J. Math. Anal. Appl. 479 (2019) 122–134

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Two identities on the mock theta function $V_0(q)$

Renrong Mao

Department of Mathematics, Soochow University, SuZhou 215006, PR China

ARTICLE INFO

Article history:
Received 1 December 2018
Available online 10 June 2019
Submitted by B.C. Berndt

Keywords:
Mock theta function
Appell-Lerch sums
Generalized Dedekind eta function
Congruence

ABSTRACT

In this paper, applying the theory of (mock) modular forms and Zwegers' results on Appell-Lerch sums, we establish two identities on the eighth-order mock theta function $V_0(q)$. Using these identities, we prove some congruences for $V_0(q)$.

1. Introduction

A partition of a positive integer n is a sequence of non-increasing positive integers whose sum equals n and $p(n)$ is defined to be the number of partitions of n while $p(0) := 1$. The following three famous congruences for $p(n)$ were found and later proved by S. Ramanujan: for all $n \in \mathbb{N}$, we have

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Indeed, Ramanujan [21] found the generating functions for $p(5n+4)$ and $p(7n+5)$,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}. \tag{1.2}$$

E-mail address: rrmao@suda.edu.cn.

https://doi.org/10.1016/j.jmaa.2019.06.018
0022-247X/

Here and for the rest of this article, we use the notations

$$(x_1, x_2, \dots, x_k; q)_m := \prod_{n=0}^{m-1} (1 - x_1 q^n)(1 - x_2 q^n) \cdots (1 - x_k q^n),$$
$$(x_1, x_2, \dots, x_k; q)_\infty := \prod_{n=0}^{\infty} (1 - x_1 q^n)(1 - x_2 q^n) \cdots (1 - x_k q^n),$$
$$[x_1, x_2, \dots, x_k; q]_\infty := (x_1, q/x_1, x_2, q/x_2, \dots, x_k, q/x_k; q)_\infty,$$

and we require $|q| < 1$ for absolute convergence. Ramanujan's work inspired the search for identities similar to (1.1) and (1.2) involving various types of special functions. For example, Hirschhorn and Hunt [16] proved identities on the generating functions for $p(5^\alpha n + \delta_\alpha)$, where α is a positive integer and δ_α is the reciprocal of 24 modulo 5^α. Using these results, they provided a simple proof of the following congruences:

$$p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha},$$

which was conjectured by Ramanujan [21] and first proved by Watson [24]. More recently, Garvan [13] established identities analogous to (1.1) and (1.2) involving Andrews' smallest parts partition function $\mathrm{spt}(n)$ [4]. As applications, families of congruences for $\mathrm{spt}(n)$ modulo powers of 5, 7 and 13 were obtained.

Throughout, we assume $\tau \in \mathbb{C}$ with $\mathrm{Re}(\tau) > 0$. Let $q := e^{2\pi\tau}$. We study the following mock theta function

$$V_0(\tau) = V_0(q) = -1 + 2\sum_{n\geq 0} \frac{q^{n^2}(-q; q^2)_n}{(q; q^2)_n} =: \sum_{n=0}^{\infty} g(n) q^n.$$

The function $V_0(q)$ is an eighth-order mock theta function first studied by Gordon and McIntosh in [14]. Applying the generalized Lambert series identities in [7], Chan and the author [9] proved some analogies of (1.1) and (1.2) for mock theta functions. As applications, congruences for many mock theta functions were established. In particular, they obtained

$$\sum_{n=0}^{\infty} g(8n+3) q^n = 4\frac{(-q, -q^3; q^4)_\infty (q^8; q^8)_\infty^4}{(q; q)_\infty^3 (-q^4; q^4)_\infty^2} \left(\frac{1}{(q, q^7; q^8)_\infty^2} + \frac{q}{(q^3, q^5; q^8)_\infty^2} \right), \tag{1.3}$$
$$\sum_{n=0}^{\infty} g(8n+6) q^n = 8\frac{(q^8; q^8)_\infty^4}{(q; q)_\infty^3} \left(\frac{1}{(q, q^7; q^8)_\infty^2} + \frac{q}{(q^3, q^5; q^8)_\infty^2} \right), \tag{1.4}$$

which imply $g(8n+3) \equiv 0 \pmod 4$ and $g(8n+6) \equiv 0 \pmod 8$, respectively. For more recent works on identities involving mock theta functions, the reader is referred to [3,8,10,18,25].

In this paper, using the theory of (mock) modular forms, we establish the following two identities for $V_0(q)$.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty} g(8n+2) q^n = \frac{4(q^4; q^4)_\infty}{[q; q^4]_\infty^6 [q^2; q^4]_\infty^2 (-q^4; q^4)_\infty} \tag{1.5}$$

and

$$\sum_{n=0}^{\infty} g(8n+5) q^n = \frac{8(q^4; q^4)_\infty}{[q; q^4]_\infty^5 (q^2; q^4)_\infty^3}. \tag{1.6}$$

In particular, we have $g(8n+2) \equiv 0 \pmod 4$ *and* $g(8n+5) \equiv 0 \pmod 8$.

Congruences for mock theta functions have been studied by many authors recently. Garthwaite [12] proved the existence of infinitely many congruences for the third-order mock theta function, $\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}$. Waldherr [23] provided two explicit congruences for $\omega(q)$:

$$a_\omega(40n+27) \equiv a_\omega(40n+35) \equiv 0 \pmod 5,$$

where $a_\omega(n)$ are coefficients of $\omega(q)$. More studies on congruences for mock theta functions can be found in [1,2,20].

Armed with (1.6), we prove two congruences for $V_0(q)$ by elementary q-series manipulation.

Corollary 1.2. *We have*

$$g(40n+13) \equiv g(40n+37) \equiv 0 \pmod{40}.$$

2. Preliminaries

For the basics of (mock) modular forms, the reader is referred to [17,19].

2.1. Appell-Lerch sums

The key in the proof of Theorem 1.1 is the mock modularity of $V_0(q)$ which is proved by applying Zwegers' results on Appell-Lerch sums.

For $u, v \in \mathbb{C} \setminus (\mathbb{Z} + \mathbb{Z}\tau)$, we define the Jacobi theta function and Appell-Lerch sums as follows:

$$\begin{aligned} \vartheta(v;\tau) &:= \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} e^{\pi \nu^2 \tau + 2\pi i \nu (v + \frac{1}{2})} \\ &= -iq^{\frac{1}{8}} e^{-\pi i v} \prod_{n=1}^{\infty} (1-q^n)(1-e^{2\pi i v} q^{n-1})(1-e^{-2\pi i v} q^n) \end{aligned} \tag{2.1}$$

and

$$\mu(u,v;\tau) := \frac{e^{\pi i u}}{\vartheta(v;\tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{2\pi i n v} q^{\frac{n(n+1)}{2}}}{1 - e^{2\pi i u} q^n}. \tag{2.2}$$

The μ-function itself does not transform as a modular form. Zwegers [26] discovered that it can be completed to a function $\widehat{\mu}$ having nice transformation properties.

To describe the completion, we define

$$\begin{aligned} R(u) &= R(u;\tau) \\ &:= \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} (-1)^{\nu - \frac{1}{2}} \left\{ sgn(\nu) - E\left(\left(\nu + \frac{\mathrm{Im}(u)}{\mathrm{Im}(\tau)} \right) \sqrt{2\mathrm{Im}(\tau)} \right) \right\} e^{-2\pi i \nu u} q^{-\frac{\nu^2}{2}}. \end{aligned} \tag{2.3}$$

Here $E(x)$ is defined by

$$E(x) := 2\int_0^x e^{-\pi u^2} du = \mathrm{sgn}(x)(1 - \beta(x^2)),$$

where for positive real x we let $\beta(x) = \int_x^\infty u^{-\frac{1}{2}} e^{-\pi u^2} du$. Let

$$\widehat{\mu}(u,v) = \widehat{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2} R(u-v;\tau). \tag{2.4}$$

To state the modular transformation formula of $\widehat{\mu}(u,v)$, we need the multiplier of the Dedekind eta-function $\eta(\tau)$. Let $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^\infty (1-q^n)$. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we define the multiplier $\nu_\eta(A)$ by

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta(A)\sqrt{(c\tau+d)}\eta(\tau). \tag{2.5}$$

Lemma 2.1. *([26, Theorem 1.11]). If $k, l, m, n \in \mathbb{Z}$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then we have*

$$\widehat{\mu}(u+k\tau+l, v+m\tau+n) = (-1)^{k+l+m+n} e^{\pi i\tau(k-m)^2 + 2\pi i(k-m)(u-v)} \widehat{\mu}(u,v), \tag{2.6}$$

and

$$\widehat{\mu}\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \nu_\eta^{-3}(A)\sqrt{c\tau+d}e^{-\pi ic(u-v)^2/(c\tau+d)} \widehat{\mu}\left(u,v;\tau\right). \tag{2.7}$$

Armed with Lemma 2.1, we prove the mock modularity of $V_0(q)$ in the following lemma.

Lemma 2.2. *Let*

$$f(\tau) := V_0(\tau) + R(0;8\tau).$$

Then, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(8)$, we have

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta^{-3}(A')\sqrt{c\tau+d} f\left(\tau\right), \tag{2.8}$$

where

$$A' = \begin{pmatrix} a & 8b \\ c/8 & d \end{pmatrix}.$$

Proof. Equation (5.41) of [15] gives

$$V_0(q) = -q^{-1}m(1,q^8,q) - q^{-1}m(1,q^8,q^3),$$

where

$$m(x,q,z) := \frac{1}{(z,q/z,q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2} z^n}{1-q^{n-1}xz}.$$

This together with (2.2) implies

$$iV_0(\tau) = \mu(\tau,\tau;8\tau) + \mu(3\tau,3\tau;8\tau).$$

Thus we have

$$\begin{aligned}f(\tau) &= V_0(\tau) + R(0; 8\tau)\\ &= -i\left\{\mu(\tau,\tau;8\tau) + \frac{i}{2}R(0;8\tau)\right\} - i\left\{\mu(3\tau,3\tau;8\tau) + \frac{i}{2}R(0;8\tau)\right\}\\ &= -i\widehat{\mu}(\tau,\tau;8\tau) - i\widehat{\mu}(3\tau,3\tau;8\tau),\end{aligned} \tag{2.9}$$

where the last equality follows from (2.4).

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(8)$, one can verify that

$$\widehat{\mu}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\tau+b}{c\tau+d}; 8\frac{a\tau+b}{c\tau+d}\right) = \widehat{\mu}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\tau+b}{c\tau+d}; A'(8\tau)\right),$$

where

$$A' = \begin{pmatrix} a & 8b \\ c/8 & d \end{pmatrix}.$$

By (2.7), we find that

$$\widehat{\mu}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\tau+b}{c\tau+d}; A'(8\tau)\right) = \nu_\eta^{-3}(A')\sqrt{c\tau+d}\,\widehat{\mu}\left(a\tau+b, a\tau+b; 8\tau\right). \tag{2.10}$$

Recall that $A \in \Gamma_1(8)$ and we have $a \equiv 1 \pmod 8$. Applying (2.6) on the left hand side of (2.10), we find that

$$\widehat{\mu}\left(a\tau+b, a\tau+b; 8\tau\right) = \widehat{\mu}\left(\tau,\tau;8\tau\right).$$

Thus we obtain

$$\widehat{\mu}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\tau+b}{c\tau+d}; 8\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta^{-3}(A')\sqrt{c\tau+d}\,\widehat{\mu}\left(\tau,\tau;8\tau\right) \tag{2.11}$$

Similarly, one can prove that

$$\widehat{\mu}\left(3\frac{a\tau+b}{c\tau+d}, 3\frac{a\tau+b}{c\tau+d}; 8\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta^{-3}(A')\sqrt{c\tau+d}\,\widehat{\mu}\left(3\tau,3\tau;8\tau\right). \tag{2.12}$$

By (2.9), we find that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = -i\widehat{\mu}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\tau+b}{c\tau+d}; 8\frac{a\tau+b}{c\tau+d}\right) - i\widehat{\mu}\left(3\frac{a\tau+b}{c\tau+d}, 3\frac{a\tau+b}{c\tau+d}; 8\frac{a\tau+b}{c\tau+d}\right). \tag{2.13}$$

Substituting (2.11) and (2.12) into (2.13), we prove (2.8). □

2.2. Generalized Dedekind eta-functions

To obtain the modular properties of the infinite products on (1.5) and (1.6), we need some results on the generalized Dedekind eta-products.

Define the generalized Dedekind eta function by

$$\eta_{\delta;g}(\tau) = q^{\frac{\delta}{2}P_2(g/\delta)} \prod_{m\equiv \pm g \pmod{\delta}} (1-q^m), \tag{2.14}$$

where $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second periodic Bernoulli polynomial, $\{t\} = t - [t]$ is the fractional part of t, $g, \delta, m \in \mathbb{Z}^+$.

Let N be a fixed positive integer. A generalized eta-product of level N is given by

$$h(\tau) = \prod_{\substack{\delta|N \\ 0<g<\delta}} \eta_{\delta;g}^{r_{\delta,g}}(\tau), \tag{2.15}$$

where

$$r_{\delta,g} \in \begin{cases} \frac{1}{2}\mathbb{Z} & \text{if } g = \delta/2, \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

Robins proved the following result.

Theorem 2.3. *([22, Theorem 3]) The function $h(\tau)$ defined in (2.15) is a modular function on $\Gamma_1(N)$ if*

(i)

$$\sum_{\substack{\delta|N \\ 0<g<\delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} \equiv 0 \pmod{2}, \text{ and}$$

(ii)

$$\sum_{\substack{\delta|N \\ 0<g<\delta}} \frac{N}{\delta} P_2(0)\, r_{\delta,g} \equiv 0 \pmod{2}.$$

To check the modularity of the infinite products on (1.5) and (1.6), we define

$$R_2(\tau) := q^{1/4} \frac{4(q^4;q^4)_\infty}{[q;q^4]_\infty^6 [q^2;q^4]_\infty^2 (-q^4;q^4)_\infty} \tag{2.16}$$

and

$$R_5(\tau) := q^{5/8} \frac{8(q^4;q^4)_\infty}{[q;q^4]_\infty^5 (q^2;q^4)_\infty^3}. \tag{2.17}$$

Rewrite $R_2(\tau)$ and $R_5(\tau)$ in terms of generalized Dedekind eta-functions as follows:

$$R_2(\tau) = \frac{4\eta(8\tau)\eta_{8;4}^{3/2}(\tau)}{\eta_{8;1}^6(\tau)\eta_{8;2}^2(\tau)\eta_{8;3}^6(\tau)}$$

and

$$R_5(\tau) = \frac{8\eta(4\tau)}{\eta_{4;1}^5(\tau)\eta_{4;2}^{3/2}}.$$

Applying Theorem 2.3 and the MAPLE program in [11], we obtain the modular properties of $\left(\frac{R_2(\tau)}{\eta(8\tau)}\right)^{24}$ and $\left(\frac{R_5(\tau)}{\eta(4\tau)}\right)^{24}$.

Lemma 2.4. *The functions* $\left(\frac{R_2(\tau)}{\eta(8\tau)}\right)^{24}$ *and* $\left(\frac{R_5(\tau)}{\eta(4\tau)}\right)^{24}$ *are weakly holomorphic modular functions on* $\Gamma_1(8)$. *The first term in the Fourier expansion of* $\left(\frac{R_2(\tau)}{\eta(8\tau)}\right)^{24}$ *(resp.* $\left(\frac{R_5(\tau)}{\eta(4\tau)}\right)^{24}$*) at the inequivalent cusps* ξ *of* $\Gamma_1(8)$*, up to a nonzero constant, is given by*

(1) $q^{-\frac{25}{8}}$ *(resp.* $q^{-\frac{13}{4}}$*) when* $\xi = 0$,
(2) q^{-2} *(resp.* q^{11}*) when* $\xi = \infty$,
(3) $q^{11/2}$ *(resp.* q^{2}*) when* $\xi = \frac{1}{2}$,
(4) $q^{-\frac{25}{8}}$ *(resp.* $q^{-\frac{13}{4}}$*) when* $\xi = \frac{1}{3}$,
(5) q^{16} *(resp.* q^{11}*) when* $\xi = \frac{1}{4}$,
(6) q^{-2} *(resp.* q^{11}*) when* $\xi = \frac{3}{8}$.

We prove the modularity of $R_2^{24}(\tau)$ and $R_5^{24}(\tau)$ in the following lemma.

Lemma 2.5. *The functions* $R_2^{24}(\tau)$ *and* $R_5^{24}(\tau)$ *are weakly holomorphic modular forms of weight* 12 *on* $\Gamma_1(8)$. *The first term in the Fourier expansion of* $R_2^{24}(\tau)$ *(resp.* $R_5^{24}(\tau)$*) at the inequivalent cusps* ξ *of* $\Gamma_1(8)$*, up to a nonzero constant, is given by*

(1) q^{-3} *(resp.* q^{-3}*) when* $\xi = 0$
(2) q^{6} *(resp.* q^{15}*) when* $\xi = \infty$
(3) q^{6} *(resp.* q^{3}*) when* $\xi = \frac{1}{2}$
(4) q^{-3} *(resp.* q^{-3}*) when* $\xi = \frac{1}{3}$
(5) q^{18} *(resp.* q^{15}*) when* $\xi = \frac{1}{4}$
(6) q^{6} *(resp.* q^{15}*) when* $\xi = \frac{3}{8}$

Proof. It is well known that $\eta(\tau)^{24}$ is a cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. By [17, Proposition 17, p. 127], we find that $\eta(8\tau)^{24}$ (resp. $\eta(4\tau)^{24}$) is a cusp form of weight 12 for $\Gamma_0(8)$ (resp. $\Gamma_0(4)$). This together with Lemma 2.4 proves the first statement of Lemma 2.5. Applying (2.5), one can obtain the q-expansions of $\eta(8\tau)^{24}$ and $\eta(4\tau)^{24}$ at the cusps of $\Gamma_1(8)$. Combining these with the q-expansions of $\left(\frac{R_2(\tau)}{\eta(8\tau)}\right)^{24}$ and $\left(\frac{R_5(\tau)}{\eta(4\tau)}\right)^{24}$ in Lemma 2.4, we can prove the second statement in Lemma 2.5. □

3. Proof of Theorem 1.1

We note that all the functions in equations (1.5) and (1.6) are holomorphic when $|q| < 1$. By analytic continuation, we only need to show that (1.5) and (1.6) are true when q is in the small interval $(-\varepsilon, \varepsilon)$. For small enough ε, it is easy to check that these functions take positive values when $q \in (-\varepsilon, \varepsilon)$. Since the infinite products on the right sides of (1.5) and (1.6) never vanish, the ratios

$$\left(\sum_{n=0}^{\infty} g(8n+2)q^n\right) \Bigg/ \frac{4(q^4;q^4)_\infty}{[q;q^4]_\infty^6 [q^2;q^4]_\infty^2 (-q^4;q^4)_\infty}$$

and

$$\left(\sum_{n=0}^{\infty} g(8n+5)q^n\right) \Bigg/ \frac{8(q^4;q^4)_\infty}{[q;q^4]_\infty^5 (q^2;q^4)_\infty^3}$$

are also holomorphic functions which are positive when $q \in (-\varepsilon, \varepsilon)$. Then we only need to prove that the 24th power of the two ratios equal to 1. Thus, recalling the definition of $R_2(\tau)$ (resp. $R_5(\tau)$) in (2.16) (resp. (2.17)), we find that, to prove Theorem 1.1, it suffices to show that

$$\left(q^{1/4} \sum_{n=0}^{\infty} g(8n+2)q^n\right)^{24} = R_2^{24}(\tau) \tag{3.1}$$

and

$$\left(q^{5/8} \sum_{n=0}^{\infty} g(8n+5)q^n\right)^{24} = R_5^{24}(\tau). \tag{3.2}$$

Let $\zeta_8 := e^{\frac{2\pi i}{8}}$. For $t = 2, 5$, we define

$$f_{\lambda,t}(\tau) := \zeta_8^{-\lambda t} f\left(\begin{pmatrix} 1 & \lambda \\ 0 & 8 \end{pmatrix} \tau\right),$$

where $f(\tau)$ is defined in Lemma 2.2 and

$$f_t(\tau) := \frac{1}{8} \sum_{\lambda=0}^{7} f_{\lambda,t}(\tau). \tag{3.3}$$

Lemma 3.1. *Equations* (3.1) *and* (3.2) *are implied by*

$$f_t^{24}(\tau) = R_t^{24}(\tau), \tag{3.4}$$

for $t = 2, 5$.

Proof. Assume that $f(\tau) =: \sum_{n=-\infty}^{\infty} a_f(n)q^n$. A straightforward calculation gives

$$f_t(\tau) = q^{\frac{t}{8}} \sum_{n=-\infty}^{\infty} a_f(8n+t)q^n. \tag{3.5}$$

By its definition and equation (2.3), we find that the function $f(\tau)$ has the following q-expansion:

$$f(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \left\{\mathrm{sgn}\left(n+\frac{1}{2}\right) - E\left(\left(n+\frac{1}{2}\right)\sqrt{2\mathrm{Im}(8\tau)}\right)\right\} q^{-(2n+1)^2}$$
$$+ \sum_{n=0}^{\infty} g(n)q^n.$$

In particular, the non-holomorphic part of $f(\tau)$ is supported on terms with exponent $-(2n+1)^2$ (which is congruent to 7 modulo 8). This fact together with (3.5) implies that, for $t = 2, 5$, we have

$$f_t(\tau) = q^{\frac{t}{8}} \sum_{n=0}^{\infty} g(8n+t)q^n, \tag{3.6}$$

which proves Lemma 3.1. $\square$

We see that $f_t(\tau)$ are holomorphic functions. The modular transformation properties of $f_t(\tau)$ are given as follows.

Lemma 3.2. *For* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(8)$, *we have*

$$f_t\left(\frac{a\tau+b}{c\tau+d}\right) = \zeta_8^{bt}\nu_\eta^{-3}(B')\sqrt{c\tau+d}f_t(\tau), \tag{3.7}$$

where

$$B' := \begin{pmatrix} a+c\lambda & (d-a)\lambda+b(1-a)-c\lambda(\lambda+b) \\ c & -c(\lambda+b)+d \end{pmatrix}.$$

Remark 1. We show that $\nu_\eta^{-3}(B')$ depends only on a, b, c, d but not on λ.

Proof. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(8)$, we have

$$\begin{pmatrix} 1 & \lambda \\ 0 & 8 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c\lambda & \frac{(d-a)\lambda+b(1-a)-c\lambda(\lambda+b)}{8} \\ 8c & -c(\lambda+b)+d \end{pmatrix}\begin{pmatrix} 1 & \lambda+b \\ 0 & 8 \end{pmatrix}.$$

Since $a \equiv d \equiv 1 \pmod 8$ and $8 \mid c$, we see that

$$B := \begin{pmatrix} a+c\lambda & \frac{(d-a)\lambda+b(1-a)-c\lambda(\lambda+b)}{8} \\ 8c & -c(\lambda+b)+d \end{pmatrix} \in \Gamma_1(8).$$

Hence

$$f_{\lambda,t}\left(\frac{a\tau+b}{c\tau+d}\right) = \zeta_8^{-\lambda t} f\left(\begin{pmatrix} 1 & \lambda \\ 0 & 8 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau\right) = \zeta_8^{-\lambda t} f\left(B\frac{\tau+\lambda+b}{8}\right). \tag{3.8}$$

Applying Lemma 2.2, we find that

$$f\left(B\frac{\tau+\lambda+b}{8}\right) = \nu_\eta^{-3}(B')\sqrt{c\tau+d}f\left(\frac{\tau+\lambda+b}{8}\right). \tag{3.9}$$

Without loss of generality, we assume $c \geq 0$. To examine $\nu_\eta(B')$, we need the following formula for ν_η (see [5, Theorem 3.4], for example): for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$,

$$\nu_\eta(A) = \exp\left\{\frac{\pi i}{12}\left(9 + \frac{a+d}{c} + 12s(-d, c)\right)\right\},$$

where

$$s(d,c) := \sum_{r=1}^{c-1}\frac{r}{c}\left(\frac{dr}{c} - \left[\frac{dr}{c}\right] - \frac{1}{2}\right). \tag{3.10}$$

Hence

$$\nu_\eta(B') = \exp\left\{\frac{\pi i}{12}\left(9 + \frac{a+d-cb}{c} + 12s(c(\lambda+b)-d, c)\right)\right\}.$$

By (3.10), one easily see that $s(c(\lambda+b)-d,c)=s(-d,c)$. This means $\nu_\eta(B')$ does not depend on λ when $c>0$. By the definition of B', it is clear that $\nu_\eta(B')$ does not depend on λ when $c=0$. With these facts, we deduce from (3.3), (3.8) and (3.9) that

$$\begin{aligned} f_t\left(\frac{a\tau+b}{c\tau+d}\right) &= \frac{1}{8}\sum_{\lambda=0}^{7}\zeta_8^{-\lambda t} f\left(\begin{pmatrix}1 & \lambda\\ 0 & 8\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\tau\right)\\ &= \frac{1}{8}\nu_\eta^{-3}(B')\sqrt{c\tau+d}\sum_{\lambda=0}^{7}\zeta_8^{-\lambda t} f\left(\frac{\tau+\lambda+b}{8}\right). \end{aligned} \tag{3.11}$$

Proceeding as in the proof of (3.6), we can show that

$$\frac{1}{8}\sum_{\lambda=0}^{7}\zeta_8^{-\lambda t} f\left(\frac{\tau+\lambda+b}{8}\right)=\zeta_8^{bt}q^{\frac{t}{8}}\sum_{n=0}^{\infty}g(8n+t)q^n=\zeta_8^{bt}f_t(\tau).$$

Substituting this into (3.11), we obtain (3.7). □

It is well known that the multiplier ν_η is a 24th root of unity. Thus we have

$$f_t^{24}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{12}f_t^{24}(\tau), \tag{3.12}$$

for $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_1(8)$. Hence $f_t^{24}(\tau)$ are weakly holomorphic modular forms of weight 12 on $\Gamma_1(8)$. We describe the behaviors of $f_t^{24}(\tau)$ at the cusps of $\Gamma_1(8)$ in the following lemma.

Lemma 3.3. *The first term in the Fourier expansion of $f_2^{24}(\tau)$ (resp. $f_5^{24}(\tau)$) at the cusp ξ, up to a nonzero constant, is given by*

(1) q^{-3} (resp. q^{-3}) when $\xi=0$
(2) q^{6} (resp. q^{15}) when $\xi=\infty$
(3) q^{6} (resp. q^{3}) when $\xi=\frac{1}{2}$
(4) q^{-3} (resp. q^{-3}) when $\xi=\frac{1}{3}$
(5) q^{18} (resp. q^{15}) when $\xi=\frac{1}{4}$
(6) q^{6} (resp. q^{15}) when $\xi=\frac{3}{8}$

Proof. The proof is a straightforward but tedious calculation involving (2.7) and (3.3). We examine the Fourier expansions of $f_t(\tau)$ at the cusp 0 and omit the others which could be obtained similarly.

By (3.3), we find that

$$f_t\left(\frac{1}{-\tau}\right)=\frac{1}{8}\sum_{\lambda=0}^{7}f_{\lambda,t}\left(\frac{1}{-\tau}\right)=\frac{1}{8}\sum_{\lambda=0}^{7}\zeta_8^{-\lambda t}f\left(\frac{\lambda\tau-1}{8\tau}\right).$$

Applying (2.9), we obtain

$$f\left(\frac{\lambda\tau-1}{8\tau}\right)=-i\widehat{\mu}\left(\frac{\lambda\tau-1}{8\tau},\frac{\lambda\tau-1}{8\tau};\frac{\lambda\tau-1}{\tau}\right)-i\widehat{\mu}\left(3\frac{\lambda\tau-1}{8\tau},3\frac{\lambda\tau-1}{8\tau};\frac{\lambda\tau-1}{\tau}\right).$$

Using (2.7), for $k=1,3$, we find that

$$\widehat{\mu}\left(k\frac{\lambda\tau-1}{8\tau},k\frac{\lambda\tau-1}{8\tau};\frac{\lambda\tau-1}{\tau}\right)=\zeta^*\sqrt{\tau}\widehat{\mu}\left(\frac{k(\lambda\tau-1)}{8},\frac{k(\lambda\tau-1)}{8};\tau\right),$$

where ζ^* is a root of unity depends on λ. Hence we have

$$f_t\left(\frac{1}{-\tau}\right)=\frac{-i\sqrt{\tau}}{8}\sum_{\lambda=0}^{7}\zeta_8^{-\lambda t}\zeta^*\sum_{k=1,3}\widehat{\mu}\left(\frac{k(\lambda\tau-1)}{8},\frac{k(\lambda\tau-1)}{8};\tau\right). \tag{3.13}$$

We see that, to examine the behavior of $f_t(\tau)$ at the cusp 0, one need to check the Fourier expansions of the functions $\widehat{\mu}\left(\frac{k(\lambda\tau-1)}{8},\frac{k(\lambda\tau-1)}{8};\tau\right)$. We also note that, since $f_t(\tau)$ are holomorphic when $t=2,5$, it suffice to consider the holomorphic part of $\widehat{\mu}\left(\frac{k(\lambda\tau-1)}{8},\frac{k(\lambda\tau-1)}{8};\tau\right)$.

Recall the definition of the Appell-Lerch sums in (2.2). We find that

$$\mu\left(\frac{k(\lambda\tau-1)}{8},\frac{k(\lambda\tau-1)}{8};\tau\right)=\frac{e^{\frac{k\pi i(\lambda\tau-1)}{8}}}{\vartheta\left(\frac{k(\lambda\tau-1)}{8};\tau\right)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n e^{\frac{2kn\pi i(\lambda\tau-1)}{8}}q^{\frac{n(n+1)}{2}}}{1-e^{\frac{2k\pi i(\lambda\tau-1)}{8}}q^n}. \tag{3.14}$$

By the product expansion (2.1), we obtain

$$\vartheta\left(\frac{k(\lambda\tau-1)}{8};\tau\right)=-iq^{\frac{1}{8}}e^{\frac{-k\pi i(\lambda\tau-1)}{8}}$$
$$\times\prod_{n=1}^{\infty}(1-q^n)(1-e^{\frac{2k\pi i(\lambda\tau-1)}{8}}q^{n-1})(1-e^{\frac{-2k\pi i(\lambda\tau-1)}{8}}q^n). \tag{3.15}$$

Substituting (3.15) into (3.14), setting $\lambda=0$ and simplifying, we find that the first term in the Fourier expansion of $\mu\left(\frac{-k}{8},\frac{-k}{8};\tau\right)$ is given by

$$\frac{i\zeta_8^k}{(1-\zeta_8^k)^2}q^{-\frac{1}{8}}.$$

For $1\le\lambda\le 7$, let $0\le\lambda^*\le 7$ such that $\lambda^*\equiv k\lambda \pmod 8$. Since $k=1,3$, it is easy to see that $\lambda^*\ge 1$. Using (2.6), we find that

$$\mu\left(\frac{k(\lambda\tau-1)}{8},\frac{k(\lambda\tau-1)}{8};\tau\right)=\mu\left(\frac{\lambda^*\tau-k}{8},\frac{\lambda^*\tau-k}{8};\tau\right).$$

Thus the first term in the Fourier expansion of $\mu\left(\frac{k(\lambda\tau-1)}{8},\frac{k(\lambda\tau-1)}{8};\tau\right)$ up to a nonzero constant is given by

$$q^{\frac{\lambda^*-1}{8}}.$$

In particular, we see that $\mu\left(\frac{k(\lambda\tau-1)}{8},\frac{k(\lambda\tau-1)}{8};\tau\right)$ is holomorphic at ∞ when $\lambda>0$.

Combining these facts, we deduce from (3.13) that the first term in the Fourier expansion of $f_t(\tau)$ at the cusp 0 is given by

$$\frac{\zeta^*}{8}\left(\frac{\zeta_8}{(1-\zeta_8)^2}+\frac{\zeta_8^3}{(1-\zeta_8^3)^2}\right)q^{-\frac{1}{8}}.$$

This proves the statement (1). $\square$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We need to prove (3.4). Since the infinite products $R_t^{24}(\tau)$ never vanish on $\mathbb{H}$, the functions $\frac{f_t^{24}(\tau)}{R_t^{24}(\tau)}$ are holomorphic on $\mathbb{H}$. Applying Lemma 2.5 and Lemma 3.3, we find that $\frac{f_t^{24}(\tau)}{R_t^{24}(\tau)}$ are holomorphic modular functions on $\Gamma_1(8)$. This means that $\frac{f_t^{24}(\tau)}{R_t^{24}(\tau)}$ must be constants. Checking the first terms of the q-expansion for $f_t^{24}(\tau)$ and $R_t^{24}(\tau)$, we find that

$$\frac{f_t^{24}(\tau)}{R_t^{24}(\tau)} = 1.$$

This completes the proof of Theorem 1.1. □

4. Proof of Corollary 1.2

Rewrite the infinite product on the right side of (1.6) as follows:

$$\frac{(q^4;q^4)_\infty}{[q;q^4]_\infty^5(q^2;q^4)_\infty^3} = \frac{(q^4;q^4)_\infty(q^2;q^4)_\infty^2}{[q;q^4]_\infty^5(q^2;q^4)_\infty^5}.$$

By the binomial theorem, we have

$$\frac{(q^4;q^4)_\infty}{[q;q^4]_\infty^5(q^2;q^4)_\infty^3} = \frac{(q^4;q^4)_\infty(q^2;q^4)_\infty^2}{[q;q^4]_\infty^5(q^2;q^4)_\infty^5} \equiv \frac{(q^4;q^4)_\infty(q^2;q^4)_\infty^2}{[q^5;q^{20}]_\infty(q^{10};q^{20})_\infty} \pmod 5.$$

Using Jacobi's triple product identity [6, pp. 33–36]

$$(-qz,-q/z,q^2;q^2)_\infty = \sum_{j=-\infty}^{\infty} z^j q^{j^2},$$

we find that

$$(q^2,q^2,q^4;q^4)_\infty = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2}.$$

Thus we obtain

$$\frac{(q^4;q^4)_\infty}{[q;q^4]_\infty^5(q^2;q^4)_\infty^3} \equiv \frac{\sum_{j=-\infty}^{\infty}(-1)^j q^{2j^2}}{[q^5;q^{20}]_\infty(q^{10};q^{20})_\infty} \pmod 5. \tag{4.1}$$

Noting that $2j^2 \equiv 0, 2$ or $3 \pmod 5$, we deduce from (4.1) that the coefficients of q^n with $n \equiv 1, 4 \pmod 5$ in the q-expansion of $\frac{(q^4;q^4)_\infty}{[q;q^4]_\infty^5(q^2;q^4)_\infty^3}$ are multiples of 5. This together with (1.6) proves Corollary 1.2.

References

[1] S. Ahlgren, B. Kim, Mock theta functions and weakly holomorphic modular forms modulo 2 and 3, Math. Proc. Cambridge Philos. Soc. 158 (1) (2015) 111–129.

[2] N. Andersen, Classification of congruences for mock theta functions and weakly holomorphic modular forms, Q. J. Math. 65 (3) (2014) 781–805.

[3] N. Andersen, Vector-valued modular forms and the mock theta conjectures, Res. Number Theory 2 (2016) 32.

[4] G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142.

[5] T.M. Apostol, Modular Functions and Dirichlet Series in Number Theory, second ed., Graduate Texts in Mathematics, vol. 41, Springer-Verlag, New York, 1990.

[6] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer–Verlag, New York, 1991.
[7] S.H. Chan, Generalized Lambert series identities, Proc. Lond. Math. Soc. (3) 91 (2005) 598–622.
[8] S.H. Chan, Congruences for Ramanujan's ϕ function, Acta Arith. 153 (2012) 161–189.
[9] S.H. Chan, R. Mao, Two congruences for Appell-Lerch sums, Int. J. Number Theory 8 (1) (2012) 111–123.
[10] A. Folsom, A short proof of the mock theta conjectures using Maass forms, Proc. Amer. Math. Soc. 136 (12) (2008) 4143–4149.
[11] J. Frye, F.G. Garvan, Automatic proof of theta-function identities, in: Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory, in: Texts Monogr. Symbol. Comput., Springer, Cham, 2019, pp. 195–258.
[12] S. Garthwaite, The coefficients of the $\omega(q)$ mock theta function, Int. J. Number Theory 4 (6) (2008) 1027–1042.
[13] F.G. Garvan, Congruences for Andrews' spt-function modulo powers of 5, 7 and 13, Trans. Amer. Math. Soc. 364 (9) (2012) 4847–4873.
[14] B. Gordon, R. McIntosh, Some eighth order mock theta functions, J. Lond. Math. Soc. (2) 62 (2000) 321–335.
[15] D. Hickerson, E. Mortenson, Hecke-type double sums, Appell-Lerch sums, and mock theta functions (I), Proc. Lond. Math. Soc. (3) 109 (2014) 382–422.
[16] M.D. Hirschhorn, D.C. Hunt, A simple proof of the Ramanujan conjecture for powers of 5, J. Reine Angew. Math. 326 (1981) 1–17.
[17] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer-Verlag, New York, 1993.
[18] J. Lovejoy, R. Osburn, On two 10th-order mock theta identities, Ramanujan J. 36 (1–2) (2015) 117–121.
[19] K. Ono, Unearthing the visions of a master: harmonic Maass forms and number theory, in: Current Developments in Mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 347–454.
[20] Y.K. Qu, Y.J. Wang, O.X.M. Yao, Generalizations of some conjectures of Chan on congruences for Appell-Lerch sums, J. Math. Anal. Appl. 460 (1) (2018) 232–238.
[21] S. Ramanujan, Some properties of $p(n)$, the number of partitions of n, Proc. Camb. Philos. Soc. 19 (1919) 207–210.
[22] S. Robins, Generalized Dedekind η-products, in: The Rademacher Legacy to Mathematics, University Park, PA, 1992, in: Contemp. Math., vol. 166, Amer. Math. Soc., Providence, RI, 1994, pp. 119–128.
[23] M. Waldherr, On certain explicit congruences for mock theta functions, Proc. Amer. Math. Soc. 139 (3) (2011) 865–879.
[24] G.N. Watson, Ramanujans Vermutung über Zerfällungsanzahlen, J. Reine Angew. Math. 179 (1938) 97–128.
[25] S. Zwegers, On two fifth order mock theta functions, Ramanujan J. 20 (2) (2009) 207–214.
[26] S. Zwegers, Mock Theta Functions, Ph.D. thesis, Universiteit Utrecht, 2002.