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On the standing waves of the NLS-log equation with a point interaction on a star graph

Nataliia Goloshchapova

University of São Paulo, Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brazil

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ABSTRACT

We study the nonlinear Schrödinger equation with logarithmic nonlinearity on a star graph $\mathcal{G}$. At the vertex an interaction occurs described by a boundary condition of delta type with strength $\alpha \in \mathbb{R}$. We investigate the orbital stability and the spectral instability of the standing wave solutions $e^{i\omega t}\mathbf{\Phi}(x)$ to the equation when the profile $\mathbf{\Phi}(x)$ has mixed structure (i.e. has bumps and tails). In our approach we essentially use the extension theory of symmetric operators by Krein–von Neumann, and the analytic perturbations theory.

1. Introduction

The logarithmic Schrödinger equation

$$i\partial_t u(t,x) + \Delta u(t,x) + u(t,x)\mathrm{Log}|u(t,x)|^2 = 0, \quad u(t,x): \mathbb{R}\times\mathbb{R}^n \to \mathbb{C},\ n \geq 1,$$

admits applications to quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose–Einstein condensation (BEC). This equation has been proposed by Bialynicki-Birula and Mycielski (see [13]) in order to obtain a nonlinear equation which helped to quantify departures from the strictly linear regime, preserving in any number of dimensions some fundamental aspects of quantum mechanics, such as separability and additivity of total energy of noninteracting subsystems.

In the present paper we study the logarithmic Schrödinger equation on a star graph $\mathcal{G}$, i.e. N half-lines $(0,\infty)$ joined at the vertex $\nu = 0$. Namely, on $\mathcal{G}$ we consider the following nonlinear Schrödinger equation with δ-interaction (NLS-log-δ equation)

$$i\partial_t \mathbf{U}(t,x) - \mathbf{H}_\delta^\alpha \mathbf{U}(t,x) + \mathbf{U}(t,x)\mathrm{Log}|\mathbf{U}(t,x)|^2 = 0, \tag{1.1}$$

E-mail address: nataliia@ime.usp.br.

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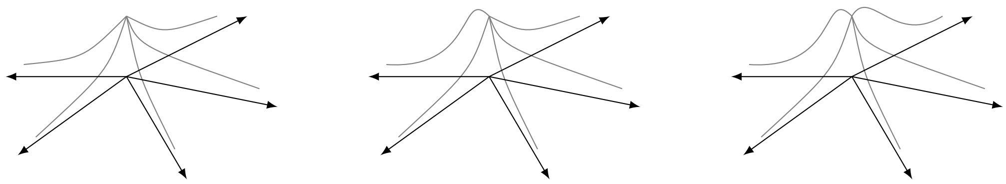

Fig. 1. $\alpha < 0$.

where $\mathbf{U}(t,x) = (u_j(t,x))_{j=1}^N : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}^N$, nonlinearity acts componentwise, i.e. $(\mathbf{U}\mathrm{Log}|\mathbf{U}|^2)_j = u_j \mathrm{Log}|u_j|^2$, and $\mathbf{H}_\delta^\alpha$ is the self-adjoint operator on $L^2(\mathcal{G})$ defined for $\mathbf{V} = (v_j)_{j=1}^N$ by

$$\begin{aligned}(\mathbf{H}_\delta^\alpha \mathbf{V})(x) &= \left(-v_j''(x)\right)_{j=1}^N, \quad x > 0,\\ \mathrm{dom}(\mathbf{H}_\delta^\alpha) &= \left\{\mathbf{V} \in H^2(\mathcal{G}) : v_1(0) = \dots = v_N(0),\ \sum_{j=1}^N v_j'(0) = \alpha v_1(0)\right\}.\end{aligned} \tag{1.2}$$

The condition (1.2) can be considered as an analog of δ-interaction condition for the Schrödinger operator on the line (see [3]), which justifies the name of the equation.

Equation (1.1) means that on each edge of the graph (i.e. on each half-line) we have

$$i\partial_t u_j(t,x) + \partial_x^2 u_j(t,x) + u_j(t,x)\mathrm{Log}|u_j(t,x)|^2 = 0,\ x > 0,\ j \in \{1,\dots,N\},$$

moreover, the vectors $\mathbf{U}(t,0) = (u_j(t,0))_{j=1}^N$ and $\mathbf{U}'(t,0) = (u_j'(t,0))_{j=1}^N$ satisfy conditions in (1.2).

The quantum graphs (star graphs equipped with a linear Hamiltonian $\mathbf{H}$) have been a very developed subject in the last couple of decades. They give simplified models in mathematics, physics, chemistry, and engineering, when one considers propagation of waves of various types through a quasi one-dimensional system that looks like a thing neighborhood of a graph (see [11,19,25,27] for details and references).

The nonlinear PDEs on graphs have been actively studied in the last ten years in the context of existence, stability, and propagation of solitary waves (see [15,29] for the references).

The main purpose of this work is to study the stability properties of the standing wave solutions

$$\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}(x) = \left(e^{i\omega t}\varphi_j(x)\right)_{j=1}^N,$$

to NLS-log-δ equation (1.1). Analogous problem has been considered for the NLS-log-δ equation on the line (see [5,8]). Similarly to the case of the NLS equation with power nonlinearity investigated in [1], it can be shown that all possible profiles $\mathbf{\Phi}(x)$ belong to the specific family of $\left[\frac{N-1}{2}\right] + 1$ vector functions $\mathbf{\Phi}_k^\alpha = (\varphi_{k,j}^\alpha)_{j=1}^N,\ k = 0,\dots,\left[\frac{N-1}{2}\right]$, given by

$$\varphi_{k,j}^\alpha(x) = \begin{cases} e^{\frac{\omega+1}{2}} e^{-\frac{(x-a_k)^2}{2}}, & j = 1,\dots,k;\\ e^{\frac{\omega+1}{2}} e^{-\frac{(x+a_k)^2}{2}}, & j = k+1,\dots,N,\end{cases} \quad \text{where } a_k = \frac{\alpha}{2k-N}. \tag{1.3}$$

In the case $\alpha < 0$ vector $\mathbf{\Phi}_k^\alpha = (\varphi_{k,j}^\alpha)_{j=1}^N$ has k *bumps* and $N-k$ *tails* (all possible profiles for $N = 5$ are given on Fig. 1). It is easily seen that $\mathbf{\Phi}_0^\alpha$ is the *N-tail profile*. Moreover, the N-tail profile is the only symmetric (i.e. invariant under permutations of the edges) profile. In the case $\alpha > 0$ vector $\mathbf{\Phi}_k^\alpha = (\varphi_{k,j}^\alpha)_{j=1}^N$ has k *tails* and $N-k$ *bumps* respectively (see Fig. 2).

In [10] the author proved (via variational approach) the orbital stability of the symmetric profile $\mathbf{\Phi}_0^\alpha$ in the energy space $W_\mathcal{E}(\mathcal{G})$ (defined in notation section below) under the restriction $\alpha < \alpha^* < 0$. Namely, the orbital stability follows from the fact that $\mathbf{\Phi}_0^\alpha$ is a minimizer of the action functional restricted to the Nehari

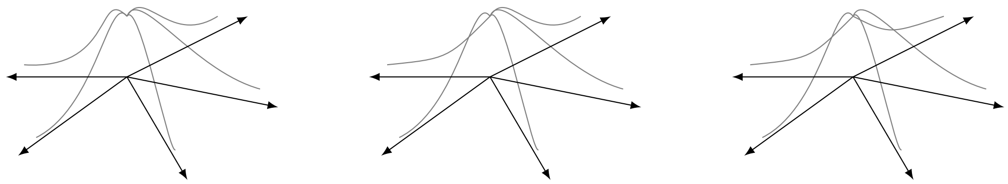

Fig. 2. $\alpha > 0$.

manifold. In the case of the NLS equation with power nonlinearity similar situation appears. In particular, the limitation on α was removed in [2]. Recently in [6] we proved orbital stability of $\mathbf{\Phi}_0^\alpha$ for $\alpha < 0$ in the weighted Hilbert space $W^1_{\mathcal{E}}(\mathcal{G})$ without restriction $\alpha < \alpha^* < 0$. The use of different space $W^1_{\mathcal{E}}(\mathcal{G})$ is due to application of the classical Lyapunov linearization procedure around the standing wave solution (i.e. the linearized operator associated to $\mathbf{\Phi}_0^\alpha$ has to be rigorously defined in an appropriate Hilbert space).

The main result of this paper is the following stability/instability theorem for the rest of the profiles $\mathbf{\Phi}_k^\alpha$, $k \neq 0$.

Theorem 1.1. *For any $k = 1, \ldots, \left[\frac{N-1}{2}\right]$, $\omega \in \mathbb{R}$, the standing wave $e^{i\omega t}\mathbf{\Phi}_k^\alpha$ is spectrally unstable for $\alpha < 0$ and orbitally stable in $W^1_{\mathcal{E},k}(\mathcal{G})$ for $\alpha > 0$.*

To our knowledge, this is the first result on the stability/instability of the profiles $\mathbf{\Phi}_k^\alpha$ in the case $k \neq 0$. For $\alpha < 0$ they are called *excited states* due to the property $S(\mathbf{\Phi}_0^\alpha) < S(\mathbf{\Phi}_k^\alpha) < S(\mathbf{\Phi}_{k+1}^\alpha)$, where S is the action functional associated to equation (1.1). Stability of the excited states is itself very interesting problem since there are only few cases when excited states of the NLS equations are explicitly known.

It is worth noting that we do not use variational techniques. Our approach is purely analytical, and it is based on the extension theory of symmetric operators, the analytic perturbations theory, and the well-known approach by Grillakis, Shatah and Strauss. In particular, we generalize to the case of the star graph the approach elaborated in [26] for the NLS equation with power nonlinearity and δ-interaction.

Notation. Let L be a densely defined symmetric operator in a Hilbert space $\mathcal{H}$. The domain of L is denoted by $\mathrm{dom}(L)$. The *deficiency subspaces* and the *deficiency numbers* of L are denoted by $\mathcal{N}_\pm(L) := \ker(L^* \mp iI)$ and $n_\pm(L) := \dim \mathcal{N}_\pm(L)$ respectively. The number of negative eigenvalues of L (counting multiplicities) is denoted by $n(L)$ (*the Morse index*). The spectrum and the resolvent set of L are denoted by $\sigma(L)$ and $\rho(L)$ respectively.

We denote by $\mathcal{G}$ the star graph constituted by N half-lines attached to a common vertex $\nu = 0$. On the graph we define

$$L^2(\mathcal{G}) = \bigoplus_{j=1}^N L^2(\mathbb{R}_+), \quad H^1(\mathcal{G}) = \bigoplus_{j=1}^N H^1(\mathbb{R}_+), \quad H^2(\mathcal{G}) = \bigoplus_{j=1}^N H^2(\mathbb{R}_+).$$

For instance, the norm in $L^2(\mathcal{G})$ is defined by

$$||\mathbf{V}||^2_{L^2(\mathcal{G})} = \sum_{j=1}^N ||v_j||^2_{L^2(\mathbb{R}_+)}, \quad \mathbf{V} = (v_j)_{j=1}^N.$$

By $||\cdot||_2$ and $(\cdot,\cdot)$ we will denote the norm and the scalar product in $L^2(\mathcal{G})$.

We also denote by $L^2_k(\mathcal{G})$ and $\mathcal{E}$ the spaces

$$L^2_k(\mathcal{G}) = \left\{ \begin{array}{c} \mathbf{V} = (v_j)_{j=1}^N \in L^2(\mathcal{G}) : v_1(x) = \cdots = v_k(x), \\ v_{k+1}(x) = \cdots = v_N(x),\, x \in \mathbb{R}_+ \end{array} \right\},$$

$$\mathcal{E} = \{\mathbf{V} \in H^1(\mathcal{G}) :\, v_1(0) = \cdots = v_N(0)\}.$$

On $\mathcal{G}$ we define the following weighted Hilbert spaces

$$W^j(\mathcal{G}) = \bigoplus_{j=1}^N W^j(\mathbb{R}_+), \quad W^j(\mathbb{R}_+) = \{f \in H^j(\mathbb{R}_+) : x^j f \in L^2(\mathbb{R}_+)\},\ j \in \{1, 2\},$$

and the Banach space

$$W(\mathcal{G}) = \bigoplus_{j=1}^N W(\mathbb{R}_+),\ \text{ where } W(\mathbb{R}_+) = \{f \in H^1(\mathbb{R}_+) : |f|^2 \mathrm{Log}|f|^2 \in L^1(\mathbb{R}_+)\}.$$

Using the above notations, we define

$$W^1_{\mathcal{E}}(\mathcal{G}) = W^1(\mathcal{G}) \cap \mathcal{E}, \quad W_{\mathcal{E}}(\mathcal{G}) = W(\mathcal{G}) \cap \mathcal{E}, \quad W^1_{\mathcal{E},k}(\mathcal{G}) = W^1_{\mathcal{E}}(\mathcal{G}) \cap L^2_k(\mathcal{G}).$$

Moreover, $W'_{\mathcal{E}}(\mathcal{G})$ denotes the dual space for $W_{\mathcal{E}}(\mathcal{G})$.

2. Well-posedness

Below we prove the well-posedness of the Cauchy problem for (1.1) in the space $W^1_{\mathcal{E}}(\mathcal{G})$. In [10] the well-posedness was proved in the Banach space $W_{\mathcal{E}}(\mathcal{G})$. Namely, the author showed the following result.

Proposition 2.1. *For any $\mathbf{U}_0 \in W_{\mathcal{E}}(\mathcal{G})$ there is a unique solution $\mathbf{U} \in C(\mathbb{R}, W_{\mathcal{E}}(\mathcal{G})) \cap C^1(\mathbb{R}, W'_{\mathcal{E}}(\mathcal{G}))$ of (1.1) such that $\mathbf{U}(0) = \mathbf{U}_0$ and $\sup\limits_{t\in\mathbb{R}} ||\mathbf{U}(t)||_{W_{\mathcal{E}}(\mathcal{G})} < \infty$. Furthermore, the conservation of energy and charge holds, that is,*

$$E(\mathbf{U}(t)) = E(\mathbf{U}_0),\ \text{ and } Q(\mathbf{U}(t)) = ||\mathbf{U}(t)||_2^2 = ||\mathbf{U}_0||_2^2,$$

where the energy E is defined by

$$E(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||_2^2 - \tfrac{1}{2}\sum_{j=1}^N \int_0^\infty |v_j|^2 \,\mathrm{Log}\,|v_j|^2 dx + \tfrac{\alpha}{2}|v_1(0)|^2, \quad \mathbf{V} = (v_j)_{j=1}^N \in W_{\mathcal{E}}(\mathcal{G}). \tag{2.1}$$

Using the above result, we obtain the well-posedness in $W^1_{\mathcal{E}}(\mathcal{G})$.

Theorem 2.2. *If $\mathbf{U}_0 \in W^1_{\mathcal{E}}(\mathcal{G})$, there is a unique solution $\mathbf{U}(t)$ of (1.1) such that $\mathbf{U}(t) \in C(\mathbb{R}, W^1_{\mathcal{E}}(\mathcal{G}))$ and $\mathbf{U}(0) = \mathbf{U}_0$. Furthermore, the conservation of energy and charge holds. Moreover, if $\mathbf{U}_0 \in W^1_{\mathcal{E},k}(\mathcal{G})$, then the solution $\mathbf{U}(t)$ to the Cauchy problem also belongs to $W^1_{\mathcal{E},k}(\mathcal{G})$.*

Proof. By [8, Lemma 3.1], we get $W^1_{\mathcal{E}}(\mathcal{G}) \subset W_{\mathcal{E}}(\mathcal{G})$, and, therefore, $\mathbf{U}_0 \in W_{\mathcal{E}}(\mathcal{G})$. By Proposition 2.1, we get the uniqueness of the solution in $W_{\mathcal{E}}(\mathcal{G})$ and the conservation of energy and charge. Note also that the solution $\mathbf{U}(t)$ belongs to $W^1_{\mathcal{E}}(\mathcal{G})$ due to [18, Lemma 7.6.2].

Let us prove the continuity in t. Assume that $t_n \underset{n\to\infty}{\longrightarrow} t$. Arguing as in the proof of [17, Theorem 2.1], we can show that $\mathbf{U}(t) \in C(\mathbb{R}, H^1(\mathcal{G}))$ (one may also use continuous embedding $W_{\mathcal{E}}(\mathcal{G}) \hookrightarrow H^1(\mathcal{G})$). Hence, $\mathbf{U}(t_n) \underset{n\to\infty}{\longrightarrow} \mathbf{U}(t)$ in $H^1(\mathcal{G})$ and we can assume that $\mathbf{U}(t_n) \underset{n\to\infty}{\longrightarrow} \mathbf{U}(t)$ a.e. in $\mathcal{G}$. It remains to prove $\mathbf{U}(t) \in C(\mathbb{R}, L^2(x^2, \mathcal{G}))$ ($L^2(x^2, \mathcal{G})$ denotes weighted L^2-space). By [18, Lemma 7.6.2], we get continuity of the function $t \mapsto ||x\mathbf{U}(t)||_2^2$ on $\mathbb{R}$. Thus, $||x\mathbf{U}(t_n)||_2^2 \underset{n\to\infty}{\longrightarrow} ||x\mathbf{U}(t)||_2^2$. Having additionally almost everywhere convergence of $x\mathbf{U}(t_n)$ to $x\mathbf{U}(t)$, we get from Brezis–Lieb Lemma in [14]

$$||x\mathbf{U}(t_n) - x\mathbf{U}(t)||_2^2 \underset{n\to\infty}{\longrightarrow} 0,$$

which implies $\mathbf{U}(t) \in C(\mathbb{R}, W^1_{\mathcal{E}}(\mathcal{G}))$.

The last assertion of the theorem follows since the solution to the Cauchy problem for (1.1) has been obtained by approximation procedure in [10] (approximating sequence consists of the solutions to the Cauchy problem for the reduced nonlinear equation with Lipschitz continuous nonlinearity) and the evolution group $e^{-it\mathbf{H}_\delta^\alpha}$ preserves the space

$$\mathcal{E}_k = \mathcal{E} \cap L^2_k(\mathcal{G}).$$

See the proof of Lemma 2.3 in [7] (or Theorem 3.4 in [6]) for the detailed explanation of this fact. □

The properties of the energy functional E are essential for the investigation of the orbital stability. For example, in $\mathcal{E}$ the energy functional is ill-defined. From [16, Lemma 2.6] it follows that E is continuously differentiable in $W_{\mathcal{E}}(\mathcal{G})$. Below we prove its continuity in the space $W^1_{\mathcal{E}}(\mathcal{G})$.

Proposition 2.3. *The energy functional E defined by (2.1) is continuous in $W^1_{\mathcal{E}}(\mathcal{G})$.*

Proof. Let $\mathbf{V}_n \underset{n\to\infty}{\longrightarrow} \mathbf{V}$ in $W^1_{\mathcal{E}}(\mathcal{G})$. It is easily seen that

$$||\mathbf{V}_n'||_2^2 + \alpha|v_{1,n}(0)|^2 \underset{n\to\infty}{\longrightarrow} ||\mathbf{V}'||_2^2 + \alpha|v_1(0)|^2.$$

Hence, we need to prove the continuity of the nonlinearity part of the functional E. Basically we will use the following inequality (see [17, Lemma 2.4.3]) for $|f| \geq |g|$

$$\left||f|^2\mathrm{Log}|f|^2 - |g|^2\mathrm{Log}|g|^2\right| \leq \left(1 + |\mathrm{Log}|f|^2|\right)\left||f|^2 - |g|^2\right|. \tag{2.2}$$

To simplify the notation for the sets we write {condition} instead of $\{x \in \mathbb{R} : \text{condition}\}$. From (2.2) we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}_+} \left||v_{j,n}|^2\mathrm{Log}|v_{j,n}|^2 - |v_j|^2\mathrm{Log}|v_j|^2\right| dx \leq & \int\limits_{\{|v_{j,n}|\geq|v_j|\}} \left(1 + |\mathrm{Log}|v_{j,n}|^2|\right)\left||v_{j,n}|^2 - |v_j|^2\right| dx \\ & + \int\limits_{\{|v_{j,n}|\leq|v_j|\}} \left(1 + |\mathrm{Log}|v_j|^2|\right)\left||v_{j,n}|^2 - |v_j|^2\right| dx.\end{aligned} \tag{2.3}$$

Let us show that the right hand side of (2.3) tends to zero. We will estimate the first expression of the right hand side of (2.3) since the analysis for the second one is analogous. By the Cauchy–Schwarz inequality we get

$$\int\limits_{\{|v_{j,n}|\geq|v_j|\}} \left(1+|\mathrm{Log}|v_{j,n}|^2|\right)\left||v_{j,n}|^2-|v_j|^2\right|dx$$
$$\leq\left(\int\limits_{\{|v_{j,n}|\geq|v_j|\}}\left(1+|\mathrm{Log}|v_{j,n}|^2|\right)^2(|v_{j,n}|+|v_j|)^2dx\right)^{\frac{1}{2}}\left(\int\limits_{\mathbb{R}_+}(|v_{j,n}|-|v_j|)^2dx\right)^{\frac{1}{2}}. \tag{2.4}$$

Let $\varepsilon>0$ be such that $|\mathrm{Log}|v_{j,n}|^2|\leq\frac{1}{|v_{j,n}|^{1/2}}$ for $|v_{j,n}|<\varepsilon$. Observing that the set $\{|v_{j,n}|\geq\varepsilon\}$ is contained in some bounded interval (due to $v_{j,n}\in H^1(\mathbb{R}_+)$) and recalling that $v_{j,n}\in L^1(\mathbb{R}_+)$ (see the proof of Lemma 3.1 in [8]), we obtain

$$\begin{aligned}\int\limits_{\{|v_{j,n}|\geq|v_j|\}}\left(1+|\mathrm{Log}|v_{j,n}|^2|\right)^2(|v_{j,n}|+|v_j|)^2dx &\leq \int\limits_{\{|v_{j,n}|\geq|v_j|\}\cap\{|v_{j,n}|\geq\varepsilon\}} C_1|v_{j,n}|^2dx\\ &+\int\limits_{\{|v_{j,n}|\geq|v_j|\}\cap\{|v_{j,n}|<\varepsilon\}}\left(1+\frac{1}{|v_{j,n}|^{1/2}}\right)^2 4|v_{j,n}|^2dx\\ &\leq C_2+4\int\limits_{\mathbb{R}_+}\left(|v_{j,n}|^2+2|v_{j,n}|^{3/2}+|v_{j,n}|\right)dx<\infty,\end{aligned}$$

where C_1 and C_2 are positive constants. It is worth noting that $\int\limits_{\mathbb{R}_+}|v_{j,n}|^{3/2}dx<\infty$ follows from the Cauchy–Schwarz inequality and the inclusion $v_{j,n}\in L^1(\mathbb{R}_+)$.

Finally, from (2.3)–(2.4) we get

$$\int\limits_{\mathbb{R}_+}|v_{j,n}|^2\mathrm{Log}|v_{j,n}|^2dx\underset{n\to\infty}{\longrightarrow}\int\limits_{\mathbb{R}_+}|v_j|^2\mathrm{Log}|v_j|^2dx. \quad\square$$

3. The proof of the main theorem

Crucial role in the orbital stability analysis is played by the symmetries of NLS equation (1.1). The basic symmetry associated to the mentioned equation is phase-invariance (in particular, translational invariance does not hold due to the defect). Thus, it is reasonable to define orbital stability as follows.

Definition 3.1. The standing wave $\mathbf{U}(t,x)=e^{i\omega t}\mathbf{\Phi}(x)$ is said to be *orbitally stable* in a Banach space X if for any $\varepsilon>0$ there exists $\eta>0$ with the following property: if $\mathbf{U}_0\in X$ satisfies $||\mathbf{U}_0-\mathbf{\Phi}||_X<\eta$, then the solution $\mathbf{U}(t)$ of (1.1) with $\mathbf{U}(0)=\mathbf{U}_0$ exists for any $t\in\mathbb{R}$, and

$$\sup_{t\in\mathbb{R}}\inf_{\theta\in\mathbb{R}}||\mathbf{U}(t)-e^{i\theta}\mathbf{\Phi}||_X<\varepsilon.$$

Otherwise, the standing wave $\mathbf{U}(t,x)=e^{i\omega t}\mathbf{\Phi}(x)$ is said to be *orbitally unstable* in X.

Below we will define spectral stability/instability of $e^{i\omega t}\mathbf{\Phi}(x)$. We assume that $\mathbf{\Phi}(x)$ belongs to the family of profiles defined by (1.3). Change of variables $\mathbf{U}(t,x)=e^{i\omega t}(\mathbf{\Phi}(x)+\mathbf{V}(t,x))$ in (1.1) leads to

$$\partial_t\mathbf{V}(t,x)=\mathbf{A}\mathbf{V}(t,x)+F(\mathbf{V}(t,x)), \tag{3.1}$$

where $\mathbf{A}$ is the linearized operator defined by

$$\mathbf{A}\mathbf{V} = -i\left\{\mathbf{H}_\delta^\alpha\mathbf{V} + \omega\mathbf{V} - \mathbf{V}\mathrm{Log}\mathbf{\Phi}^2 - \mathbf{V} - \overline{\mathbf{V}}\right\},$$

and $F(\mathbf{V})$ is the nonlinear term given by

$$F(\mathbf{V}) = i\left\{(\mathbf{\Phi}+\mathbf{V})\mathrm{Log}|\mathbf{\Phi}+\mathbf{V}|^2 - \mathbf{V}\mathrm{Log}\mathbf{\Phi}^2 - \mathbf{V} - \overline{\mathbf{V}} - \mathbf{\Phi}\mathrm{Log}\mathbf{\Phi}^2\right\}.$$

Definition 3.2. The standing wave $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}(x)$ is said to be *spectrally stable* if

$$\sigma(\mathbf{A}) \subset i\mathbb{R}.$$

Otherwise, the standing wave $\mathbf{U}(t,x) = e^{i\omega t}\mathbf{\Phi}(x)$ is said to be *spectrally unstable*.

It is standard to show that $\sigma(\mathbf{A})$ is symmetric with respect to the real and imaginary axes (see, for instance, [22, Lemma 5.6]). Hence, it is equivalent to say that $e^{i\omega t}\mathbf{\Phi}(x)$ is spectrally unstable if $\sigma(\mathbf{A})$ contains some point λ with $\Re(\lambda) > 0$.

It is widely known that the spectral instability is a key prerequisite to show nonlinear (orbital) instability in numerous works (see [22,31] and references therein). However, it is highly nontrivial problem whether spectral instability implies orbital instability.

3.1. Stability framework

In this subsection we introduce basic objects of the stability framework. The action functional for the NLS-log-δ equation is given by

$$S(\mathbf{V}) = \tfrac{1}{2}||\mathbf{V}'||_2^2 + \tfrac{(\omega+1)}{2}||\mathbf{V}||_2^2 - \tfrac{1}{2}\sum_{j=1}^N\int_0^\infty |v_j|^2\mathrm{Log}|v_j|^2 dx + \tfrac{\alpha}{2}|v_1(0)|^2, \quad \mathbf{V}\in W_{\mathcal{E}}(\mathcal{G}).$$

The profiles $\mathbf{\Phi}_k^\alpha$ defined by (1.3) are the critical points of the action functional. Indeed, for $\mathbf{U} = (u_j)_{j=1}^N, \mathbf{V} = (v_j)_{j=1}^N \in W_{\mathcal{E}}(\mathcal{G})$,

$$\begin{aligned} S'(\mathbf{U})\mathbf{V} &= \frac{d}{dt}S(\mathbf{U}+t\mathbf{V})|_{t=0} \\ &= \mathrm{Re}\left[\sum_{j=1}^N\left(\int_{\mathbb{R}_+} u_j'\overline{v_j'}dx - \int_{\mathbb{R}_+} u_j\overline{v_j}(\mathrm{Log}|u_j|^2-\omega)dx\right) + \alpha u_1(0)\overline{v_1(0)}\right]. \end{aligned}$$

Obviously $S'(\mathbf{\Phi}_k^\alpha) = \mathbf{0}$. Below we will use the notation $\mathbf{\Phi}_k := \mathbf{\Phi}_k^\alpha$.

The basic ingredient of stability study is the operator $\mathbf{A}_k$ introduced in (3.1) (index k means that we need to linearize equation around each $\mathbf{\Phi}_k$). To express $\mathbf{A}_k$ it is convenient to split $\mathbf{V}\in W_{\mathcal{E}}^1(\mathcal{G})$ into real and imaginary parts: $\mathbf{V} = \mathbf{V}_1 + i\mathbf{V}_2$. Then we get

$$\mathbf{A}_k\mathbf{V} = \mathbf{T}_{2,k}^\alpha\mathbf{V}_2 - i\mathbf{T}_{1,k}^\alpha\mathbf{V}_1,$$

where

$$\mathbf{T}^\alpha_{1,k} = \operatorname{diag}\left(\underset{\mathbf{1}}{-\frac{d^2}{dx^2} + V_k^-(x)}, \dots, \underset{\mathbf{k}}{-\frac{d^2}{dx^2} + V_k^-(x)}, \underset{\mathbf{k+1}}{-\frac{d^2}{dx^2} + V_k^+(x)}, \dots, \underset{\mathbf{N}}{-\frac{d^2}{dx^2} + V_k^+(x)}\right),$$

$$\mathbf{T}^\alpha_{2,k} = \operatorname{diag}\left(\underset{\mathbf{1}}{-\frac{d^2}{dx^2} + W_k^-(x)}, \dots, \underset{\mathbf{k}}{-\frac{d^2}{dx^2} + W_k^-(x)}, \underset{\mathbf{k+1}}{-\frac{d^2}{dx^2} + W_k^+(x)}, \dots, \underset{\mathbf{N}}{-\frac{d^2}{dx^2} + W_k^+(x)}\right),$$

$$\operatorname{dom}(\mathbf{T}^\alpha_{1,k}) = \operatorname{dom}(\mathbf{T}^\alpha_{2,k}) = \left\{\mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \dots = v_N(0), \sum_{j=1}^N v_j'(0) = \alpha v_1(0)\right\},$$

with $V_k^\pm(x) = \left(x \pm \frac{\alpha}{2k-N}\right)^2 - 3$ and $W_k^\pm(x) = \left(x \pm \frac{\alpha}{2k-N}\right)^2 - 1$. Finally, we get formally

$$\mathbf{A}_k = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \mathbf{H}^\alpha_k,$$

where $\mathbf{H}^\alpha_k = \begin{pmatrix} \mathbf{T}^\alpha_{1,k} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^\alpha_{2,k} \end{pmatrix}$, moreover, $\mathbf{0}$ and $\mathbf{I}$ are zero and identity $N \times N$ matrices. Observe also that $\mathbf{H}^\alpha_k$ is the self-adjoint operator associated with $S''(\mathbf{\Phi}_k)$ (see [1, Section 6] for details).

Noting that $\partial_\omega ||\mathbf{\Phi}_k||_2^2 > 0$, and combining [21, Theorem 3.5] with [22, Theorem 5.1], we can formulate the stability/instability theorem for the NLS-log-δ equation.

Theorem 3.3. *Let $\alpha \neq 0$, $k = 1, \dots, \left[\frac{N-1}{2}\right]$, and $n(\mathbf{H}^\alpha_k|_{L^2_k(\mathcal{G})})$ be the number of negative eigenvalues of $\mathbf{H}^\alpha_k$ in $L^2_k(\mathcal{G})$. Suppose also that*

1) $\ker(\mathbf{T}^\alpha_{2,k}) = \operatorname{span}\{\mathbf{\Phi}_k\}$,
2) $\ker(\mathbf{T}^\alpha_{1,k}) = \{\mathbf{0}\}$,
3) *the negative spectrum of $\mathbf{T}^\alpha_{1,k}$ and $\mathbf{T}^\alpha_{2,k}$ consists of a finite number of negative eigenvalues (counting multiplicities),*
4) *the rest of the spectrum of $\mathbf{T}^\alpha_{1,k}$ and $\mathbf{T}^\alpha_{2,k}$ is positive and bounded away from zero.*

Then the following assertions hold.

(i) *If $n(\mathbf{H}^\alpha_k|_{L^2_k(\mathcal{G})}) = 1$, then the standing wave $e^{i\omega t}\mathbf{\Phi}_k$ is orbitally stable in $W^1_{\mathcal{E},k}(\mathcal{G})$.*
(ii) *If $n(\mathbf{H}^\alpha_k|_{L^2_k(\mathcal{G})}) = 2$, then the standing wave $e^{i\omega t}\mathbf{\Phi}_k$ is spectrally unstable.*

Remark 3.4.

(i) Note that for the proof of item (i) continuity of the energy functional E shown in Proposition 2.3 is essential (see the proof of Theorem 3.5 in [21]).
(ii) In item (ii) we affirm only spectral instability since to show orbital instability we need to prove some additional nontrivial properties of the NLS-log-δ equation, for instance, some estimates for the semigroup $e^{t\mathbf{A}_k}$ generated by $\mathbf{A}_k$ (see [31, Lemmas 2 and 3], [22, Theorem 6.1]), or the property that the mapping data-solution associated to the NLS-log-δ equation is of class C^2 around $\mathbf{\Phi}_k$ (see [23, Section 2] for the general idea and [9] for the particular application). However, we conjecture that for the operator $\mathbf{A}_k$ so-called spectral mapping theorem holds (that is, $\sigma(e^{\mathbf{A}_k}) = e^{\sigma(\mathbf{A}_k)}$) which would imply mentioned estimates (see, for instance, the discussion in [20]).

3.2. Spectral properties of $\mathbf{T}^{\alpha}_{1,k}$ and $\mathbf{T}^{\alpha}_{2,k}$

Below we describe the spectrum of the operators $\mathbf{T}^{\alpha}_{1,k}$ and $\mathbf{T}^{\alpha}_{2,k}$ which will help us to verify the conditions of Theorem 3.3. Our ideas are based on the extension theory of symmetric operators and the perturbation theory.

The main result of this subsection is the following.

Theorem 3.5. *Let $\alpha \neq 0$, $k = 1, \dots, \left[\frac{N-1}{2}\right]$. Then the following assertions hold.*

(*i*) *If $\alpha < 0$, then $n(\mathbf{H}^{\alpha}_{k}|_{L^2_k(\mathcal{G})}) = 2$.*
(*ii*) *If $\alpha > 0$, then $n(\mathbf{H}^{\alpha}_{k}|_{L^2_k(\mathcal{G})}) = 1$.*

The proof of Theorem 3.5 is an immediate consequence of Propositions 3.6 and 3.12 below.

Proposition 3.6. *Let $\alpha \neq 0$, $k = 1, \dots, \left[\frac{N-1}{2}\right]$. Then the following assertions hold.*

(*i*) $\ker(\mathbf{T}^{\alpha}_{2,k}) = \mathrm{span}\{\mathbf{\Phi}_k\}$ *and* $\mathbf{T}^{\alpha}_{2,k} \geq 0$.
(*ii*) $\ker(\mathbf{T}^{\alpha}_{1,k}) = \{\mathbf{0}\}$.
(*iii*) *The spectrum of the operators $\mathbf{T}^{\alpha}_{1,k}$ and $\mathbf{T}^{\alpha}_{2,k}$ is discrete.*

Proof. (*i*) It is clear that $\mathbf{\Phi}_k \in \ker(\mathbf{T}^{\alpha}_{2,k})$. To show the equality $\ker(\mathbf{T}^{\alpha}_{2,k}) = \mathrm{span}\{\mathbf{\Phi}_k\}$ let us note that any $\mathbf{V} = (v_j)_{j=1}^N \in W^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + ((x \mp \tfrac{\alpha}{2k-N})^2 - 1)v_j = \frac{-1}{\varphi_{k,j}} \frac{d}{dx}\left[\varphi_{k,j}^2 \frac{d}{dx}\left(\frac{v_j}{\varphi_{k,j}}\right)\right], \quad x > 0,$$

where $\varphi_{k,j} = \varphi^{\alpha}_{k,j}$ is defined by (1.3), and the sign $-$ $(+)$ corresponds to $j = 1, \dots, k$ $(j = k+1, \dots, N)$. Thus, for $\mathbf{V} \in \mathrm{dom}(\mathbf{T}^{\alpha}_{2,k})$ we obtain

$$\begin{aligned}(\mathbf{T}^{\alpha}_{2,k}\mathbf{V}, \mathbf{V}) &= \sum_{j=1}^{N} \int_0^{\infty} (\varphi_{k,j})^2 \left[\frac{d}{dx}\left(\frac{v_j}{\varphi_{k,j}}\right)\right]^2 dx + \sum_{j=1}^{N}\left[-v_j' v_j + v_j^2 \frac{\varphi_{k,j}'}{\varphi_{k,j}}\right]_0^{\infty} \\ &= \sum_{j=1}^{N} \int_0^{\infty} (\varphi_{k,j})^2 \left[\frac{d}{dx}\left(\frac{v_j}{\varphi_{k,j}}\right)\right]^2 dx + \sum_{j=1}^{N}\left[v_j'(0)v_j(0) - v_j^2(0)\frac{\varphi_{k,j}'(0)}{\varphi_{k,j}(0)}\right].\end{aligned}$$

Using boundary conditions (1.2), we get

$$\sum_{j=1}^{N}\left[v_j'(0)v_j(0) - v_j^2(0)\frac{\varphi_{k,j}'(0)}{\varphi_{k,j}(0)}\right] = 0,$$

which induces $(\mathbf{T}^{\alpha}_{2,k}\mathbf{V}, \mathbf{V}) > 0$ for $\mathbf{V} \in \mathrm{dom}(\mathbf{T}^{\alpha}_{2,k}) \setminus \mathrm{span}\{\mathbf{\Phi}_k\}$. Therefore, $\ker(\mathbf{T}^{\alpha}_{2,k}) = \mathrm{span}\{\mathbf{\Phi}_k\}$.

(*ii*) Concerning the kernel of $\mathbf{T}^{\alpha}_{1,k}$, the only $L^2(\mathbb{R}_+)$-solution of the equation

$$-v_j'' + ((x \mp \tfrac{\alpha}{2k-N})^2 - 3)v_j = 0$$

is $v_j = \varphi_{k,j}'$ up to a factor. Thus, any element of $\ker(\mathbf{T}^{\alpha}_{1,k})$ has the form $\mathbf{V} = (v_j)_{j=1}^N = (c_j\varphi_{k,j}')_{j=1}^N$, $c_j \in \mathbb{R}$. Continuity condition $v_1(0) = \dots = v_N(0)$ induces that $c_1 = \dots = c_N$, i.e.

$$v_j(x) = c\begin{cases} -\varphi'_{k,j}, & j = 1, \dots, k; \\ \varphi'_{k,j}, & j = k+1, \dots, N, \end{cases} \qquad c \in \mathbb{R}.$$

Condition $\sum\limits_{j=1}^{N} v'_j(0) = \alpha v_j(0)$ is equivalent to the equality $c\left(\frac{\alpha^2}{(N-2k)^2} - 1\right) = c\frac{\alpha^2}{(N-2k)^2}$. The last one induces that $c = 0$, and, therefore, $\mathbf{V} \equiv \mathbf{0}$.

(iii) With slight modifications we can repeat the proof of [12, Theorem 3.1, Chapter II] to show that the spectrum of $\mathbf{T}^{\alpha}_{1,k}$ is discrete since $\lim\limits_{x\to+\infty}(x \mp \frac{\alpha}{2k-N})^2 - 3 = +\infty$, i.e. $\sigma(\mathbf{T}^{\alpha}_{1,k}) = \sigma_p(\mathbf{T}^{\alpha}_{1,k}) = \{\mu_{1,j}\}_{j\in\mathbb{N}}$. In particular, we have the following distribution of the eigenvalues

$$\mu_{1,1} < \mu_{1,2} < \cdots < \mu_{1,j} < \cdots,$$

with $\mu_{1,j} \to +\infty$ as $j \to +\infty$. $\square$

Below using the perturbation theory we will study $n(\mathbf{T}^{\alpha}_{1,k})$ in the space $L^2_k(\mathcal{G})$ for any $k = 1, \dots, \left[\frac{N-1}{2}\right]$. The following lemma states the analyticity of the family of operators $\mathbf{T}^{\alpha}_{1,k}$.

Lemma 3.7. *As a function of α, $(\mathbf{T}^{\alpha}_{1,k})$ is real-analytic family of self-adjoint operators of type (B) in the sense of Kato.*

Proof. By [24, Theorem VII-4.2], it suffices to note that the family of bilinear forms $(B^{\alpha}_{1,k})$ defined by

$$B^{\alpha}_{1,k}(\mathbf{F}, \mathbf{G}) = \sum_{j=1}^{N} \int\limits_{\mathbb{R}_+} f'_j g'_j dx + \sum_{j=1}^{k} \int\limits_{\mathbb{R}_+} f_j g_j \left(\left(x - \tfrac{\alpha}{2k-N}\right)^2 - 3\right) dx$$
$$+ \sum_{j=k+1}^{N} \int\limits_{\mathbb{R}_+} f_j g_j \left(\left(x + \tfrac{\alpha}{2k-N}\right)^2 - 3\right) dx + \alpha f_1(0) g_1(0)$$

is real-analytic of type (B). $\square$

Observe that $\mathbf{T}^{\alpha}_{1,k}$ tends (in the generalized sense) to the following self-adjoint matrix Schrödinger operator on $L^2(\mathcal{G})$ with the Kirchhoff condition at $\nu = 0$ as $\alpha \to 0$

$$\begin{aligned} &\mathbf{T}^0_1 = \left(\left(-\frac{d^2}{dx^2} + x^2 - 3\right)\delta_{i,j}\right), \\ &\mathrm{dom}(\mathbf{T}^0_1) = \left\{\mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0),\ \sum_{j=1}^{N} v'_j(0) = 0\right\}. \end{aligned} \tag{3.2}$$

As we intend to study negative spectrum of $\mathbf{T}^{\alpha}_{1,k}$, we first need to describe spectral properties of $\mathbf{T}^0_1$.

Theorem 3.8. *Let $\mathbf{T}^0_1$ be defined by (3.2) and $k = 1, \dots, \left[\frac{N-1}{2}\right]$. Then*

(i) $\ker(\mathbf{T}^0_1) = \mathrm{span}\{\hat{\mathbf{\Phi}}_{0,1}, \dots, \hat{\mathbf{\Phi}}_{0,N-1}\}$, *where*

$$\hat{\mathbf{\Phi}}_{0,j} = (0, \dots, 0, \underset{\mathbf{j}}{\varphi'_0}, \underset{\mathbf{j+1}}{-\varphi'_0}, 0, \dots, 0),\ \varphi_0 = e^{-\frac{x^2}{2}}.$$

(*ii*) *In the space* $L^2_k(\mathcal{G})$ *we have* $\ker(\mathbf{L}_1^0)=\operatorname{span}\{\widetilde{\mathbf{\Phi}}_{0,k}\}$, *i.e.* $\ker(\mathbf{T}_1^0|_{L^2_k(\mathcal{G})})=\operatorname{span}\{\widetilde{\mathbf{\Phi}}_{0,k}\}$, *where*

$$\widetilde{\mathbf{\Phi}}_{0,k}=\Big(\underset{\mathbf{1}}{\tfrac{N-k}{k}\varphi_0'},\dots,\underset{\mathbf{k}}{\tfrac{N-k}{k}\varphi_0'},\underset{\mathbf{k+1}}{-\varphi_0'},\dots,\underset{\mathbf{N}}{-\varphi_0'}\Big). \tag{3.3}$$

(*iii*) *The operator* $\mathbf{T}_1^0$ *has one simple negative eigenvalue in* $L^2(\mathcal{G})$, *i.e.* $n(\mathbf{T}_1^0)=1$. *Moreover,* $\mathbf{T}_1^0$ *has one simple negative eigenvalue in* $L^2_k(\mathcal{G})$ *for any* k, *i.e.* $n(\mathbf{T}_1^0|_{L^2_k(\mathcal{G})})=1$.

(*iv*) *The positive part of the spectrum of* $\mathbf{T}_1^0$ *is bounded away from zero.*

Proof. The proof can be found in [6]. We repeat it for the reader's convenience.

(*i*) The only $L^2(\mathbb{R}_+)$-solution to the equation

$$-v_j''+(x^2-3)v_j=0$$

is $v_j=\varphi_0'$ (up to a factor). Thus, any element of $\ker(\mathbf{T}_1^0)$ has the form $\mathbf{V}=(v_j)_{j=1}^N=(c_j\varphi_0')_{j=1}^N$, $c_j\in\mathbb{R}$. It is easily seen that the continuity condition is satisfied since $\varphi_0'(0)=0$. Condition $\sum\limits_{j=1}^N v_j'(0)=0$ gives rise to $(N-1)$-dimensional kernel of $\mathbf{T}_1^0$. Finally, note that functions $\hat{\mathbf{\Phi}}_{0,j}$, $j=1,\dots,N-1$, form basis there.

(*ii*) Arguing as in the previous item, we can show that $\ker(\mathbf{T}_1^0)$ is one-dimensional in $L^2_k(\mathcal{G})$, and it is spanned by $\widetilde{\mathbf{\Phi}}_{0,k}$.

(*iii*) Denote $\boldsymbol{t}_0=\Big(\Big(-\frac{d^2}{dx^2}+x^2-3\Big)\delta_{k,j}\Big)$. First, we will show that the operator $\mathbf{T}_0$ defined by

$$\mathbf{T}_0=\boldsymbol{t}_0,\ \operatorname{dom}(\mathbf{T}_0)=\left\{\mathbf{V}\in W^2(\mathcal{G}): v_1(0)=\dots=v_N(0)=0,\sum_{j=1}^N v_j'(0)=0\right\}$$

is non-negative. The proof follows from the identity

$$-v_j''+(x^2-3)v_j=\frac{-1}{\varphi_0'}\frac{d}{dx}\left[(\varphi_0')^2\frac{d}{dx}\left(\frac{v_j}{\varphi_0'}\right)\right],\quad x>0,$$

for any $\mathbf{V}=(v_j)_{j=1}^N\in W^2(\mathcal{G})$. Using the above equality and integrating by parts, we get for $\mathbf{V}\in\operatorname{dom}(\mathbf{T}_0)$

$$\begin{aligned}(\mathbf{T}_0\mathbf{V},\mathbf{V})&=\sum_{j=1}^N\int_0^\infty(\varphi_0')^2\left[\frac{d}{dx}\left(\frac{v_j}{\varphi_0'}\right)\right]^2dx+\sum_{j=1}^N\left[-v_j'v_j+v_j^2\frac{\varphi_0''}{\varphi_0'}\right]_0^\infty\\&=\sum_{j=1}^N\int_0^\infty(\varphi_0')^2\left[\frac{d}{dx}\left(\frac{v_j}{\varphi_0'}\right)\right]^2dx\geq 0.\end{aligned}$$

Note that the equality

$$\sum_{j=1}^N\left[-v_j'v_j+v_j^2\frac{\varphi_0''}{\varphi_0'}\right]_0^\infty=0$$

follows from the condition $v_j(0)=0$ and the fact that $x=0$ is the first-order zero for $\varphi_0'(x)$ (i.e. $\varphi_0''(0)\neq 0$).

Next we need to prove that $n_\pm(\mathbf{T}_0)=1$. First, we establish the scale of Hilbert spaces associated with the self-adjoint non-negative operator (see [4, Section I, §1.2.2])

$$\mathbf{T} = \boldsymbol{t}_0 + 3\mathbf{I}, \quad \mathrm{dom}(\mathbf{T}) = \left\{ \mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0), \sum_{j=1}^{N} v_j'(0) = 0 \right\}.$$

Define for $s \geq 0$ the space

$$\mathfrak{H}_s(\mathbf{T}) = \left\{ \mathbf{V} \in L^2(\mathcal{G}) : \|\mathbf{V}\|_{s,2} = \left\| (\mathbf{T} + \mathbf{I})^{s/2} \mathbf{V} \right\|_2 < \infty \right\}.$$

The space $\mathfrak{H}_s(\mathbf{T})$ with norm $\|\cdot\|_{s,2}$ is complete. The dual space of $\mathfrak{H}_s(\mathbf{T})$ is denoted by $\mathfrak{H}_{-s}(\mathbf{T}) = \mathfrak{H}_s(\mathbf{T})'$. The norm in the space $\mathfrak{H}_{-s}(\mathbf{T})$ is defined by the formula

$$\|\mathbf{V}\|_{-s,2} = \left\| (\mathbf{T} + \mathbf{I})^{-s/2} \mathbf{V} \right\|_2.$$

The spaces $\mathfrak{H}_s(\mathbf{T})$ form the following chain

$$\cdots \subset \mathfrak{H}_2(\mathbf{T}) \subset \mathfrak{H}_1(\mathbf{T}) \subset L^2(\mathcal{G}) = \mathfrak{H}_0(\mathbf{T}) \subset \mathfrak{H}_{-1}(\mathbf{T}) \subset \mathfrak{H}_{-2}(\mathbf{T}) \subset \ldots$$

The norm of the space $\mathfrak{H}_1(\mathbf{T})$ can be calculated as follows

$$\begin{aligned} \|\mathbf{V}\|_{1,2}^2 &= ((\mathbf{T} + \mathbf{I})^{1/2} \mathbf{V}, (\mathbf{T} + \mathbf{I})^{1/2} \mathbf{V}) \\ &= \sum_{j=1}^{N} \int_0^\infty \left(|v_j'(x)|^2 + |v_j(x)|^2 + x^2 |v_j(x)|^2 \right) dx. \end{aligned}$$

Therefore, we have the embedding $\mathfrak{H}_1(\mathbf{T}) \hookrightarrow H^1(\mathcal{G})$ and, by the Sobolev embedding, $\mathfrak{H}_1(\mathbf{T}) \hookrightarrow L^\infty(\mathcal{G})$. From the former remark we obtain that the functional $\delta_1 : \mathfrak{H}_1(\mathbf{T}) \to \mathbb{C}$ acting as $\delta_1(\mathbf{V}) = v_1(0)$ belongs to $\mathfrak{H}_1(\mathbf{T})' = \mathfrak{H}_{-1}(\mathbf{T})$ and consequently $\delta_1 \in \mathfrak{H}_{-2}(\mathbf{T})$. Therefore, using [4, Lemma 1.2.3], it follows that the restriction $\hat{\mathbf{T}}_0$ of the operator $\mathbf{T}$ onto the domain

$$\mathrm{dom}(\hat{\mathbf{T}}_0) = \{ \mathbf{V} \in \mathrm{dom}(\mathbf{T}) : \delta_1(\mathbf{V}) = v_1(0) = 0 \} = \mathrm{dom}(\mathbf{T}_0)$$

is a densely defined symmetric operator with equal deficiency indices $n_\pm(\hat{\mathbf{T}}_0) = 1$. By [28, Chapter IV, Theorem 6], the operators $\hat{\mathbf{T}}_0$ and $\mathbf{T}_0$ have equal deficiency indices. Since $\mathbf{T}_1^0$ is the self-adjoint extension of $\mathbf{T}_0$, by Proposition A.1, we get $n(\mathbf{T}_1^0) \leq 1$. Using $\mathbf{T}_1^0 \mathbf{\Phi}_0 = -2\mathbf{\Phi}_0$, where $\mathbf{\Phi}_0 = (\varphi_0, \ldots, \varphi_0)$, we arrive at $n(\mathbf{T}_1^0) = 1$. Since $\mathbf{\Phi}_0 \in L_k^2(\mathcal{G})$ for any k, one concludes $n(\mathbf{T}_1^0|_{L_k^2(\mathcal{G})}) = 1$.

(iv) See the proof of Proposition 3.6(iii). □

Remark 3.9. Observe that, when we deal with the deficiency indices, the operator $\mathbf{T}_0$ is assumed to act on complex-valued functions which however does not affect the analysis of the negative spectrum of $\mathbf{T}_1^0$ acting on real-valued functions.

Combining Lemma 3.7 and Theorem 3.8, in the framework of the perturbation theory we obtain the following proposition.

Proposition 3.10. *Let $k = 1, \ldots, \left[\frac{N-1}{2}\right]$. Then there exist $\alpha_0 > 0$ and two analytic functions $\mu_k : (-\alpha_0, \alpha_0) \to \mathbb{R}$ and $\mathbf{E}_k : (-\alpha_0, \alpha_0) \to L_k^2(\mathcal{G})$ such that*

- (i) *$\mu_k(0) = 0$ and $\mathbf{E}_k(0) = \widetilde{\mathbf{\Phi}}_{0,k}$, where $\widetilde{\mathbf{\Phi}}_{0,k}$ is defined by (3.3).*
- (ii) *For all $\alpha \in (-\alpha_0, \alpha_0)$, $\lambda_k(\alpha)$ is the simple isolated second eigenvalue of $\mathbf{T}_{1,k}^\alpha$ in $L_k^2(\mathcal{G})$, and $\mathbf{E}_k(\alpha)$ is the associated eigenvector for $\lambda_k(\alpha)$.*

(iii) *α_0 can be chosen small enough to ensure that for $\alpha \in (-\alpha_0, \alpha_0)$ the spectrum of $\mathbf{T}^{\alpha}_{1,k}$ in $L^2_k(\mathcal{G})$ is positive, except at most the first two eigenvalues.*

Proof. Using the structure of the spectrum of the operator $\mathbf{T}^0_1$ given in Theorem 3.8(*ii*)–(*iv*), we can separate the spectrum $\sigma(\mathbf{T}^0_1)$ in $L^2_k(\mathcal{G})$ into two parts $\sigma_0 = \{\mu^0_1, 0\}$, $\mu^0_1 < 0$, and σ_1 by a closed curve Γ (for example, a circle), such that σ_0 belongs to the inner domain of Γ and σ_1 to the outer domain of Γ (note that $\sigma_1 \subset (\epsilon, +\infty)$ for $\epsilon > 0$). Next, Lemma 3.7 and the analytic perturbations theory imply that $\Gamma \subset \rho(\mathbf{T}^{\alpha}_{1,k})$ for sufficiently small $|\alpha|$, and $\sigma(\mathbf{T}^{\alpha}_{1,k})$ is likewise separated by Γ into two parts, such that the part of $\sigma(\mathbf{T}^{\alpha}_{1,k})$ inside Γ consists of a finite number of eigenvalues with total multiplicity (algebraic) two. Therefore, we obtain from the Kato–Rellich Theorem (see [30, Theorem XII.8]) the existence of two analytic functions $\mu_k, \mathbf{E}_k$ defined in a neighborhood of zero such that items (*i*), (*ii*) and (*iii*) hold. $\square$

Now we investigate how the perturbed second eigenvalue moves depending on the sign of α.

Proposition 3.11. *There exists $0 < \alpha_1 < \alpha_0$ such that $\lambda_k(\alpha) < 0$ for any $\alpha \in (-\alpha_1, 0)$, and $\lambda_k(\alpha) > 0$ for any $\alpha \in (0, \alpha_1)$. Thus, for α close to 0, we have $n(\mathbf{T}^{\alpha}_{1,k}|_{L^2_k(\mathcal{G})}) = 2$ as $\alpha < 0$, and $n(\mathbf{T}^{\alpha}_{1,k}|_{L^2_k(\mathcal{G})}) = 1$ as $\alpha > 0$.*

Proof. Using analyticity of $(\mathbf{T}^{\alpha}_{1,k})$, we obtain for sufficiently small α the following expansions of the second eigenvalue μ_k and the corresponding eigenfunction of $\mathbf{T}^{\alpha}_{1,k}$ in $L^2_k(\mathcal{G})$

$$\mu_k(\alpha) = \mu_{0,k}\alpha + O(\alpha^2) \quad \text{and} \quad \mathbf{E}_k(\alpha) = \widetilde{\mathbf{\Phi}}_{0,k} + \alpha\mathbf{E}_{0,k} + \mathbf{O}(\alpha^2). \tag{3.4}$$

From (3.4) we get

$$(\mathbf{T}^{\alpha}_{1,k}\mathbf{E}_k(\alpha), \widetilde{\mathbf{\Phi}}_{0,k}) = \mu_{0,k}\alpha||\widetilde{\mathbf{\Phi}}_{0,k}||^2_2 + O(\alpha^2). \tag{3.5}$$

Using

$$(\mathbf{T}^{\alpha}_{1,k}\widetilde{\mathbf{\Phi}}_{0,k})_j = \begin{cases} \frac{N-k}{k}\left(-\frac{2\alpha}{2k-N}x + \frac{\alpha^2}{(2k-N)^2}\right)\varphi_0', & j = 1, \dots, k; \\ -\left(\frac{2\alpha}{2k-N}x + \frac{\alpha^2}{(2k-N)^2}\right)\varphi_0', & j = k+1, \dots, N, \end{cases}$$

we obtain

$$\begin{aligned}(\mathbf{T}^{\alpha}_{1,k}\mathbf{E}_k(\alpha), \widetilde{\mathbf{\Phi}}_{0,k}) &= (\mathbf{E}_k(\alpha), \mathbf{T}^{\alpha}_{1,k}\widetilde{\mathbf{\Phi}}_{0,k}) = (\widetilde{\mathbf{\Phi}}_{0,k}, \mathbf{T}^{\alpha}_{1,k}\widetilde{\mathbf{\Phi}}_{0,k}) + O(\alpha^2) \\ &= \frac{2\alpha(N-k)}{k}\int\limits_0^{\infty} x(\varphi_0')^2dx + O(\alpha^2).\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\mu_{0,k} = \frac{\frac{2(N-k)}{k}\int\limits_0^{\infty} x(\varphi_0')^2dx}{||\widetilde{\mathbf{\Phi}}_{0,k}||^2_2} + O(\alpha).$$

It is easily seen that $\mu_{0,k} > 0$ for small α. Therefore, $n(\mathbf{T}^{\alpha}_{1,k}|_{L^2_k(\mathcal{G})}) = 2$ for $\alpha < 0$, and $n(\mathbf{T}^{\alpha}_{1,k}|_{L^2_k(\mathcal{G})}) = 1$ for $\alpha > 0$. $\square$

Now we can count the number of negative eigenvalues of $\mathbf{T}^{\alpha}_{1,k}$ in $L^2_k(\mathcal{G})$ for any α, using a classical continuation argument based on the Riesz-projection.

Proposition 3.12. *Let $k = 1, \dots, \left[\frac{N-1}{2}\right]$. Then the following assertions hold.*

(*i*) *If $\alpha > 0$, then $n(\mathbf{T}^{\alpha}_{1,k}|_{L^2_k(\mathcal{G})}) = 1$.*
(*ii*) *If $\alpha < 0$, then $n(\mathbf{T}^{\alpha}_{1,k}|_{L^2_k(\mathcal{G})}) = 2$.*

Proof. We consider the case $\alpha < 0$. Recall that $\ker(\mathbf{T}^{\alpha}_{1,k}) = \{\mathbf{0}\}$ by Proposition 3.6. Define α_∞ by

$$\alpha_\infty = \inf\{\tilde{\alpha} < 0 : \mathbf{T}^{\alpha}_{1,k} \text{ has exactly two negative eigenvalues in } L^2_k(\mathcal{G}) \text{ for all } \alpha \in (\tilde{\alpha}, 0)\}.$$

Proposition 3.11 implies that α_∞ is well defined and $\alpha_\infty \in [-\infty, 0)$. We claim that $\alpha_\infty = -\infty$. Suppose that $\alpha_\infty > -\infty$. Let $M = n(\mathbf{T}^{\alpha_\infty}_{1,k}|_{L^2_k(\mathcal{G})})$ and Γ be a closed curve (for example, a circle or a rectangle) such that $0 \in \Gamma \subset \rho(\mathbf{T}^{\alpha_\infty}_{1,k})$, and all the negative eigenvalues of $\mathbf{T}^{\alpha_\infty}_{1,k}$ belong to the inner domain of Γ. The existence of such Γ can be deduced from the lower semi-boundedness of the quadratic form associated to $\mathbf{T}^{\alpha_\infty}_{1,k}$.

Next, from Lemma 3.7 it follows that there is $\epsilon > 0$ such that for $\alpha \in [\alpha_\infty - \epsilon, \alpha_\infty + \epsilon]$ we have $\Gamma \subset \rho(\mathbf{T}^{\alpha}_{1,k})$ and for $\xi \in \Gamma$, $\alpha \to (\mathbf{T}^{\alpha}_{1,k} - \xi)^{-1}$ is analytic. Therefore, the existence of an analytic family of Riesz-projections $\alpha \to P(\alpha)$ given by

$$P(\alpha) = -\frac{1}{2\pi i}\int\limits_{\Gamma} (\mathbf{T}^{\alpha}_{1,k} - \xi)^{-1} d\xi$$

implies that $\dim(\operatorname{ran} P(\alpha)) = \dim(\operatorname{ran} P(\alpha_\infty)) = M$ for all $\alpha \in [\alpha_\infty - \epsilon, \alpha_\infty + \epsilon]$. Next, by definition of α_∞, $\mathbf{T}^{\alpha_\infty+\epsilon}_{1,k}$ has two negative eigenvalues and $M = 2$, hence, $\mathbf{T}^{\alpha}_{1,k}$ has two negative eigenvalues for $\alpha \in (\alpha_\infty - \epsilon, 0)$, which contradicts with the definition of α_∞. Therefore, $\alpha_\infty = -\infty$. $\square$

Remark 3.13. In Proposition A.2 we show the following estimates of $n(\mathbf{T}^{\alpha}_{1,k})$ in the whole space $L^2(\mathcal{G})$:

- $n(\mathbf{T}^{\alpha}_{1,k}) \le k + 1$ for $\alpha < 0$;
- $n(\mathbf{T}^{\alpha}_{1,k}) \le N - k$ for $\alpha > 0$.

We believe that these estimates might be useful for the investigation of the orbital instability of the standing waves $e^{i\omega t}\mathbf{\Phi}^{\alpha}_k$ in $W^1_{\mathcal{E}}(\mathcal{G})$.

Proof of Theorem 1.1. Due to Theorem 3.5, we have $n(\mathbf{H}^{\alpha}_k|_{L^2_k(\mathcal{G})}) = 2$ for $\alpha < 0$, and $n(\mathbf{H}^{\alpha}_k|_{L^2_k(\mathcal{G})}) = 1$ for $\alpha > 0$. Using Theorem 3.3, we obtain the orbital stability of $e^{i\omega t}\mathbf{\Phi}_k$ in $W^1_{\mathcal{E},k}(\mathcal{G})$ for $\alpha > 0$ and the spectral instability for $\alpha < 0$.

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Appendix A

Below we show the estimates from Remark 3.13. We use the following abstract result (see [28]).

Proposition A.1. *Let A be a densely defined lower semi-bounded symmetric operator (that is, $A \ge mI$) with finite deficiency indices $n_\pm(A) = n < \infty$ in the Hilbert space $\mathcal{H}$, and let $\widetilde{A}$ be a self-adjoint extension of A. Then the spectrum of $\widetilde{A}$ in $(-\infty, m)$ is discrete and consists of at most n eigenvalues counting multiplicities.*

Proposition A.2. *Let $\alpha \neq 0$, $k = 1, \dots, \left[\frac{N-1}{2}\right]$. Then the following assertions hold.*

(*i*) *If $\alpha < 0$, then $n(\mathbf{T}^{\alpha}_{1,k}) \leq k+1$.*
(*ii*) *If $\alpha > 0$, then $n(\mathbf{T}^{\alpha}_{1,k}) \leq N-k$.*

Proof. (*i*) First, note that $\mathbf{T}^{\alpha}_{1,k}$ is the self-adjoint extension of the symmetric operator

$$\mathbf{T}_{0,k} = \left(\left(-\frac{d^2}{dx^2} + (x \pm \tfrac{\alpha}{2k-N})^2 - 3\right)\delta_{i,j}\right),$$

$$\mathrm{dom}(\mathbf{T}_{0,k}) = \left\{ \begin{array}{l} \mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0) = 0, \\ \sum\limits_{j=1}^{N} v_j'(0) = 0,\ v_1(a_k) = \cdots = v_k(a_k) = 0 \end{array} \right\},\ a_k = \frac{\alpha}{2k-N}.$$

Below we show that the operator $\mathbf{T}_{0,k}$ is non-negative and $n_{\pm}(\mathbf{T}_{0,k}) = k+1$ (when $\mathbf{T}_{0,k}$ is assumed to act on complex-valued functions).

First, note that every component of the vector $\mathbf{V} = (v_j)_{j=1}^N \in W^2(\mathcal{G})$ satisfies the following identity

$$-v_j'' + \omega v_j + \left((x \pm \tfrac{\alpha}{2k-N})^2 - 3\right) v_j = \frac{-1}{\varphi'_{k,j}} \frac{d}{dx}\left[(\varphi'_{k,j})^2 \frac{d}{dx}\left(\frac{v_j}{\varphi'_{k,j}}\right)\right], \quad x \in \mathbb{R}_+ \setminus \{a_k\}. \tag{A.1}$$

Moreover, for $j \in \{k+1, \dots, N\}$ the above equality holds also for a_k since $\varphi'_{k,j}(a_k) \neq 0$, $j \in \{k+1, \dots, N\}$. Using the above equality and integrating by parts, we get for $\mathbf{V} \in \mathrm{dom}(\mathbf{T}_{0,k})$

$$\begin{aligned}
(\mathbf{T}_{0,k}\mathbf{V}, \mathbf{V}) &= \sum_{j=1}^{k}\Big(\int_0^{a_k-} + \int_{a_k+}^{+\infty}\Big)(\varphi'_{k,j})^2\left[\frac{d}{dx}\left(\frac{v_j}{\varphi'_{k,j}}\right)\right]^2 dx + \sum_{j=k+1}^{N}\int_0^{\infty}(\varphi'_{k,j})^2\left[\frac{d}{dx}\left(\frac{v_j}{\varphi'_{k,j}}\right)\right]^2 dx \\
&\quad + \sum_{j=1}^{N}\left[-v_j'v_j + v_j^2\frac{\varphi''_{k,j}}{\varphi'_{k,j}}\right]_0^{\infty} + \sum_{j=1}^{k}\left[v_j'v_j - v_j^2\frac{\varphi''_{k,j}}{\varphi'_{k,j}}\right]_{a_k-}^{a_k+} \\
&= \sum_{j=1}^{k}\Big(\int_0^{a_k-} + \int_{a_k+}^{+\infty}\Big)(\varphi'_{k,j})^2\left[\frac{d}{dx}\left(\frac{v_j}{\varphi'_{k,j}}\right)\right]^2 dx + \sum_{j=k+1}^{N}\int_0^{\infty}(\varphi'_{k,j})^2\left[\frac{d}{dx}\left(\frac{v_j}{\varphi'_{k,j}}\right)\right]^2 dx \geq 0.
\end{aligned}$$

The equality $\sum\limits_{j=1}^{k}\left[v_j'v_j - v_j^2\frac{\varphi''_{k,j}}{\varphi'_{k,j}}\right]_{a_k-}^{a_k+} = 0$ needs an additional explanation. Indeed, since a_k is a zero of the first order for $\varphi'_{k,j}$ (i.e. $\varphi''_{k,j}(a_k) \neq 0$), $v_j \in H^2(\mathbb{R}_+)$ and $v_j(a_k) = 0$, we get $\lim\limits_{x \to a_k}\frac{v_j^2(x)}{\varphi'_{k,j}(x)} = \lim\limits_{x \to a_k}\frac{2v_j(x)v_j'(x)}{\varphi''_{k,j}(x)} = 0$. To prove $n_{\pm}(\mathbf{T}_{0,k}) = k+1$ we will use the idea of the proof of Theorem 3.8(*iii*). Consider the following non-negative self-adjoint operator

$$\mathbf{T}_k = \left(\left(-\frac{d^2}{dx^2} + (x \pm \tfrac{\alpha}{2k-N})^2\right)\delta_{i,j}\right),$$

$$\mathrm{dom}(\mathbf{T}_k) = \left\{\mathbf{V} \in W^2(\mathcal{G}) : v_1(0) = \cdots = v_N(0), \sum_{j=1}^{N} v_j'(0) = 0\right\}.$$

As in the proof of Theorem 3.8(*iii*), we define chain of the Hilbert spaces

$$\dots \subset \mathfrak{H}_2(\mathbf{T}_k) \subset \mathfrak{H}_1(\mathbf{T}_k) \subset L^2(\mathcal{G}) = \mathfrak{H}_0(\mathbf{T}_k) \subset \mathfrak{H}_{-1}(\mathbf{T}_k) \subset \mathfrak{H}_{-2}(\mathbf{T}_k) \subset \dots$$

We have the embedding $\mathfrak{H}_1(\mathbf{T}_k) \hookrightarrow H^1(\mathcal{G})$ and, by the Sobolev embedding, $\mathfrak{H}_1(\mathbf{T}_k) \hookrightarrow L^\infty(\mathcal{G})$. From the former remark we obtain that the functionals

$$\begin{aligned} &\delta_1 : \mathfrak{H}_1(\mathbf{T}_k) \to \mathbb{C}, \quad \delta_{j,a_k} : \mathfrak{H}_1(\mathbf{T}_k) \to \mathbb{C}, \\ &\delta_1(\mathbf{V}) = v_1(0), \quad \delta_{j,a_k}(\mathbf{V}) = v_j(a_k),\ j \in \{1, \dots, k\}, \end{aligned}$$

belong to $\mathfrak{H}_1(\mathbf{T}_k)' = \mathfrak{H}_{-1}(\mathbf{T}_k)$ and consequently $\delta_1, \delta_{j,a_k} \in \mathfrak{H}_{-2}(\mathbf{T}_k)$. Therefore, using [4, Lemma 3.1.1], it follows that the restriction $\hat{\mathbf{T}}_{0,k}$ of the operator $\mathbf{T}_k$ onto the domain

$$\operatorname{dom}(\hat{\mathbf{T}}_{0,k}) = \left\{ \begin{array}{c} \mathbf{V} \in \operatorname{dom}(\mathbf{T}_k) : \delta_1(\mathbf{V}) = v_1(0) = 0, \\ \delta_{j,a_k}(\mathbf{V}) = v_j(a_k) = 0,\ j \in \{1, \dots, k\} \end{array} \right\} = \operatorname{dom}(\mathbf{T}_{0,k})$$

is a densely defined symmetric operator with equal deficiency indices $n_\pm(\hat{\mathbf{T}}_{0,k}) = k+1$. By [28, Chapter IV, Theorem 6], the operators $\hat{\mathbf{T}}_{0,k}$ and $\mathbf{T}_{0,k}$ have equal deficiency indices. Therefore, $n(\mathbf{T}^\alpha_{1,k}) \le k+1$.

(ii) The proof is similar. In particular, we need to consider the operator $\mathbf{T}^\alpha_{1,k}$ as the self-adjoint extension of the non-negative symmetric operator

$$\begin{aligned} &\mathbf{T}_{0,N-k} = \left(\left(-\frac{d^2}{dx^2} + (x \pm \tfrac{\alpha}{2k-N})^2 - 3 \right) \delta_{i,j} \right), \\ &\operatorname{dom}(\mathbf{T}_{0,N-k}) = \left\{ \mathbf{V} \in \operatorname{dom}(\mathbf{T}^\alpha_{1,k}) : v_{k+1}(a_k) = \dots = v_N(a_k) = 0 \right\}. \end{aligned}$$

The deficiency indices of $\mathbf{T}_{0,N-k}$ equal $N-k$ (since basically $\mathbf{T}_{0,N-k}$ is the restriction of the operator $\mathbf{T}^\alpha_{1,k}$ onto the subspace of codimension $N-k$). To show the non-negativity of $\mathbf{T}_{0,N-k}$, we need to use formula (A.1). It induces

$$\begin{aligned} (\mathbf{T}_{0,N-k}\mathbf{V}, \mathbf{V}) &= \sum_{j=k+1}^{N} \Big(\int_0^{a_k-} + \int_{a_k+}^{+\infty} \Big) (\varphi'_{k,j})^2 \left[\frac{d}{dx} \left(\frac{v_j}{\varphi'_{k,j}} \right) \right]^2 dx + \sum_{j=1}^{k} \int_0^\infty (\varphi'_{k,j})^2 \left[\frac{d}{dx} \left(\frac{v_j}{\varphi'_{k,j}} \right) \right]^2 dx \\ &\quad + \sum_{j=1}^{N} \left[-v'_j v_j + v_j^2 \frac{\varphi''_{k,j}}{\varphi'_{k,j}} \right]_0^\infty + \sum_{j=k+1}^{N} \left[v'_j v_j - v_j^2 \frac{\varphi''_{k,j}}{\varphi'_{k,j}} \right]_{a_k-}^{a_k+} \\ &= \sum_{j=k+1}^{N} \Big(\int_0^{a_k-} + \int_{a_k+}^{+\infty} \Big) (\varphi'_{k,j})^2 \left[\frac{d}{dx} \left(\frac{v_j}{\varphi'_{k,j}} \right) \right]^2 dx + \sum_{j=1}^{k} \int_0^\infty (\varphi'_{k,j})^2 \left[\frac{d}{dx} \left(\frac{v_j}{\varphi'_{k,j}} \right) \right]^2 dx \\ &\quad + \sum_{j=1}^{N} \left[v'_j(0) v_j(0) - v_j^2(0) \frac{\varphi''_{k,j}(0)}{\varphi'_{k,j}(0)} \right] \ge 0. \end{aligned}$$

Indeed, $\sum\limits_{j=k+1}^{N} \left[v'_j v_j - v_j^2 \frac{\varphi''_{k,j}}{\varphi'_{k,j}} \right]_{a_k-}^{a_k+} = 0$ (see the proof of item (i)). Moreover,

$$\sum_{j=1}^{N} \left[v'_j(0) v_j(0) - v_j^2(0) \frac{\varphi''_{k,j}(0)}{\varphi'_{k,j}(0)} \right] = v_1^2(0) \frac{(N-2k)^2}{\alpha} \ge 0.$$

Finally, due to Proposition A.1, we get the result. □

Remark A.3.

(*i*) It is easily seen that

$$\sum_{j=1}^{N}\left[v_j'(0)v_j(0) - v_j^2(0)\frac{\varphi_{k,j}''(0)}{\varphi_{k,j}'(0)}\right] = v_1^2(0)\frac{(N-2k)^2}{\alpha} \leq 0$$

for $\alpha < 0$, and, therefore, the restriction of $\mathbf{T}_{1,k}^{\alpha}$ onto the subspace

$$\{\mathbf{V} \in W^2(\mathcal{G}) : v_1(a_k) = \cdots = v_k(a_k) = 0\}$$

is not a non-negative operator as $\alpha < 0$. Thus, we need to assume additionally that $v_1(0) = \cdots = v_N(0) = 0$.

(*ii*) The result of the item (*ii*) (for $\alpha > 0$) of the above Proposition can be extended to the case of $k = 0$, i.e. $n(\mathbf{T}_{1,0}^{\alpha}) \leq N$.

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