mu)<R(\mu), \tag{4.76}$$

is not new, it was proved by G. Bastien and M. Rogalsk [2, Proposition 3]. *This, the right hand side of inequalities (4.73) give a generalization of inequality (4.76) and Theorem 11 gives the converse of (4.76). Moreover, in* [3, Remark 4.1], *Batir proved the following inequality*

$$e^{-2(\mu-1)}R(\mu)\le\zeta(\mu). \tag{4.77}$$

So, its clear that the left hand side of inequalities (4.74) is better than the above inequality. This is justified by the following inequality $1<4^{\mu-1}$.

Remark 11. *In* [32, Theorem 6, Eq. (4.3)] *Srivastava et al. was proved the following upper bound for the Riemann zeta function*

$$\zeta(2\mu)\le R_2(\mu)=\sqrt{\frac{3\pi}{2}}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1/2)},\ \mu\ge1. \tag{4.78}$$

The numerical computation in Table 1 shows the superiority of (4.74) over (4.78).

μ	$\zeta(2\mu)$	$R(2\mu)$	$R_2(\mu)$
1	1.64	1.76	2.44
3/2	1.20	1.56	2.88
2	*1.08*	*1.84*	*3.26*
5/2	1.06	2.44	3.60

Table. 1

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