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The hyperrigidity of tensor algebras of C*-correspondences

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ABSTRACT

Given a C*-correspondence X, we give necessary and sufficient conditions for the tensor algebra $\mathcal{T}_X^+$ to be hyperrigid. In the case where X is coming from a topological graph we obtain a complete characterization.

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1. Introduction

A not necessarily unital operator algebra $\mathcal{A}$ is said to be *hyperrigid* if given any non-degenerate $*$-homomorphism

$$\tau\colon \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \longrightarrow B(\mathcal{H}),$$

the only completely positive, completely contractive extension of the restricted map $\tau_{|\mathcal{A}}$ is τ itself. Arveson coined the term hyperrigid in [1] but he was not the only one considering properties similar to this at the time, e.g. [5].

There are many examples of hyperrigid operator algebras such as those which are Dirichlet, but the situation was not very clear in the case of tensor algebras of C*-correspondences. It was known that the tensor algebra of a row-finite graph is hyperrigid [5], [6] and Dor-On and Salomon [4] showed that row-finiteness completely characterizes hyperrigidity for such graph correspondences. These approaches, while successful, did not lend themselves to a more general characterization.

The authors, in a previous work [12], developed a sufficient condition for hyperrigidity in tensor algebras. In particular, if Katsura's ideal acts non-degenerately on the left then the tensor algebra is hyperrigid. The

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motivation was to provide a large class of hyperrigid C*-correspondence examples as crossed products of operator algebras behave in a very nice manner when the operator algebra is hyperrigid. This theory was in turn leveraged to provide a positive confirmation to the Hao-Ng isomorphism problem in the case of graph correspondences and arbitrary groups. For further reading on the subject please see [10–12].

In this paper, we provide a necessary condition for the hyperrigidity of a tensor algebra, that a C*-correspondence cannot be σ-degenerate, and show that this completely characterizes the situation where the C*-correspondence is coming from a topological graph, which generalizes both the graph correspondence case and the semicrossed product arising from a multivariable dynamical system [3].

1.1. Regarding hyperrigidity

The reader familiar with the literature recognizes that in our definition of hyperrigidity, we are essentially asking that the restriction on $\mathcal{A}$ of any non-degenerate representation of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ possesses the *unique extension property* (abbr. UEP). According to [4, Proposition 2.4] a representation $\rho : \mathcal{A} \to B(\mathcal{H})$, degenerate or not, has the UEP if and only if ρ is a maximal representation of $\mathcal{A}$, i.e., whenever π is a representation of $\mathcal{A}$ dilating ρ, then $\pi = \rho \oplus \pi'$ for some representation π'. Our definition of hyperrigidity is in accordance with Arveson's nomenclature [1], our earlier work [8,12] and the works of Dor-On and Salomon [4] and Salomon [17], who systematized quite nicely the non-unital theory.

An alternative definition of hyperrigidity for $\mathcal{A}$ may ask that *any* representation of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$, not just the non-degenerate ones, possesses the UEP when restricted on $\mathcal{A}$. It turns out that for operator algebras with a positive contractive approximate unit,[1] such a definition would be equivalent to ours [17, Proposition 3.6 and Theorem 3.9]. However when one moves beyond operator algebras with an approximate unit, there are examples to show that the two definitions differ. One such example is the non-unital operator algebra $\mathcal{A}_V$ generated by the unilateral forward shift V. It is easy to see that $\mathcal{A}_V$ is hyperrigid according to our definition and yet the zero map, as a representation on $\mathcal{H} = \mathbb{C}$, does not have the UEP. (See for instance [17, Example 3.4].)

2. Main results

A C*-correspondence $(X, \mathcal{C}, \varphi_X)$ (often just $(X, \mathcal{C})$) consists of a C*-algebra $\mathcal{C}$, a Hilbert $\mathcal{C}$-module $(X, \langle\, , \,\rangle)$ and a (non-degenerate) $*$-homomorphism $\varphi_X : \mathcal{C} \to \mathcal{L}(X)$ into the C*-algebra of adjointable operators on X.

An isometric (Toeplitz) representation $(\rho, t, \mathcal{H})$ of a C*-correspondence $(X, \mathcal{C})$ consists of a non-degenerate $*$-homomorphism $\rho : \mathcal{C} \to B(\mathcal{H})$ and a linear map $t : X \to B(\mathcal{H})$, such that

$$\rho(c)t(x) = t(\varphi_X(c)(x)), \quad \text{and}$$
$$t(x)^* t(x') = \rho(\langle x, x' \rangle),$$

for all $c \in \mathcal{C}$ and $x, x' \in X$. These relations imply that the C*-algebra generated by this isometric representation equals the closed linear span of

$$\{\rho(c) \mid c \in \mathcal{C}\} \cup \{t(x_1) \cdots t(x_n) t(y_1)^* \cdots t(y_m)^* \mid x_i, y_j \in X\}.$$

Moreover, there exists a $*$-homomorphism $\psi_t : \mathcal{K}(X) \to B(\mathcal{H})$, such that

$$\psi_t(\theta_{x,y}) = t(x)t(y)^*,$$

[1] Which includes all operator algebras appearing in this paper.

where $\mathcal{K}(X) \subset \mathcal{L}(X)$ is the subalgebra generated by the operators $\theta_{x,y}(z) = x\langle y, z\rangle,\ x, y, x \in X$, which are called by analogy the compact operators.

The Cuntz-Pimsner-Toeplitz C*-algebra $\mathcal{T}_X$ is defined as the C*-algebra generated by the image of (ρ_∞, t_∞), the universal isometric representation. This algebra is universal in the sense that for any other isometric representation $(\rho, t, \mathcal{H})$ of $(X, \mathcal{C})$, there exists a $*$-homomorphism $\rho \rtimes t\colon \mathcal{T}_X \to B(\mathcal{H})$ onto the C*-algebra generated by $(\rho, t, \mathcal{H})$ in the most natural way.

The *tensor algebra* $\mathcal{T}_X^+$ of a C*-correspondence $(X, \mathcal{C})$ is the norm-closed subalgebra of $\mathcal{T}_X$ generated by $\rho_\infty(\mathcal{C})$ and $t_\infty(X)$. See [15] for more on these constructions.

Consider Katsura's ideal

$$\mathcal{J}_X := \ker \varphi_X^\perp \cap \varphi_X^{-1}(\mathcal{K}(X)).$$

An isometric representation (ρ, t) of $(X, \mathcal{C}, \varphi_X)$ is said to be covariant (or Cuntz-Pimsner) if and only if

$$\psi_t(\varphi_X(c)) = \rho(c),$$

for all $c \in \mathcal{J}_X$. The Cuntz-Pimsner algebra $\mathcal{O}_X$ is the universal C*-algebra for all isometric covariant representations of $(X, \mathcal{C})$, see [14] for further details. Furthermore, the first author and Kribs [9, Lemma 3.5] showed that $\mathcal{O}_X$ contains a completely isometric copy of $\mathcal{T}_X^+$ and $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_X^+) \simeq \mathcal{O}_X$.

We turn now to the hyperrigidity of tensor algebras. In [12] a sufficient condition for hyperrigidity was developed, Katsura's ideal acting non-degenerately on the left of X. To be clear, non-degeneracy here means that $[\varphi_X(\mathcal{J}_X)X] = X$, where $[\,\cdot\,]$ denotes closed linear span. However Cohen's factorization theorem implies that we actually have $\varphi_X(\mathcal{J}_X)X = X$.

Theorem 2.1 *(Theorem 3.1, [12]). Let $(X, \mathcal{C})$ be a* C*-*correspondence. If $\varphi_X(\mathcal{J}_X)$ acts non-degenerately on X, then $\mathcal{T}_X^+$ is a hyperrigid operator algebra.*

The proof shows that if $\tau'\colon \mathcal{O}_X \longrightarrow B(\mathcal{H})$ is a completely contractive and completely positive map that agrees with a $*$-homomorphism of $\mathcal{O}_X$ on $\mathcal{T}_X^+$ then the multiplicative domain of τ' must be everything. This is accomplished through the multiplicative domain arguments of [2, Proposition 1.5.7] and the use of Kasparov's Stabilization Theorem. In earlier versions of [12], Theorem 2.1 was claimed only for countably generated C*-correspondences but a slight modification of the earlier proof makes it work for arbitrary C*-correspondences.

A C*-correspondence $(X, \mathcal{C})$ is called *regular* if and only if $\mathcal{C}$ acts faithfully on X by compact operators, i.e., $\mathcal{J}_X = \mathcal{C}$. We thus obtain the following which also appeared in [12].

Corollary 2.2. *The tensor algebra of a regular* C*-*correspondence is necessarily hyperrigid.*

We seek a converse to Theorem 2.1.

Definition 2.3. Let $(X, \mathcal{C})$ be a C*-correspondence, let $\mathcal{J}_X$ be Katsura's ideal and let $\sigma\colon \mathcal{C} \to B(\mathcal{H})$ be a representation of $\mathcal{C}$. We say that $\varphi_X(\mathcal{J}_X)$ acts *σ-degenerately* on X if

$$\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H} \neq X \otimes_\sigma \mathcal{H}.$$

Remark 2.4. In particular, if there exists $n \in \mathbb{N}$ so that

$$(\varphi_X(\mathcal{J}_X) \otimes \mathrm{id})X^{\otimes n} \otimes_\sigma \mathcal{H} \neq X^{\otimes n} \otimes_\sigma \mathcal{H},$$

then by considering the Hilbert space $\mathcal{K} := X^{\otimes n-1} \otimes_\sigma \mathcal{H}$, we see that

$$\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{K} \neq X \otimes_\sigma \mathcal{K},$$

and so $\varphi_X(\mathcal{J}_X)$ acts σ-degenerately on X.

The following gives a quick example of a σ-degenerate action. Note that this is possibly stronger than having a not non-degenerate action.

Proposition 2.5. *Let $(X, \mathcal{C})$ be a* C*-*correspondence. If $(\varphi_X(\mathcal{J}_X)X)^\perp \neq \{0\}$, then there exists a representation $\sigma \colon \mathcal{C} \to B(\mathcal{H})$ so that $\varphi_X(\mathcal{J}_X)$ acts σ-degenerately on X.*

Proof. Let $0 \neq f \in (\varphi_X(\mathcal{J}_X)X)^\perp$. Let $\sigma : \mathcal{C} \to B(\mathcal{H})$ be a $*$-representation and $h \in \mathcal{H}$ so that $\sigma\big(\langle f, f\rangle^{1/2}\big)h \neq 0$. Then,

$$\langle f \otimes_\sigma h, f \otimes_\sigma h\rangle = \langle h, \sigma((\langle f, f\rangle)h\rangle = \|\sigma\big(\langle f, f\rangle^{1/2}\big)h\| \neq 0.$$

A similar calculation shows that

$$0 \neq f \otimes_\sigma h \in (\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})^\perp$$

and we are done. $\square$

We need the following

Lemma 2.6. *Let $(X, \mathcal{C})$ be a* C*-*correspondence and (ρ, t) an isometric representation of $(X, \mathcal{C})$ on $\mathcal{H}$.*

(i) *If $\mathcal{M} \subseteq \mathcal{H}$ is an invariant subspace for $(\rho \rtimes t)(\mathcal{T}_X^+)$, then the restriction $(\rho_{|\mathcal{M}}, t_{|\mathcal{M}})$ of (ρ, t) on $\mathcal{M}$ is an isometric representation.*
(ii) *If $\rho(c)h = \psi_t(\varphi_X(c))h$, for all $c \in \mathcal{J}_X$ and $h \in [t(X)\mathcal{H}]^\perp$, then (ρ, t) is a Cuntz-Pimsner representation.*

Proof. (i) If p is the orthogonal projection on $\mathcal{M}$, then p commutes with $\rho(\mathcal{C})$ and so $\rho_{|\mathcal{M}}(\cdot) = p\rho(\cdot)p$ is a $*$-representation of $\mathcal{C}$.

Furthermore, for $x, y \in X$, we have

$$\begin{aligned} t_{|\mathcal{M}}(x)^* t_{|\mathcal{M}}(y) &= pt(x)^* pt(y)p \\ &= pt(x)^* t(y)p \\ &= p\rho(\langle x, y\rangle)p = \rho_{|\mathcal{M}}(\langle x, y\rangle) \end{aligned}$$

and the conclusion follows.

(ii) It is easy to see on rank-one operators and therefore by linearity and continuity on all compact operators $K \in \mathcal{K}(X)$ that

$$t(Kx) = \psi_t(K)t(x), \quad x \in X.$$

Now if $c \in \mathcal{J}_X$, then for any $x \in X$ and $h \in \mathcal{H}$ we have

$$\rho(c)t(x)h = t(\varphi_X(c)x)h = \psi_t(\varphi_X(c))t(x)h.$$

By assumption $\rho(c)h = \psi_t(\varphi_X(c))h$, for any $h \in [t(X)\mathcal{H}]^\perp$ and the conclusion follows. $\square$

Theorem 2.7. *Let $(X,\mathcal{C})$ be a C*-correspondence and assume that there exists a representation $\sigma\colon \mathcal{C} \to B(\mathcal{H})$ so that $\varphi_X(\mathcal{J}_X)$ acts σ-degenerately on X. Then the tensor algebra $\mathcal{T}_X^+$ is not hyperrigid.*

Proof. Let $\sigma\colon \mathcal{C} \to B(\mathcal{H})$ so that

$$\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H} \neq X \otimes_\sigma \mathcal{H}$$

and let $\mathcal{M}_0 := (\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})^\perp$.

We claim that

$$(\varphi_X(\mathcal{J}_X) \otimes I)\mathcal{M}_0 = \{0\}. \tag{1}$$

Indeed for any $f \in \mathcal{M}_0$ and $j \in \mathcal{J}_X$ we have

$$\big\langle (\varphi_X(j) \otimes I)f \, , \, (\varphi_X(j) \otimes I)f \big\rangle = \langle f, (\varphi_X(j^*j) \otimes I)f\rangle = 0$$

since $f \in (\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})^\perp$. This proves the claim.

We also claim that

$$(\varphi_X(\mathcal{C}) \otimes I)\mathcal{M}_0 = \mathcal{M}_0. \tag{2}$$

Indeed, since $\varphi_X(\mathcal{C})$ acts non-degenerately on X, we have

$$(\varphi_X(\mathcal{C}) \otimes I)(X \otimes_\sigma \mathcal{H}) = \varphi_X(\mathcal{C})X \otimes_\sigma \mathcal{H} = X \otimes_\sigma \mathcal{H}. \tag{3}$$

Now $\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H}$ is invariant and therefore reducing for $\varphi_X(\mathcal{C}) \otimes I$. Since $\mathcal{M}_0 = (\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})^\perp$, we obtain

$$(\varphi_X(\mathcal{C}) \otimes I)\mathcal{M}_0 \subseteq \mathcal{M}_0. \tag{4}$$

Now

$$\begin{aligned}(\varphi_X(\mathcal{C}) \otimes I)(X \otimes_\sigma \mathcal{H}) &= \big((\varphi_X(\mathcal{C}) \otimes I)(\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})\big) \oplus (\varphi_X(\mathcal{C}) \otimes I)\mathcal{M}_0 \\ &\subseteq (\varphi_X(\mathcal{C})\mathcal{J}_X \otimes_\sigma \mathcal{H}) \oplus \mathcal{M}_0.\end{aligned}$$

If the inclusion in (4) was proper, then the above inclusion would also be proper and this would contradict (3). Therefore the inclusion in (4) is actually an equality and the proof of the claim is complete.

Using the subspace $\mathcal{M}_0$ we produce a Cuntz-Pimsner representation (ρ, t) of $(X,\mathcal{C})$ as follows. Let (ρ_∞, t_∞) be the universal representation of $(X,\mathcal{C})$ on the Fock space $\mathcal{F}(X) = \oplus_{n=0}^\infty X^{\otimes n}$, $X^{\otimes 0} := \mathcal{C}$. Let

$$\begin{aligned}\rho_0 \colon \mathcal{C} &\longrightarrow B(\mathcal{F}(X) \otimes_\sigma \mathcal{H}); c \longmapsto \rho_\infty(c) \otimes I \\ t_0 \colon X &\longrightarrow B(\mathcal{F}(X) \otimes_\sigma \mathcal{H}); x \longmapsto t_\infty(x) \otimes I.\end{aligned}$$

Define

$$\begin{aligned}\mathcal{M} :&= 0 \oplus \mathcal{M}_0 \oplus (X \otimes \mathcal{M}_0) \oplus (X^{\otimes 2} \otimes \mathcal{M}_0) \oplus \dots \\ &= [(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)(0 \oplus \mathcal{M}_0 \oplus 0 \oplus 0 \oplus \dots)] \subseteq \mathcal{F}(X) \otimes_\sigma \mathcal{H},\end{aligned}$$

with the second equality following from (2). Clearly, $\mathcal{M}$ is an invariant subspace for $(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)$.

Let $\rho := \rho_{0|_{\mathcal{M}}}$ and $t := t_{0|_{\mathcal{M}}}$. By Lemma 2.6(i), (ρ, t) is an isometric representation of $(X, \mathcal{C})$. We claim that (ρ, t) is actually Cuntz-Pimsner.

Indeed by Lemma 2.6(ii) it suffices to examine whether $\psi_t(\varphi_X(j))h = \rho(j)h$, for any $h \in \mathcal{M} \ominus t(X)\mathcal{M}$. Note that since

$$[t(X)\mathcal{M}] = 0 \oplus 0 \oplus (X \otimes \mathcal{M}_0) \oplus (X^{\otimes 2} \otimes \mathcal{M}_0) \oplus ...,$$

we have that

$$\mathcal{M} \ominus t(X)\mathcal{M} = 0 \oplus \mathcal{M}_0 \oplus 0 \oplus 0 \oplus \dots.$$

From this it follows that for any $h \in \mathcal{M} \ominus t(X)\mathcal{M}$ we have

$$t_0(x)^*h \in (\mathcal{C} \otimes_\sigma \mathcal{H}) \oplus 0 \oplus 0 \oplus ..., \quad x \in X$$

and so in particular for any $j \in \mathcal{J}_X$ we obtain

$$\psi_t(\varphi_X(j))h \in t_{0|_{\mathcal{M}}}(X)(t_{0|_{\mathcal{M}}})(X)^*h = \{0\}.$$

On the other hand,

$$\rho(j)h \in 0 \oplus (\varphi_X(\mathcal{J}_X) \otimes I)\mathcal{M}_0 \oplus 0 \oplus 0 \oplus \cdots = \{0\},$$

because of (2). Hence (ρ, t) is Cuntz-Pimsner.

At this point by restricting on $\mathcal{T}_X^+$, we produce the representation $\rho \rtimes t\mid_{\mathcal{T}_X^+}$ of $\mathcal{T}_X^+$ coming from a $*$-representation of its C*-envelope $\mathcal{O}_X$, which admits a dilation, namely $\rho_0 \rtimes t_0\mid_{\mathcal{T}_X^+}$. If we show now that $\rho_0 \rtimes t_0\mid_{\mathcal{T}_X^+}$ is a non-trivial dilation of $\rho \rtimes t\mid_{\mathcal{T}_X^+}$, i.e. $\mathcal{M}$ is not reducing for $(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)$, then $\rho \rtimes t\mid_{\mathcal{T}_X^+}$ is not a maximal representation of $\mathcal{T}_X^+$. Proposition 2.4 [4] shows $\rho \rtimes t\mid_{\mathcal{T}_X^+}$ does not have the UEP and so $\mathcal{T}_X^+$ is not hyperrigid, as desired.

Towards this end, note that

$$\mathcal{M}^\perp = \mathcal{C} \oplus (\varphi_X(\mathcal{J}_X)X \otimes_\sigma H) \oplus (X \otimes \mathcal{M}_0)^\perp \oplus \dots$$

and so

$$t_0(X)\mathcal{M}^\perp = 0 \oplus (X\mathcal{C} \otimes_\sigma \mathcal{H}) \oplus 0 \oplus 0 \oplus \cdots \not\subseteq \mathcal{M}^\perp$$

Therefore $\mathcal{M}^\perp$ is not an invariant subspace for $(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)$ and so $\mathcal{M}$ is not a reducing subspace for $(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)$. This completes the proof. $\square$

The previous result raises the question whether $\varphi_X(\mathcal{J}_X)$ acting σ-nondegenerately on X, for all possible representations σ of $\mathcal{C}$, is actually equivalent to $\varphi_X(\mathcal{J}_X)$ acting non-degenerately on X. In the next section we will see that this is indeed the case for C*-correspondences coming from topological graphs. We suspect that the same is true for arbitrary C*-correspondences but we have not been able to establish it.

3. Topological graphs

A broad class of C*-correspondences arises naturally from the concept of a topological graph. For us, a topological graph $G = (G^0, G^1, r, s)$ consists of two σ-locally compact[2] spaces G^0, G^1, a continuous proper

[2] Locally compact spaces which are a countable union of compact subspaces.

map $r : G^1 \to G^0$ and a local homeomorphism $s : G^1 \to G^0$. The set G^0 is called the base (vertex) space and G^1 the edge space. When G^0 and G^1 are both equipped with the discrete topology, we have a discrete countable graph.

With a given topological graph $G = (G^0, G^1, r, s)$ we associate a C*-correspondence X_G over $C_0(G^0)$. The right and left actions of $C_0(G^0)$ on $C_c(G^1)$ are given by

$$(fFg)(e) = f(r(e))F(e)g(s(e))$$

for $F \in C_c(G^1)$, $f, g \in C_0(G^0)$ and $e \in G^1$. The inner product is defined for $F, H \in C_c(G^1)$ by

$$\langle F \mid H \rangle (v) = \sum_{e \in s^{-1}(v)} \overline{F(e)} H(e)$$

for $v \in G^0$. Finally, X_G denotes the completion of $C_c(G^1)$ with respect to the norm

$$\|F\| = \sup_{v \in G^0} \langle F \mid F \rangle (v)^{1/2}. \tag{5}$$

See [13] and [16, Chapter 9] for further reading on topological graphs and the associated C*-algebras.

When G^0 and G^1 are both equipped with the discrete topology, then the tensor algebra $\mathcal{T}_G^+ := \mathcal{T}_{X_G}^+$ associated with G coincides with the quiver algebra of Muhly and Solel [15].

Given a topological graph $G = (G^0, G^1, r, s)$, we can describe the ideal $\mathcal{J}_{X_G}$ as follows. Let

$$\begin{aligned} G^0_{\text{sce}} &:= \{v \in G^0 \mid v \text{ has a neighborhood } V \text{ such that } r^{-1}(V) = \emptyset\} \\ &= \big(r(G^1)^c\big)^\circ \end{aligned} \tag{6}$$

and

$$G^0_{\text{fin}} := \{v \in G^0 \mid v \text{ has a neighborhood } V \text{ such that } r^{-1}(V) \text{ is compact}\}$$

Both sets are easily seen to be open and in [13, Proposition 1.24] Katsura shows that

$$\ker \varphi_{X_G} = C_0(G^0_{\text{sce}}) \;\text{ and }\; \varphi_{X_G}^{-1}(\mathcal{K}(X_G)) = C_0(G^0_{\text{fin}}).$$

From the above it is easy to see that $\mathcal{J}_{X_G} = C_0(G^0_{\text{reg}})$, where

$$G^0_{\text{reg}} := G^0_{\text{fin}} \backslash \overline{G^0_{\text{sce}}}.$$

We need the following

Lemma 3.1. *Let $G = (G^0, G^1, r, s)$ be a topological graph. Then $r^{-1}\big(G^0_{\text{reg}}\big) = G^1$ if and only if $r : G^1 \to G^0$ is a proper map satisfying $r(G^1) \subseteq \big(\overline{r(G^1)}\big)^\circ$.*

Proof. Notice that

$$r^{-1}(G^0_{\text{reg}}) = r^{-1}(G^0_{\text{fin}}) \cap r^{-1}(\overline{G^0_{\text{sce}}})^c$$

and so $r^{-1}\big(G^0_{\text{reg}}\big) = G^1$ is equivalent to $r^{-1}(G^0_{\text{fin}}) = r^{-1}(\overline{G^0_{\text{sce}}})^c = G^1$.

First we claim that $r^{-1}(G^0_{\text{fin}}) = G^1$ if and only if r is a proper map. Indeed, assume that $r^{-1}(G^0_{\text{fin}}) = G^1$ and let $K \subseteq r(G^1)$ compact in the relative topology. For every $x \in K$, let V_x be a compact neighborhood of x such that $r^{-1}(V_x)$ is compact and so $r^{-1}(V_x \cap K))$ is also compact. By compactness, there exist $x_1, x_2, \dots, x_n \in K$ so that $K = \cup_{i=1}^n (V_{x_i} \cap K)$ and so

$$r^{-1}(K) = \cup_{i=1}^n r^{-1}(V_{x_i} \cap K)$$

and so $r^{-1}(K)$ is compact.

Conversely, if r is proper then any compact neighborhood V of any point in G^0 is inverted by r^{-1} to a compact set and so $r^{-1}(G^0_{\text{fin}}) = G^1$.

We now claim that $r^{-1}(\overline{G^0_{\text{sce}}}) = \emptyset$ if and only if $r(G^1) \subseteq \left(\overline{r(G^1)}\right)^\circ$.

Indeed, $r^{-1}(\overline{G^0_{\text{sce}}}) = \emptyset$ is equivalent to $r(G^1) \subseteq \left(\overline{G^0_{\text{sce}}}\right)^c$. From (6) we now have that

$$\overline{G^0_{\text{sce}}} = \overline{(r(G^1)^c)^\circ}$$

and so $r^{-1}(\overline{G^0_{\text{sce}}}) = \emptyset$ is equivalent to

$$r(G^1) \subseteq \left(\overline{G^0_{\text{sce}}}\right)^c = \left(\overline{(r(G^1)^c)^\circ}\right)^c = \left(\overline{r(G^1)}\right)^\circ,$$

as desired. □

If $G = (G^0, G^1, r, s)$ is a topological graph and $S \subseteq G^1$, then $N(S)$ denotes the collection of continuous functions $F \in X_G$ with $F_{|S} = 0$, i.e., vanishing at S. The following appears as Lemma 4.3(ii) in [7].

Lemma 3.2. *Let $G = (G^0, G^1, r, s)$ be a topological graph. If $S_1 \subseteq G^0$, $S_2 \subseteq G^1$ closed, then*

$$N(r^{-1}(S_1) \cup S_2) = \overline{\text{span}}\{(f \circ r)F \mid f_{|S_1} = 0, F_{|S_2} = 0\}$$

Theorem 3.3. *Let $G = (G^0, G^1, r, s)$ be a topological graph and let X_G the* C*-*correspondence associated with G. Then the following are equivalent*

(i) *the tensor algebra $\mathcal{T}^+_{X_G}$ is hyperrigid*
(ii) *$\varphi(\mathcal{J}_{X_G})$ acts non-degenerately on X_G*
(iii) *$r : G^1 \to G^0$ is a proper map satisfying $r(G^1) \subseteq \left(\overline{r(G^1)}\right)^\circ$*

Proof. If $\varphi(\mathcal{J}_{X_G})$ acts non-degenerately on X_G, then Theorem 2.1 shows that $\mathcal{T}^+_{X_G}$ is hyperrigid. Thus (ii) implies (i).

For the converse, assume that $\varphi(\mathcal{J}_{X_G})$ acts degenerately on X_G. If we verify that $\varphi(\mathcal{J}_{X_G})$ acts σ-degenerately on X_G, then Theorem 2.7 shows that $\mathcal{T}^+_{X_G}$ is not hyperrigid and so (i) implies (ii).

Towards this end note that $\mathcal{J}_{X_G} = \mathcal{C}_0(\mathcal{U})$ for some proper open set $\mathcal{U} \subseteq G^0$. (Actually we know that $\mathcal{U} = G^0_{\text{reg}}$ but this is not really needed for this part of the proof!) Hence

$$\begin{aligned} \varphi(\mathcal{J}_{X_G})X_G &= \overline{\text{span}}\{(f \circ r)F \mid f_{|\mathcal{U}^c} = 0\} \\ &= N(r^{-1}(\mathcal{U})^c), \end{aligned} \tag{7}$$

according to Lemma 3.2.

Since $\varphi(\mathcal{J}_{X_G})$ acts degenerately on X_G, (7) shows that $r^{-1}(\mathcal{U})^c \neq \emptyset$. Let $e \in r^{-1}(\mathcal{U})^c$ and let $F \in C_c(G^1) \subseteq X_G$ with $F(e) = 1$ and $F(e') = 0$, for any other $e' \in G^1$ with $s(e') = s(e)$. Consider the one dimensional representation $\sigma : C_0(G_0) \to \mathbb{C}$ coming from evaluation at $s(e)$. We claim that

$$\varphi_{X_G}(\mathcal{J}_{X_G})X_G \otimes_\sigma \mathbb{C} \neq X_G \otimes_\sigma \mathbb{C}.$$

Indeed for any $G \in \varphi(\mathcal{J}_{X_G})X_G = N(r^{-1}(\mathcal{U})^c)$ we have

$$\begin{aligned}\langle F \otimes_\sigma 1, G \otimes_\sigma 1\rangle &= \langle 1, \sigma(\langle F, G\rangle 1) = \langle F, G\rangle s(e) \\ &= \sum_{s(e')=s(e)} \overline{F(e')}G(e') \\ &= \overline{F(e)}G(e) = 0.\end{aligned}$$

Furthermore,

$$\langle F \otimes_\sigma 1, F \otimes_\sigma 1\rangle s(e) = |F(e)|^2 = 1$$

and so $0 \neq F \otimes_\sigma 1 \in (\varphi_{X_G}(\mathcal{J}_{X_G})X_G \otimes_\sigma \mathbb{C})^\perp$. This establishes the claim and finishes the proof of (i) implies (ii).

Finally we need to show that (ii) is equivalent to (iii). Notice that (7) implies that $\varphi(\mathcal{J}_{X_G})$ acts degenerately on X_G if and only if

$$r^{-1}(\mathcal{U})^c = r^{-1}(G^0_{\mathrm{reg}})^c = \emptyset.$$

The conclusion now follows from Lemma 3.1. □

The statement of the previous Theorem takes its most pleasing form when G^0 is a compact space. In that case $\mathcal{T}_X^+$ is hyperrigid if and only if is G^1 is compact and $r(G^1) \subseteq G^0$ is clopen.

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