Generic non-extendability and total unboundedness in function spaces

V. Nestoridis [a], A.G. Siskakis [b,*], A. Stavrianidi [a], S. Vlachos [a]

[a] *National and Kapodistrian University of Athens, Department of Mathematics, Panepistemiopolis, 15784, Athens, Greece*
[b] *Department of Mathematics, University of Thessaloniki, 54124 Thessaloniki, Greece*

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For a function space $X(\Omega)$ satisfying weak assumptions we prove that the generic function in $X(\Omega)$ is totally unbounded, hence non-extendable. We provide several examples of such spaces; they are mainly localized versions of classical function spaces and intersections of them.

1. Introduction

For a domain Ω in $\mathbb{C}^d$ and a function space $X = X(\Omega)$ consisting of holomorphic functions on Ω it was proved in [7] that under certain weak assumptions on X, the set of non-extendable functions in X is either empty or a G_δ and dense subset of X. Similar results were proved for the infinite dimensional case in [1].

In the present paper we consider a stronger notion than non-extendability; It is the notion of *total unboundedness* for holomorphic functions $f : \Omega \to \mathbb{C}$. Such a function is called totally unbounded if for every ball $B(\zeta, \varepsilon)$ centered at $\zeta \in \partial\Omega$ with radius $\varepsilon > 0$ and every component Y of $\Omega \cap B(\zeta, \varepsilon)$ the restriction $f|_Y$ is unbounded, see [3]. If f or one of its derivatives is totally unbounded then f is non-extendable beyond Ω. The converse is not true since there exist non-extendable functions on the unit disc D of the complex plane whose derivatives of every order have continuous, thus bounded, extensions on the closed disc, see [6].

We prove that if X satisfies some weak assumptions, then the set of totally unbounded functions in X is a G_δ and dense subset of X if and only if X contains sufficiently many "locally unbounded" functions, i.e.

* Corresponding author.
E-mail addresses: vnestor@math.uoa.gr (V. Nestoridis), siskakis@math.auth.gr (A.G. Siskakis), aleksandrastavrianidi@yahoo.gr (A. Stavrianidi), steliosgerrard8@hotmail.gr (S. Vlachos).

if the following holds: For every ball $B(\zeta, \varepsilon)$ with radius $\varepsilon > 0$ centered at $\zeta \in \partial\Omega$ and every component Y of $\Omega \cap B(\zeta, \varepsilon)$, there exists a function h_Y in X such that its restriction $h_Y|_Y$ is unbounded. We also obtain a version of this result for the derivatives of functions in X.

The assumption for the space $X = X(\Omega)$ is that it is a topological vector space of holomorphic functions on Ω whose topology is induced by a complete metric such that convergence in X implies uniform convergence on compact subsets of Ω. The details and the main result is in section 2.

In the rest of the article we give examples of function spaces that satisfy the above assumption. Most of these spaces are localized versions of classical function spaces. Very often a classical function space consists of all holomorphic in Ω functions satisfying a property P when we approach from Ω the whole boundary $\partial\Omega$. For the localized version we consider a denumerable set of properties P_i, $i \in I$, and we demand that each one of them holds when we approach only a part J_i, $i \in I$, of the boundary $\partial\Omega$, from inside Ω. These spaces endowed with their natural topology are often Fréchét spaces and satisfy our requirements. In order to assure the existence of the functions h_Y in X we impose some geometrical or topological assumptions on Ω. One such assumption is that for some $\alpha \in \partial\Omega$ there exists β so that the segment $(\alpha, \beta]$ is disjoint from $\overline{\Omega}$ or Ω. The essential point is that $\arg\frac{z-\alpha}{z-\beta}$ has a bounded on Ω branch. Thus, the segment $(\alpha, \beta]$ can be replaced by other curves not turning around very much.

In the last section we give an example of a space X which is not a localization of a classical function space, but is itself a classical function space. This is the space of holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the open unit disc D such that $\sum_{n=0}^{\infty} |a_n|^p < \infty$. We prove that for the generic function f in these spaces its derivative f' is totally unbounded and f is non-extendable.

In the future we will consider localized versions of the Nevanlinna class or of subclasses of it and of Dirichlet type spaces. We mention that the localized Hardy space H^1 appears naturally with the results of [4]. More precisely let Ω be a Jordan domain in $\mathbb{C}$ and $\Phi : D \to \Omega$ a Riemann map. Assume that for a closed arc $J_{[A,B]} = \{e^{i\theta} : A \le \theta \le B\} \subset \partial D$, $A < B \le A + 2\pi$, the curve $\Phi(J_{[A,B]})$ has finite length, then

$$\sup_{0 \le r < 1} \int_{\alpha}^{\beta} |\Phi'(re^{i\theta})| d\theta < \infty$$

for all α, β with $A < \alpha < \beta < B$; that is, the derivative Φ' belongs to the localized Hardy space $H^1(D, J_{[\alpha,\beta]})$. In order to obtain an "if and only if" statement is suffices to consider the localized Hardy space $H^1(D, O)$, where $O \subset \mathbb{T}$ is relatively open. Then $\Phi' \in H^1(D, O)$ if and only if the curve $\Phi(J_{[\alpha,\beta]})$ has finite length for all α, β such that $J_{[\alpha,\beta]} \subset O$. For the definitions of the localized Hardy spaces see section 6 below.

Finally it is interesting to investigate properties of the new localized spaces or of the functions belonging to them. For example what can be said about the growth or the zero sets of such functions? How about the existence of non-tangential limits? Are polynomials or rational functions dense in these spaces?

We also mention that some of the results established in this paper also hold if $X = X(\Omega)$ is a space of harmonic functions on a domain Ω in $\mathbb{R}^d$.

2. Extendability and total unboundedness

In this section we present some facts about non extendable and totally unbounded functions that will be central in the paper.

Definition 2.1. Let $\Omega \subset \mathbb{C}^d$ be a domain and f a holomorphic function in Ω. The function f is called extendable if there exist a domain $U \subset \mathbb{C}^d$, $\partial\Omega \cap U \ne \emptyset$ and a holomorphic function F in U such that for some connected component Y of $U \cap \Omega$ we have $f|_Y = F|_Y$. Otherwise f is called non-extendable.

We will also need the following fact:

Lemma 2.2. *([3], [7]). Let $\Omega \subset \mathbb{C}^d$ be a domain and $U \subset \mathbb{C}^d$ a domain such that $U \cap \partial\Omega \neq \emptyset$. Let Y denote a connected component of $U \cap \Omega$. Then $\overline{Y} \cap U \cap \partial\Omega \neq \emptyset$.*

Proposition 2.3. *([7]). Let $\Omega \subset \mathbb{C}^d$ be a domain and $f : \Omega \to \mathbb{C}$ a holomorphic function. Then the following are equivalent.*

   (i) f is extendable

   (ii) There exist two balls b, B in $\mathbb{C}^d$ such that $\overline{b} \subset B \cap \Omega$, $B \cap \Omega^c \neq \emptyset$ and a bounded holomorphic function $F : B \to \mathbb{C}$ such that $F|_b = f|_b$.

Proposition 2.4. *Let $\Omega \subset \mathbb{C}^d$ be a domain and $f : \Omega \to \mathbb{C}$ a holomorphic function. Let T denote the identity or an operator of mixed partial derivation. If Tf is non-extendable, then f is non-extendable as well.*

Proof. If f is extendable, then according to Definition 2.1 there exists a domain $U \subset \mathbb{C}^d$, $\partial\Omega \cap U = \emptyset$ and a function F holomorphic in U such that $F|_Y = f|_Y$ for some component Y of $U \cap \Omega$. Then obviously $TF|_Y = Tf|_Y$ since Y is open. It follows that Tf is extendable, which contradicts our hypothesis. $\square$

Theorem 2.5. *([7]). Let $\Omega \subset \mathbb{C}^d$ be a domain and $X = X(\Omega)$ be a topological vector space consisting of holomorphic functions on Ω, under the usual operations of addition and scalar multiplication of functions, whose topology is induced by a complete metric and such that convergence in X implies uniform convergence on compacta of Ω. Then the set of non-extendable functions in X is either empty or it is a dense and G_δ subset of X.*

A slightly stronger notion is that of totally unbounded function ([3]).

Definition 2.6. Let $\Omega \subset \mathbb{C}^d$ be a domain. A function $g : \Omega \to \mathbb{C}$ is called totally unbounded on Ω, if for every ball $B(\zeta, \varepsilon) \subset \mathbb{C}^d$ with radius $\varepsilon > 0$ and centered at $\zeta \in \partial\Omega$ and every component Y of $B(\zeta, \varepsilon) \cap \Omega$ the restriction $g|_Y$ is unbounded.

Proposition 2.7. *([3]). Let $\Omega \subset \mathbb{C}^d$ be a domain and $g : \Omega \to \mathbb{C}$ a holomorphic function. If g is totally unbounded, then g is non-extendable.*

Theorem 2.8. *Let $\Omega \subset \mathbb{C}^d$ be a domain and $X = X(\Omega)$ be a topological vector space of holomorphic functions on Ω, under the usual operations of addition and scalar multiplication of functions, whose topology is induced by a complete metric, such that convergence in X implies uniform convergence on compacta of Ω. Let T be the identity or an operator of mixed partial derivation. We assume that for every ball $B(\zeta, \varepsilon) \subset \mathbb{C}^d$ with radius $\varepsilon > 0$ centered at $\zeta \in \partial\Omega$ and for every component Y of $\Omega \cap B(\zeta, \varepsilon)$, there exists a function h_Y in X such that its restriction $Th_Y|_Y$ is unbounded. Then there exists a function $h \in X$ such that Th is totally unbounded on Ω. The set of $f \in X$ such that Tf is totally unbounded on Ω is a dense and G_δ subset of X. Furthermore, the set of non-extendable functions in X is a dense and G_δ subset of X.*

Proof. Let $Z \subset \partial\Omega$ be a countable set which is dense in $\partial\Omega$. We write

$$\Gamma = \{B(z, q) : z \in Z, q \in \mathbb{Q}^+\}$$

for the collection of all balls with centers in Z and rational radii, a denumerable family. For $B(z, q) \in \Gamma$ let $J_{B(z,q)} = \{Y \text{ connected component of } B(z, q) \cap \Omega\}$ and

$$J = \bigcup_{B(z,q) \in \Gamma} J_{B(z,q)}.$$

For a subset $Y \subset \Omega$ we write $S_Y = \{f \in X : Tf \text{ is unbounded in } Y\}$.

Claim. Tf is totally unbounded if and only if $f \in \bigcap_{Y \in J} S_Y$.

Proof of the claim. It is clear that if Tf is totally unbounded then $f \in \cap_{Y \in J} S_Y$. To prove the converse suppose $B = B(x_0, \varepsilon_0)$ is a ball with $x_0 \in \partial\Omega$ and $\varepsilon_0 > 0$, and let Y_0 be a connected component of $B(x_0, \varepsilon_0) \cap \Omega$. According to Lemma 2.2 we have $\overline{Y_0} \cap \partial\Omega \cap B \neq \emptyset$; let $x_1 \in \overline{Y_0} \cap \partial\Omega \cap B$, then since B is open, there exists $\varepsilon_1 > 0$ such that $B(x_1, \varepsilon_1) \subset B$. Let $q \in \mathbb{Q}^+$ such that $0 < q < \varepsilon_1$ then $B(x_1, q) \subset B(x_1, \varepsilon_1) \subset B$. Since $x_1 \in \partial\Omega = \overline{Z}$ there exists $z_1 \in Z$ such that $d(x_1, z_1) < \frac{q}{2}$, thus

$$x_1 \in B\left(z_1, \frac{q}{2}\right) \subset B(x_1, q) \subset B.$$

Setting $\varepsilon_2 = \frac{q}{2} - d(x_1, z_1) > 0$ we have $B(x_1, \varepsilon_2) \subset B(z_1, \frac{q}{2})$. Since $x_1 \in \overline{Y_0}$ there exists $x_2 \in Y_0 \cap B(x_1, \varepsilon_2)$. Thus

$$x_2 \in Y_0 \cap B\left(z_1, \frac{q}{2}\right) \subset \Omega \cap B\left(z_1, \frac{q}{2}\right).$$

Let A be the connected component of $\Omega \cap B(z_1, \frac{q}{2})$ containing x_2. Then $A \in J$, and since $A \subset \Omega \cap B(z_1, \frac{q}{2}) \subset \Omega \cap B$ and $x_2 \in A$, it follows that $A \subseteq Y_0$ as A is connected and Y_0 is a connected component.

Now let $f \in \cap_{Y \in J} S_Y$ then $f \in S_A$ and thus Tf is unbounded on A. Since $A \subseteq Y_0$ it follows that $Tf|_{Y_0}$ is unbounded. This finishes the proof of the claim. $\square$

We continue with the proof of Theorem. If we prove that each S_Y, $Y \in J$, is G_δ and dense in the complete metric space X, since J is denumerable, then also $\cap_{Y \in J} S_Y$ is G_δ and dense, according to Baire's theorem. Thus it suffices to prove that each S_Y is G_δ and dense in X.

From [8, Proposition 5.2] we know that S_Y is either empty or G_δ and dense. By assumption there exists $h_Y \in S_Y$. Thus each S_Y is G_δ and dense in X.

We have proved that the set of $f \in X$, such that Tf is totally unbounded is a G_δ and dense subset of X. Proposition 2.7 implies that there exists $h \in X$ such that Th is non-extendable. Proposition 2.4 implies that $h \in X$ is non extendable. Finally, Theorem 2.5 implies that the set of non-extendable functions in X is G_δ and dense. This completes the proof. $\square$

3. Localization of $H^\infty(\Omega)$

In this section we present a first application of Theorem 2.8. Let $\Omega \subset \mathbb{C}$ be a domain and $V \subset \Omega$ an open set. Let $H(\Omega)$ denote the space of all holomorphic $f : \Omega \to \mathbb{C}$. We consider the set

$$X(\Omega, V) = H(\Omega) \cap H^\infty(V) = \{f \in H(\Omega) : f|_V \text{ is bounded}\},$$

see [5]. If $\overline{V} \subset \Omega$ and V is bounded, then obviously $X(\Omega, V) = H(\Omega)$ and the space is endowed with its usual Fréchét topology. More generally, independently of whether $\overline{V} \subset \Omega$ or $\overline{V} \cap \partial\Omega \neq \emptyset$, the topology of $X(\Omega, V)$ is the Fréchét topology induced by the seminorms $\|f|_V\|_\infty$ and $\|f|_{K_m}\|_\infty$, $m = 1, 2, \ldots$ where $\{K_m\}_{m=1}^\infty$ is an exhaustive sequence of compact subsets of Ω. Here and below $\| \|_\infty$ is the usual supremum norm.

Obviously $X = X(\Omega, V)$ satisfies the requirements of Theorem 2.8. Therefore, in order to prove that the set of functions with totally unbounded derivative is G_δ and dense in this space it suffices to prove that for each ball $B(x_0, \varepsilon)$ centered at $x_0 \in \partial\Omega$ with radius $\varepsilon > 0$ and Y connected component of $B(x_0, \varepsilon) \cap \Omega$ there exists an $h_Y \in X(\Omega, V)$ such that h'_Y is unbounded on Y. This requires some extra hypothesis on Ω.

Theorem 3.1. *Let $\Omega \subset \mathbb{C}$ be a domain and $V \subset \Omega$ an open set such that for all $\alpha \in \overline{V} \cap \partial\Omega$ there exists $\beta \in \Omega^c$ such that $\beta \neq \alpha$ and $[\alpha, \beta] \subseteq \Omega^c$. Then the set $\{f \in X(\Omega, V) : f'$ totally unbounded$\}$ is dense and G_δ in $X(\Omega, V)$; furthermore, the set of non-extendable functions in $X(\Omega, V)$ is a G_δ and dense subset.*

Proof. Consider a ball $B(x_0, \varepsilon)$ such that $x_0 \in \partial\Omega$ and Y a connected component of $B(x_0, \varepsilon) \cap \Omega$. If $\overline{V} \cap \partial\Omega \cap \overline{Y} \neq \emptyset$ then for $\alpha \in \overline{V} \cap \partial\Omega \cap \overline{Y} \subset \overline{V} \cap \partial\Omega$ there exists $\beta \in \Omega^c$ such that $[\alpha, \beta] \subseteq \Omega^c$. As a result a branch of logarithm can be chosen such that $f(z) = \log\left(\frac{z-\alpha}{z-\beta}\right)$ is a holomorphic function on Ω and $h_Y(z) = e^{-if(z)}$ has the desired property; that is, $h_Y(z) \in X(\Omega, V)$ and h'_Y is unbounded on Y. This follows easily from the fact that $|h_Y(z)| \in (e^{-\pi}, e^\pi)$ for all $z \in \Omega$ and $h'_Y(z) = (-i)h_Y(z) \cdot \frac{\alpha-\beta}{(z-\alpha)(z-\beta)}$ and $\alpha \in \overline{Y}$.

If $\overline{V} \cap \partial\Omega \cap \overline{Y} = \emptyset$, then according to Lemma 2.2 there exists $\alpha \in \overline{Y} \cap \partial\Omega$ which implies that $\alpha \notin \overline{V}$ and the function $h_Y = \frac{1}{z-\alpha}$ belongs to $X(\Omega, V)$ and h'_Y is unbounded on Y. The proof is completed as in Theorem 2.8. $\square$

4. Localization of $A^p(\Omega)$

This section contains a second application of Theorem 2.8. Let Ω be a Jordan domain in $\mathbb{C}$ and $J \subseteq \partial\Omega$ a relatively open subset of its boundary. The space $A(\Omega, J)$ consists of all functions $f \in H(\Omega)$ such that f can be continuously extended on $\Omega \cup J$. The topology in this space is induced by the seminorms $\|f|_{\Delta_m}\|_\infty$ where

$$\Delta_m = \left\{z \in \Omega \cup J : \mathrm{dist}(z, \partial\Omega \setminus J) \geq \frac{1}{m}\right\}, \quad m = 1, 2, \ldots,$$

is a sequence of compact subsets of $\Omega \cup J$, see [5]. In a similar way we define the space $A^p(\Omega, J)$ to contain all functions f holomorphic on Ω such that for every $\ell = 0, 1, 2, ..., p$, the derivative of order ℓ belongs to $A(\Omega, J)$. The topology in this space is induced by the seminorms $\|f^{(\ell)}|_{\Delta_m}\|_\infty$, $0 \leq \ell \leq p$, $m = 1, 2, \ldots$, where $\{\Delta_m\}_{m=1}^\infty$ are as above.

Obviously $A^p(\Omega, J)$ satisfies the requirements of Theorem 2.8. Therefore, in order to prove that the set of functions with a totally unbounded derivative of order $p + 1$ is G_δ and dense in this space it suffices to prove that for each ball $B(x_0, \varepsilon)$ centered at $x_0 \in \partial\Omega$ with radius $\varepsilon > 0$ and each connected component Y of $B(x_0, \varepsilon) \cap \Omega$ there exists a function $h_Y \in A^p(\Omega, J)$ such that $h_Y^{(p+1)}$ is unbounded on Y.

Theorem 4.1. *Let Ω be a Jordan domain and $J \subset \partial\Omega$ a relatively open subset of $\partial\Omega$ such that for all $\alpha \in J$ there exists $\beta \in (\overline{\Omega})^c$ such that $(\alpha, \beta] \subseteq (\overline{\Omega})^c$. Then the set:*

$$\{f \in A^p(\Omega, J) : f^{(p+1)} \text{ is totally unbounded}\}$$

is dense and G_δ in $A^p(\Omega, J)$. Furthermore, the set of non-extendable functions in $A^p(\Omega, J)$ is G_δ and dense in $A^p(\Omega, J)$.

Proof. Consider a ball $B(x_0, \varepsilon)$ such that $x_0 \in \partial\Omega$, $\varepsilon > 0$ and Y a connected component of $B(x_0, \varepsilon) \cap \Omega$.

If $J \cap \overline{Y} \neq \emptyset$, we pick $\alpha \in J \cap \overline{Y}$. By hypothesis there exists $\beta \in (\overline{\Omega})^c$ such that $(\alpha, \beta] \subseteq (\overline{\Omega})^c$. As a result the function $f(z) = \log\left(\frac{z-\alpha}{z-\beta}\right)$ is a holomorphic function on Ω and its imaginary part $\mathrm{Arg}\left(\frac{z-\alpha}{z-\beta}\right)$ stays bounded on Ω.

Claim. *The function*

$$h_Y(z) = \frac{(z-\alpha)^{p+1}}{(p+1)!} \log\left(\frac{z-\alpha}{z-\beta}\right)$$

has the desired property; that is, $h_Y \in A^p(\Omega, J)$ and $h_Y^{(p+1)}$ is unbounded on Y.

Proof of the claim. A computation gives

$$h_Y'(z) = \frac{(z-\alpha)^p}{p!} \log\left(\frac{z-\alpha}{z-\beta}\right) + \frac{(\alpha-\beta)}{(p+1)!} \frac{(z-\alpha)^p}{z-\beta}$$

$$= \frac{(z-\alpha)^p}{p!} \cdot \log\left(\frac{z-\alpha}{z-\beta}\right) + \frac{(\alpha-\beta)}{(p+1)!}\left(q_1(z) + \frac{c_1}{z-\beta}\right),$$

where $q_1(z)$ is a polynomial of degree $p-1$ and c_1 is a constant. By induction,

$$h_Y^{(k)}(z) = \frac{(z-\alpha)^{p+1-k}}{(p+1-k)!} \cdot \log\left(\frac{z-\alpha}{z-\beta}\right)$$

$$+ \sum_{n=1}^{k} \frac{(\alpha-\beta)}{(p+1-k+n)!} \cdot \left[q_{k-n+1}^{(n-1)}(z) + (-1)^{n+1}\frac{(n-1)!c_{k-n+1}}{(z-\beta)^n}\right]$$

where q_{k-n+1} is the quotient of $(z-\alpha)^{p-k+n}$ divided by $z-\beta$ and c_{k-n+1} is the remainder of the division; thus, for $k = p+1$ we have

$$h_Y^{(p+1)}(z) = \log\left(\frac{z-\alpha}{z-\beta}\right) + \sum_{n=1}^{p+1} \frac{(\alpha-\beta)}{n!}\left[q_{p+2-n}^{(n-1)}(z) + (-1)^{n+1}\frac{(n-1)!c_{p+2-n}}{(z-\beta)^n}\right]$$

which is the sum of $\log\left(\frac{z-\alpha}{z-\beta}\right)$ and a rational function bounded near α. Therefore $h_Y^{(p+1)}$ is unbounded on Y because $\alpha \in \overline{Y}$, while $h_Y^{(k)} \in A(\Omega, J)$ since it can be continuously extended on J by setting $h_Y^{(k)}(\alpha) = 0$ for $0 \le k \le p$, by using the fact that $\operatorname{Im}\log\frac{z-\alpha}{z-\beta} = \operatorname{Arg}\frac{z-\alpha}{z-\beta}$ is bounded on Ω. This completes the proof of the claim. □

If $J \cap \overline{Y} = \emptyset$, then according to Lemma 2.2, there exists $\alpha \in \overline{Y} \cap \partial\Omega$ which implies $\alpha \notin J$. We consider the function $h_Y = \frac{1}{z-\alpha}$ which has the desired property. This completes the proof. □

5. Localization of Bergman spaces

We now proceed to a third application of Theorem 2.8. Let $p \in (0,\infty)$, $\Omega \subset \mathbb{C}$ a domain and $V \subset \Omega$ a bounded open subset of Ω. We define $OL^p(\Omega, V)$ to be the space of holomorphic functions on Ω such that $\iint_V |f|^p dxdy < \infty$, see [3], [5].

For $1 \le p < \infty$ the topology of $OL^p(\Omega, V)$ is the Fréchét topology induced by the seminorms

$$\|f\|_{1,p,V} = \left(\iint_V |f|^p dxdy\right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{m,p,V} = \|f|_{K_m}\|_\infty$$

for $m = 2,3,\dots$ where K_m is an exhaustive sequence of compact subsets of Ω. The distance function is

$$d_{p,V}(f,g) = \sum_{m=1}^\infty \frac{1}{2^m} \cdot \frac{\|f-g\|_{m,p,V}}{1+\|f-g\|_{m,p,V}}.$$

In a similar way for $0 < p < 1$ we set

$$d_{1,p,V}(f,g) = \iint_V |f - g|^p dx dy$$

and

$$d_{m,p,V}(f,g) = \sup_{z \in K_m} |f(z) - g(z)|, \quad m = 2, 3, \ldots$$

and the distance function in $OL^p(\Omega, V)$ is given by

$$d_{p,V}(f,g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{d_{m,p,V}(f,g)}{1 + d_{m,p,V}(f,g)}.$$

It is clear that for all $0 < p < \infty$, $OL^p(\Omega, V)$ satisfies the requirements of Theorem 2.8. In addition we consider the spaces X_q for $0 < q \le \infty$ which are defined by

$$X_q = \bigcap_{p<q} OL^p(\Omega, V).$$

Convergence in X_q is taken to coincide with convergence in $OL^p(\Omega, V)$ for all $p < q$. It is easy to see that if $\{p_n\}$, $n = 1, 2, 3, \ldots$, is a strictly increasing sequence converging to q, then convergence in X_q is equivalent to convergence in $OL^{p_n}(\Omega, V)$ for all $n = 1, 2, \ldots$, i.e. it is induced by the distance function

$$d_{X_q,V}(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_{p_n,V}(f,g)}{1 + d_{p_n,V}(f,g)}.$$

One can check that the space X_q satisfies the requirements of Theorem 2.8. Thus, in order to prove that in each of the previously mentioned spaces the set of totally unbounded functions and the set of non-extendable functions are G_δ and dense it suffices to find the function h_Y of Theorem 2.8.

Theorem 5.1. *Let $\Omega \subset \mathbb{C}$ be a domain and $V \subset \Omega$ a bounded open set, such that for each $\alpha \in \overline{V} \cap \partial\Omega$ there exists $\beta \in \Omega^c$ with $\beta \neq \alpha$ and $[\alpha, \beta] \subseteq \Omega^c$. Let X be any of the spaces $OL^p(\Omega, V)$ or X_q as above. Then the set of totally unbounded functions in X is a G_δ and dense subset of X. Furthermore, the set of non-extendable functions in X is also a G_δ and dense subset of X.*

Proof. Let $B(\zeta, \varepsilon)$ be a ball centered at $\zeta \in \partial\Omega$ with radius $\varepsilon > 0$ and let Y be a connected component of $\Omega \cap B(\zeta, \varepsilon)$. According to Theorem 2.8 it suffices to find a function $h_Y \in X$ whose restriction on Y is unbounded.

If $\overline{Y} \cap \overline{V} \cap \partial\Omega \neq \emptyset$ we take $\alpha \in \overline{Y} \cap \overline{V} \cap \partial\Omega$ and by assumption there exists $\beta \in \Omega^c$ such that $[\alpha, \beta] \subseteq \Omega^c$. Hence the function $f(z) = \log\left(\frac{z-\alpha}{a-\beta}\right)$ is holomorphic on Ω. We set $h_Y(z) = \log\left(\frac{z-\alpha}{z-\beta}\right)$. This function belongs to all spaces X mentioned above. This can be easily seen using the following facts:

1) the triangle inequality of the p-norm
2) the fact that the $\arg\left(\frac{z-\alpha}{z-\beta}\right)$ is bounded on V
3) the fact that the set V is bounded which implies $(\log|z - \alpha|)|z - \alpha|^{\frac{1}{p}} \le C_{p,R} < \infty$ for all z belonging to a ball $B(\alpha, R)$ containing V
4) the fact that $\iint_{B(\alpha,R)} |\log|z - \alpha||^p dx dy < \infty$ for each $p \in (0, \infty)$ which follows easily using polar coordinates with center α.

If $\overline{Y} \cap \overline{V} \cap \partial\Omega = \emptyset$, then according to Lemma 2.2 we have $\overline{Y} \cap \partial\Omega \neq \emptyset$. In this case we take $\alpha \in \overline{Y} \cap \partial\Omega$ which implies $\alpha \notin \overline{V}$ and the function $h_Y = \frac{1}{z-\alpha}$ is unbounded on Y and belongs to X, because it is bounded on V. This completes the proof according to Theorem 2.8. $\square$

6. Localization of Hardy spaces in the disc

We now consider localized Hardy spaces in the unit disc D in $\mathbb{C}$. Let $0 < p < \infty$ and $J = J_{[\alpha,\beta]} = \{e^{i\theta} : \alpha \leq \theta \leq \beta\}$, $\alpha < \beta \leq \alpha + 2\pi$, a closed arc of the unit circle $\mathbb{T}$. We define $H^p(D, J)$ to be the space of holomorphic functions on D such that

$$\sup_{0 \leq r < 1} \int_\alpha^\beta |f(re^{i\theta})|^p d\theta < \infty.$$

For $1 \leq p < \infty$ the topology of $H^p(D, J)$ is the Fréchét topology induced by the seminorms

$$\|f\|_{1,p,J} = \sup_{0 \leq r < 1} \left(\int_\alpha^\beta |f(re^{i\theta})|^p d\theta \right)^{1/p}$$

and

$$\|f\|_{m,p,J} = \sup_{|z| \leq 1 - \frac{1}{m}} |f(z)| \quad m = 2, 3, \dots ,$$

and the distance function is

$$d_{p,J}(f, g) = \sum_{m=1}^\infty \frac{1}{2^m} \frac{\|f - g\|_{m,p,J}}{1 + \|f - g\|_{m,p,J}}.$$

For $0 < p < 1$ we set

$$d_{1,p,J}(f, g) = \sup_{0 \leq r < 1} \int_\alpha^\beta |f(re^{i\theta}) - g(re^{i\theta})|^p d\theta,$$

$$d_{m,p,J}(f, g) = \sup_{|z| \leq 1 - \frac{1}{m}} |f(z) - g(z)|, \quad m = 2, 3, \dots$$

so the distance function is given by

$$d_{p,J}(f, g) = \sum_{m=1}^\infty \frac{1}{2^m} \frac{d_{m,p,J}(f, g)}{1 + d_{m,p,J}(f, g)}.$$

It is easy to check that for $0 < p < \infty$ the space $H^p(D, J)$ satisfies the requirements of the Theorem 2.8.

We also consider the spaces $X_q(D, J)$ for $q \in (0, \infty]$ defined as $X_q(D, J) = \cap_{p<q} H^p(D, J)$. Convergence in this space is equivalent to convergence in $H^p(D, J)$ for all $p < q$, which coincides with convergence in $H^{p_n}(D, J)$ for all n, where $\{p_n\}$, $n = 1, 2, \dots$, is a strictly increasing sequence convergent to q. The distance function in $X_q(D, J)$ is given by

$$d_{X_q,J}(f, g) = \sum_{n=1}^\infty \frac{1}{2^n} \frac{d_{p_n,J}(f, g)}{1 + d_{p_n,J}(f, g)}.$$

One can check that the space $X_q(D, J)$ satisfies the requirements of Theorem 2.8. If we consider a denumerable family of spaces of the previous type with varying J, p or q then the intersection of these spaces endowed with a distance function as above also satisfies the requirements of Theorem 2.8.

In addition if $O \subset \mathbb{T}$ is a relatively open set and J_m are defined,

$$J_m = \left\{ z \in O : \operatorname{dist}(z, \mathbb{T} - O) \geq \frac{1}{m} \right\}, \quad m = 1, 2, \dots ,$$

then J_m are an exhaustive family of compact subsets of O, and we define the space $H^p(D, O)$ by

$$H^p(D, O) = \left\{ f \in H(D) : \sup_{0 \leq r < 1} \int_{J_m} |f(re^{i\theta})|^p d\theta < \infty \text{ for all } m = 1, 2, \dots \right\}.$$

We notice that each J_m is a finite disjoint union of closed subarcs of $\mathbb{T}$. Thus $H^p(D, O)$ coincides with the set of f holomorphic on D such that $f \in H^p(D, J)$ for all closed arcs $J \subset O$ and endowed with the natural topology induced by $H^p(D, J_m)$, $m = 1, 2, \dots$, and satisfies the requirements of Theorem 2.8. Furthermore, spaces that are denumerable intersections of such spaces, satisfy the requirements of Theorem 2.8.

Thus, in order to prove that in each one of the previous spaces the set of totally unbounded functions and the set of non extendable functions are G_δ and dense it suffices to find the function h_Y of Theorem 2.8.

Let $Y = D \cap \{z : |z - 1| < \varepsilon\}$, $\varepsilon > 0$. It is well known that the function $h_Y(z) = \log(1 - z)$ belongs to all Hardy spaces $H^p(D)$, $0 < p < \infty$, therefore it also belongs to all localized Hardy spaces or their intersections as considered above. This function h_Y is unbounded on Y, and its rotations $h_{Y,\varphi}(z) = \log(1 - e^{-i\varphi}z)$ are unbounded on $Y_\varphi = D \cap \{z : |z - e^{i\varphi}| < \varepsilon\}$. Thus, Theorem 2.8 implies the following.

Theorem 6.1. *Let X be any localized $H^p(D, J)$, $J \subset \mathbb{T}$ closed arc, $0 < p < \infty$ or any denumerable intersection of them. Then the set of totally unbounded functions in X is a G_δ and dense subset of X. Furthermore, the set of non-extendable functions in X is also a G_δ and dense subset of X.*

It is interesting to observe that, as in the case of classical Hardy spaces, functions in a localized Hardy space posses nontangential limits on the appropriate arc.

Proposition 6.2. *Let $J = \{e^{i\theta} : A \leq \theta \leq B\} \subset \mathbb{T}$, $A < B \leq A + 2\pi$, be a closed arc and $0 < p < \infty$. If $f \in H^p(D, J)$ then for almost all $e^{i\theta}$ in J with respect to arc length measure the non-tangential limit of $f(z)$ at $e^{i\theta}$ exists.*

Proof. Since $\sup_{0 \leq r < 1} \int_A^B |f(re^{i\theta})|^p d\theta = M < \infty$ we have

$$\int_0^1 \int_A^B |f(re^{i\theta})|^p d\theta dr < \infty.$$

By Fubini's theorem for almost all α and β, $A < \alpha < \beta < B$ we have

$$\int_0^1 |f(re^{i\alpha})|^p dr = M_\alpha < \infty \quad \text{and} \quad \int_0^1 |f(re^{i\beta})|^p dr = M_\beta < \infty.$$

For such α and β it suffices to prove that for almost all θ in (α, β) the non-tangential limit of $f(z)$ at $e^{i\theta}$ exists.

We consider the sector $\Omega = \{re^{i\theta} : 0 < r < 1, \ \theta \in (\alpha, \beta)\}$ whose boundary has finite length. According to the discussion in [2] it suffices to show that $f|_\Omega$ belongs to the Hardy space $E^p(\Omega)$, consisting of functions on Ω whose integral of p-powers along a sequence of rectifiable Jordan curves tending to the boundary remains bounded.

Let $\varepsilon > 0$ and $r_\varepsilon \in (1 - \frac{\varepsilon}{2}, 1)$. Then

$$\int_\alpha^\beta |f(r_\varepsilon e^{i\theta})|^p d\theta \leq \sup_{0 \leq r < 1} \int_\alpha^\beta |f(re^{i\theta})|^p d\theta = M < \infty.$$

The function f is uniformly continuous on the compact set $\{re^{i\theta} : 0 \leq r \leq r_\varepsilon, a \leq \theta \leq \beta\} \subset D$. We can find segments $[P,Q], [P,R]$ with $P, Q, R \in \Omega$ and $|Q| = |R| = r_\varepsilon$, such that $[P,Q]$ is parallel to $[0, e^{i\alpha}]$, $[P,R]$ is parallel to $[0, e^{i\beta}]$ and such that the curve $[P,Q] \cup \widehat{QR} \cup [R,P]$ is ε-close to $\partial\Omega$, where $\widehat{QR}$ is an arc in the circle with center 0 and radius r_ε, in such a way that $\int_{[P,Q]} |f(z)|^p|dz| < M_\alpha + 1$ and $\int_{[P,R]} |f(z)|^p|dz| < M_\beta + 1$. It follows that

$$\int_{[P,Q] \cup \widehat{QR} \cup [R,P]} |f(z)|^p|dz| < M + M_\alpha + M_\beta + 2.$$

Since $M + M_\alpha + M_\beta + 2$ is independent of ε, it follows that $f|_\Omega$ belongs to the Hardy space $E^p(\Omega)$ and the proof is complete. □

7. Functions with Taylor coefficients in l^p, c_o and l^∞

We next consider the spaces l_a^p, $c_{0,a}$ and l_a^∞ consisting of holomorphic functions in the unit disc whose sequence of Taylor coefficients is p-summable, a null sequence, or a bounded sequence respectively. More precisely for $0 < p \leq \infty$ let

$$l_a^p = \{f(z) = \sum_{n=0}^\infty a_n z^n : (a_n) \in l^p\}$$

where l^p is the classical space of complex sequences (a_n) such that $\|(a_n)\|_p = (\sum_{n=0}^\infty |a_n|^p)^{1/p} < \infty$, and $\|(a_n)\|_\infty = \sup_n |a_n|$ for $p = \infty$. For $c_{0,a}$ the sequence (a_n) tends to 0. It is clear that each power series in these spaces has radius of convergence at least 1 since in all cases (a_n) is a bounded sequence, so the series defines an analytic function f on the unit disc D. For $1 \leq p \leq \infty$ the spaces l_a^p are Banach spaces with the norm inherited from l^p, and the same is true for $c_{0,a}$ with norm inherited from l_a^∞. For $0 < p < 1$ the distance function

$$d_p(f, g) = \sum_{n=0}^\infty |f(n) - g(n)|^p$$

(where we use the notation $f(n)$ for the nth Taylor coefficient of an f), is a complete metric and makes l_a^p into a complete linear topological space.

If X is any of the above spaces then it can be checked easily that convergence in X of a sequence (f_n), $f_n(z) = \sum_{k=0}^\infty f_n(k)z^k$, to a function $f(z) = \sum_{k=0}^\infty f(k)z^k$ implies

$$\sup_k |f_n(k) - f(k)| \to 0, \text{ as } n \to \infty.$$

Thus if $0 < r < 1$ then for $|z| < r$,

$$|f_n(z) - f(z)| \leq \sum_{k=0}^{\infty} |f_n(k) - f(k)||z|^k \leq \sup_k |f_n(k) - f(k)| \sum_{k=0}^{\infty} r^k$$

$$= \frac{1}{1-r} \sup_k |f_n(k) - f(k)|,$$

and it follows that convergence in X implies uniform convergence on compact subsets of D. Thus X satisfies the requirements of Theorem 2.8.

We may also consider the spaces $X_q = \cap_{p>q} \ell_a^p$ for $0 \leq q < \infty$ with the distance function $d_q(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_{p_n}}{1 + d_{p_n}}$ where (p_n) is a strictly decreasing sequence of indices converging to q, and d_{p_n} is the distance function in $l_a^{p_n}$. Then it is easily verified that the space X_q, $0 \leq q < \infty$, also satisfies the requirements of Theorem 2.8.

Theorem 7.1. *Let X be any of the spaces l_a^p, $(1 < p \leq \infty)$, $c_{0,a}$ or X_q, $(1 \leq q < \infty)$. Then the set of totally unbounded functions in X is a G_δ and dense subset of X, and the same is true for the set of non-extendable functions in X.*

Proof. To apply Theorem 2.8 it suffices to find the function h_Y in X. Let $Y = D \cap \{z : |z - 1| < \varepsilon\}$, $\varepsilon > 0$. Then function $h_Y(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^k$ satisfies $\lim_{r \to 1^-} h_Y(r) = \infty$ and h_Y is unbounded on Y. It belongs to the space X because $\sum_{k=1}^{\infty} \frac{1}{k^p} < \infty$ for all $1 < p < \infty$. If $Y = D \cap \{z : |z - e^{i\theta}| < \varepsilon\}$ then the rotated function $h_Y(z) = \sum_{k=1}^{\infty} \frac{1}{k} e^{-ik\theta} z^k$ has the same properties on Y, and Theorem 2.8 applies and gives the assertion. □

If $f \in l_a^p$ with $0 < p \leq 1$ then f extends continuously on the closed disc $\overline{D}$ and is bounded, but the derivative f' can be totally unbounded.

Theorem 7.2. *Let X be any of the space l_a^p, $(0 < p \leq \infty)$, $c_{0,a}$ or X_q, $0 \leq q < \infty$. Then the set of functions $f \in X$ such that the derivative f' is totally unbounded is a G_δ and dense subset of X. The same holds for the set of non-extendable functions in X.*

Proof. We will apply again Theorem 2.8. Let $Y = D \cap \{z : |z - 1| < \varepsilon\}$. The function $h_Y(z) = \sum_{n=0}^{\infty} \frac{1}{2^n} z^{2^n}$ belongs to X since $\sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right)^p = \frac{2^p}{2^p - 1} < \infty$ for all $p > 0$. Its derivative $h_Y'(z) = \sum_{n=0}^{\infty} z^{2^n - 1}$ satisfies $\lim_{r \to 1^-} h_Y'(r) = \infty$ and h_Y' is unbounded on Y. If $Y = D \cap \{z : |z - e^{i\theta}| < \varepsilon\}$ then $h_Y(z) = \sum_{n=0}^{\infty} \frac{e^{-i2^n \theta}}{2^n} z^{2^n}$ has the same property on Y. Thus Theorem 2.8 applies. □

Remark. Since the sets $D \cap \{z : |z - e^{i\theta}| < \varepsilon\}$ are convex, it follows easily that, if g is holomorphic on D and the derivative $g^{(k)}$ for some $k \geq 0$ is totally unbounded on D then for every $\ell \geq k$ the derivative $g^{(\ell)}$ is also totally unbounded. Thus the conclusion of Theorem 7.1 implies that the set of $f \in X$ such that every derivative $f^{(\ell)}$, $\ell \geq 0$, is totally unbounded is a G_δ and dense subset of X. Similarly in Theorem 7.2 the set of $f \in X$ such that every derivative $f^{(\ell)}$, $\ell \geq 1$, is totally unbounded is a G_δ and dense subset of X.

Finally, we mention that in the spaces X considered in this section a necessary and sufficient condition so that the set of totally unbounded functions f in X is G_δ and dense is that there exists an unbounded function g in X. One can easily prove the above statement not only for the functions themselves but for their derivatives, as well, but we do not need this fact. We mention it because it furnishes interesting information.

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