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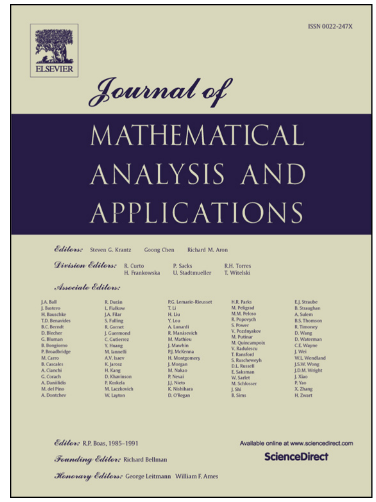

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Boundedness and asymptotic behavior to a chemotaxis-fluid system with singular sensitivity and logistic source

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Abstract: In this paper, we consider the following chemotaxis-fluid model with singular sensitivity and logistic source

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \chi\nabla\cdot(\frac{n}{c}\nabla c) + rn - \mu n^k, & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t + \lambda(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0 \end{cases}$$

in a bounded domain $\Omega\subset \mathrm{R}^N$ $(N=2,3)$ with smooth boundary $\partial\Omega$. Under the non-flux boundary conditions for n and c, and the non-slip boundary condition for u, we establish the global boundedness and the time-decay rates of the classical solutions for any $k>1$ provided that χ satisfies suitable restrictions.

Keywords: Chemotaxis-fluid; Singular sensitivity; Logistic source; Asymptotic behavior.

AMS (2000) Subject Classifications: 35K55, 35Q92, 35Q35, 92C17

1 Introduction

In biology, one of the most important chemotactic models is the Keller-Segel system, which was introduced by Keller and Segel [15] to describe the aggregation of certain types of bacteria. In mathematics, the Keller-Segel system consists of two parabolic equations

$$\begin{cases} n_t = \Delta n - \nabla\cdot(n\chi(n,c)\nabla c), & x\in\Omega,\ t>0,\\ c_t = \Delta c - c + n, & x\in\Omega,\ t>0. \end{cases} \tag{1.1}$$

Here the unknowns $n=n(t,x)$ and $c=c(t,x)$ denote the cell density and chemical signal concentration, respectively. Usually, the physical domain $\Omega\subset\mathbb{R}^N$ is assumed to be a bounded domain with smooth boundary $\partial\Omega$. The given function χ denotes the chemotactic sensitivity.

In recent years, many scholars have done a lot of nice works on system (1.1) in different spatial dimensions and with different assumptions on the sensitive function χ. For example, for the singular chemotactic sensitivity of the form $\chi(n,c) := \frac{\chi_0}{c}$ with $\chi_0>0$, which can be derived from the Weber-Fechner laws, it was shown in [26] that all solutions of the non-flux initial-boundary value problem for system (1.1) are global in time when $N=1$. The same conclusion holds for $N=2$ [25] and $\chi_0<\frac{5}{2}$ if the initial data is radial or is non-radial under the further restriction $\chi_0<1$. In case of $N\geq 2$, there are bounded global classical solutions if $0<\chi_0<\sqrt{\frac{2}{N}}$ [10], and there exists at least a global weak solution [51] if $0<\chi_0<\sqrt{\frac{N+2}{3N+4}}$, while there possesses a global generalized solution if

$$0<\chi_0<\begin{cases}\infty, & N=2,\\ \sqrt{8}, & N=3,\\ \dfrac{N}{N-2}, & N\geq 4\end{cases}$$

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[18]. Under the fast signal diffusion assumption, system (1.1) could be simplified to a parabolic-elliptic system

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\chi(n,c)\nabla c), & x \in \Omega,\ t > 0, \\ 0 = \Delta c - c + n, & x \in \Omega,\ t > 0 \end{cases} \tag{1.2}$$

(for the rigorous verification of this limit process, we may refer to the recent pioneering work Wang-Winkler-Xiang [38]). For the non-flux initial-boundary value problem of system (1.2) with $\chi(n,c) := \frac{\chi_0}{c}$, all radial classical solutions are global-in-time if either $N = 2$ and $\chi_0 > 0$ or $N \geq 3$ and $\chi_0 < \frac{2}{N-2}$ [24], and there exists a unique global bounded classical solution if $N \geq 1$ and $\chi_0 < \frac{2}{N}$ [11], while there exist generalized solutions if $N \geq 2$ and $\chi < \frac{N}{N-2}$ [2]. The finite-time blow-up in low-dimensional Keller-Segel system (1.1) in the ball with logistic-type superlinear degradation has also been investigated by Winkler [49].

When the bacteria or microorganisms live in the fluid, the dynamics of chemotaxis is intimately related to the surrounding environment. More than a decade ago, Tuval et al. [35] proposed a coupled cell-fluid model to describe the dynamics of swimming bacteria, which in particular takes into account the transport effect of the viscous fluids:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(n,c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c + g(n,c), & x \in \Omega,\ t > 0, \\ u_t + \lambda(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0. \end{cases} \tag{1.3}$$

Here, n and c are given as before, while $u = u(t,x)$ and $P = P(t,x)$ denote the velocity field and the pressure of the fluid, respectively. The given functions $\chi(n,c)$ and $g(n,c)$ describe the chemotactic sensitivity and the signal consumption or production. The scalar valued function ϕ is a given potential function. It can be produced by the different physical mechanism such as gravity. The parameter $\lambda = 0$ or $\lambda = 1$ denotes the Stokes or Navier-Stokes flows, respectively. In the case of signal consumption (e.g., $g(n,c) := -nc$), for the Cauchy problem of system (1.3), the global existence of classical solutions around a constant steady state in two dimensional case with $\lambda = 1$ or of weak solutions in three-dimensional setting with $\lambda = 0$ was first proved by Duan-Lorz-Markowich [7], while for the no-flux and no-slip initial-boundary value problem, Winkler [50] removed the smallness assumptions on the initial data and Winkler [52, 53] further investigated the eventual smoothness of weak solutions. The stabilization and convergence rate were also investigated by Winkler [54] and Zhang-Li [56]. Then Duan-Li-Xiang [6] established the global existence of weak or classical solutions for both the Cauchy problem and initial-boundary value problem with relaxed restrictions on χ and g. Recently, Wang-Winkler-Xiang [39] obtained the first rigorous mathematical result on a small-convection limit (i.e. $\lambda \to 0$) in chemotaxis-fluid system (1.3) and supplemented the previously gained knowledge mainly based on numerical experiments. Some variant models have also been studied by many scholars, who still mainly concern the global solvability and stabilization of system (1.3) with the logistic source terms [17, 21, 22], rotational sensitivity functions [4, 5, 36], and nonlinear cell diffusion [8, 40, 41]. Meanwhile, Peng and Xiang [28, 29] introduced several new technique to investigate the global existence of classical solutions to system (1.3) in a three-dimensional unbounded domain with boundary.

On the other hand, a particular motivation for the signal production mechanism in system (1.3) (e.g., $g(n,c) = n - c$) comes from the phenomenon of broadcast spawning. In [23], it was shown that initial-boundary problem of the two dimensional system (1.3) possesses a global bounded solution for $\chi(n,c) \equiv 1$ and $\lambda = 0$. Due to the fact of finite time blow-up in the fluid-free system for the case of $\chi(n,c) \equiv 1$, the suitable saturation is introduced in chemotaxis-fluid system. For example, when $\chi(n,c)$ is a non-constant scalar function, Winkler [48] showed the existence of a global bounded classical solution to the three dimensional system (1.3) with $g(n,c) = n - c$ and $\chi(n,c) = (1+n)^{-\alpha}$ ($\alpha > \frac{1}{3}$), while when $\chi(n,c)$ is a tensor-valued function with saturation, we may refer to [19, 37, 42, 43] for the global solvability of classical solutions in two or three dimensional setting. We also mention the more complicated variants, e.g., involving logistic source terms [33, 45] as well as nonlinear diffusion and rotational flux [20, 27].

Recent years, many scholars have paid their attention to system (1.3) with singular sensitivity function by taking $\chi(n,c)$ as $\frac{\chi}{c}$ (the latter χ is a constant), which is motivated more or less by the fluid-free work Wang-Xiang-Yu [44], where the global existence, asymptotic decay rates and diffusion convergence rate of solutions have been investigated by the method of energy estimates. Taking into account the effect of fluid, Black [1] proved that the global generalized solution will be eventual smoothness provided that the initial data is appropriately small. Under the condition that $0 < \chi < \sqrt{\frac{2}{N}}$, Black-Lankeit-Mizukam [3] recently investigated the signal production case and proved the global existence of classical solutions for $\lambda = 0$ and $N = 3$ or $\lambda \in \{0, 1\}$ and $N = 2$.

Motivated by the above works, in this paper, we let $\Omega \subset \mathbb{R}^N (N = 2, 3)$ be a bounded domain with smooth boundary with outer normal vector ν and investigate the following chemotaxis-fluid system with singular sensitivity and logistic source:

$$
\begin{cases}
n_t + u\cdot\nabla n = \Delta n - \chi\nabla\cdot(\frac{n}{c}\nabla c) + rn - \mu n^k, & x\in\Omega,\ t>0,\\
c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega,\ t>0,\\
u_t + \lambda(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, \quad \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\
\partial_\nu n = \partial_\nu c = 0,\ u = 0, & x\in\partial\Omega,\ t>0
\end{cases}
\tag{1.4}
$$

together with initial data

$$
n(x,0) = n_0(x),\ c(x,0) = c_0(x),\ u(x,0) = u_0(x), \tag{1.5}
$$

where $\chi,\ r,\ \mu > 0$ and $k > 1$ are constants and n_0, c_0, u_0 and ϕ satisfy

$$
\begin{cases}
0 \le n_0(x) \in C^0(\bar\Omega) \quad \text{and} \quad n_0(x) \not\equiv 0,\ x\in\bar\Omega,\\
c_0(x)\in W^{1,\vartheta}(\Omega),\ \inf_{x\in\Omega} c_0(x) > 0,\\
u_0 \in D(A^\alpha),\ \phi\in C^2(\bar\Omega)
\end{cases}
\tag{1.6}
$$

for some $\vartheta > N$ and $\alpha\in(\frac{N}{4},1)$ with $A := -\mathcal{P}\Delta$ denoting the Stokes operator in $L^2_\sigma(\Omega) := \{\varphi\in L^2(\Omega)\,|\,\nabla\cdot\varphi = 0\}$ under homogeneous Dirichlet boundary conditions. The chemotactic sensitivity parameter χ satisfies

$$
\begin{cases}
0 < \chi < \min\left\{2\sqrt{r},\ \sqrt{\frac{2}{N}}\right\}, & \text{if } 1 < k \le 2,\\
0 < \chi < \min\left\{\frac{\sqrt{4r(r+1)k + r^2} - r}{k},\ \frac{2}{\sqrt{k(k-1)(k-2)}},\ \sqrt{\frac{2}{N}}\right\}, & \text{if } k > 2.
\end{cases}
\tag{1.7}
$$

Under these assumptions, we can establish the global well-posedness and time-decay estimates as follows.

Theorem 1.1 *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that (1.6) and (1.7) hold. If $N = 3$ and $\lambda = 0$ or $N = 2$ and $\lambda\in\{0,1\}$, then system (1.4)-(1.5) possesses a global classical solution (n, c, u, P) which enjoys the regularity properties*

$$
\begin{cases}
n\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega)\times(0,\infty)),\\
c\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}((\bar\Omega)\times(0,\infty))\cap L^\infty([0,\infty);\ W^{1,\vartheta}(\Omega)),\\
u\in C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega)\times(0,\infty)),\\
P\in C^{1,0}(\bar\Omega\times[0,\infty)).
\end{cases}
\tag{1.8}
$$

Moreover, this solution is uniformly bounded in the sense that

$$
\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le M \quad \text{for all } t\in(0,\infty) \tag{1.9}
$$

with some positive constant M.

Theorem 1.2 *Let $k\ge 2$. Suppose that the assumptions in Theorem 1.1 hold. If*

$$
\mu > \left(\frac{\sqrt{2}\chi}{4\eta}\right)^{k-1} r^{\frac{3-k}{2}}, \tag{1.10}
$$

then there exist some constants $t_0 > 0$, $\gamma = \gamma(\chi,\eta,\mu,r,k,N) > 0$ and $C > 0$ such that the classical solution (n, c, u) satisfies

$$
\|n(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le Ce^{-\gamma t} \tag{1.11}
$$

and

$$
\|c(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le Ce^{-\gamma t} \tag{1.12}
$$

for all $t > t_0$, where $\eta > 0$ is a constant from the lower bound estimate of c. Furthermore, if

$$
\mu > \left(\frac{\chi^2}{8\eta^2} + 2\kappa_1\right)^{\frac{k-1}{2}} r^{\frac{3-k}{2}}, \tag{1.13}
$$

there is a positive constant $\gamma_\star = \gamma_\star(\chi, \eta, \mu, r, k, \kappa_1, \kappa_2, N)$ *such that the solution* (n, c, u) *fulfills*

$$\|n(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le Ce^{-\gamma_\star t} \tag{1.14}$$

and

$$\|c(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le Ce^{-\gamma_\star t} \tag{1.15}$$

as well as

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\gamma_\star t} \tag{1.16}$$

for all $t > t_0$*, where* κ_1 *and* κ_2 *are positive constants from Poincaré's inequality.*

Theorem 1.3 *Let* $1 < k < 2$*. Suppose that the assumptions in Theorem 1.1 hold and that*

$$\mu \ge \max\left\{\left(\frac{\chi^2 M_0^{2-k}}{4(k-1)\eta^2}\right)^{\frac{k-1}{k}} r^{\frac{1}{k}}, \left(\frac{\chi}{2\eta\sqrt{k-1}}\right)^{k-1} r^{\frac{3-k}{2}}\right\}, \tag{1.17}$$

where $M_0 := \sup_t \|n(\cdot,t)\|_{L^\infty(\Omega)}$*. Then there exist some constants* $t_1 > 0$*,* $\tilde{\gamma} = \tilde{\gamma}(\chi, \eta, \mu, r, k, N) > 0$*, and* $C > 0$ *such that the solution* (n, c, u) *satisfies*

$$\|n(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le Ce^{-\tilde{\gamma} t} \tag{1.18}$$

and

$$\|c(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le Ce^{-\tilde{\gamma} t} \tag{1.19}$$

for all $t > t_1$*. Furthermore, if*

$$\mu \ge \max\left\{\left(\frac{\chi^2 M_0^{2-k}}{4(k-1)\eta^2}\right)^{\frac{k-1}{k}} r^{\frac{1}{k}}, \left(\frac{\chi}{2\eta\sqrt{k-1}}\right)^{k-1} r^{\frac{3-k}{2}}\right\} \quad \text{and} \quad \mu > \max\left\{\left(\frac{30\kappa_1}{k-1}\right)^{k-1} r^{2-k}, \frac{30\kappa_1 M_0^{2-k}}{k-1}, (24\kappa_1)^{k-1} r\right\}, \tag{1.20}$$

there exists a positive constant $\tilde{\gamma}_\star = \tilde{\gamma}_\star(\chi, \eta, \mu, r, k, \kappa_1, \kappa_2, N)$ *such that the solution* (n, c, u) *fulfills*

$$\|n(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le Ce^{-\tilde{\gamma}_\star t}$$

and

$$\|c(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le Ce^{-\tilde{\gamma}_\star t}$$

as well as

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le Ce^{-\tilde{\gamma}_\star t}$$

for all $t > t_1$*.*

Remark. For system (1.4) without logistic source, Black-Lankeit-Mizukam [3] established the global existence of classical solutions in case of $0 < \chi < \sqrt{\frac{2}{N}}$. When the logistic source is included, Zhao-Zheng [57] studied the global existence and boundedness of very weak solutions under the assumption that χ satisfies

$$\begin{cases} 0 < \chi < 2\sqrt{r} \text{ and } \chi \le 2, & k \in (2 - \frac{1}{N}, 2], \\ 2 < \chi < \dfrac{2\sqrt{r}\,(1 + k(2-k))}{\sqrt{k(k+1)(2-k)(3-k)}}, & k \in (2 - \frac{1}{N}, 2], \\ 0 < \chi < \min\left\{\sqrt{\dfrac{2r(1+r)}{k}}, \dfrac{2}{\sqrt{k(k-1)(k-2)}}\right\}, & k \in (2, \infty). \end{cases}$$

Comparing with [3] and [57], we established the global existence and boundedness of classical solutions as well as time decay for any $k > 1$ provided that χ satisfies (1.7).

2 Preliminaries and bounded estimates

We first give a local existence result. The proof is based on the using of Banach's fixed point theorem in a closed bounded set in $L^\infty((0,T); C^0(\bar{\Omega}) \times W^{1,\vartheta}(\Omega) \times D(A^\alpha))$ for suitably small $T > 0$ and we omit the details here.

Lemma 2.1 *For $N \in \{2,3\}, \lambda \in \{0,1\}, \chi > 0, \vartheta > N, \alpha \in (\frac{N}{4}, 1)$, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that n_0, c_0, u_0, ϕ satisfy (1.6). Then there exist $T_{\max} \in (0, \infty]$ and a classical solution (n, c, u, P) to system (1.4)-(1.5) in $\Omega \times (0, T_{\max})$ such that*

$$\begin{cases} n \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega}) \times (0, T_{\max})), \\ c \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}((\bar{\Omega}) \times (0, T_{\max})) \cap L^\infty_{loc}([0, T_{\max});\ W^{1,\vartheta}(\Omega)), \\ u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega}) \times (0, T_{\max})), \\ P \in C^{1,0}(\bar{\Omega} \times [0, T_{\max})) \end{cases} \tag{2.1}$$

and

$$T_{\max} = \infty \ \text{ or } \lim_{t \to T_{\max}} (\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}) = \infty. \tag{2.2}$$

Also, the solution is unique, up to addition of spatially constant function to P and, moreover, has the properties $n(x,t) \geq 0$, $c(x,t) \geq (\min_{x \in \bar{\Omega}} c_0(x)) e^{-t}$ for all $t \in (0, T_{\max})$.

The following lemma is very basic but important and will be frequently used in the sequel.

Lemma 2.2 *For $k > 1$, it holds that*

$$\int_\Omega n(x,t)dx \leq m_* := \max\left\{\int_\Omega n_0 dx,\ |\Omega| \left(\frac{r}{\mu}\right)^{\frac{1}{k-1}}\right\} \ \text{ for all } t \in (0, T_{\max}) \tag{2.3}$$

and

$$\int_\Omega c(x,t)dx \leq \max\left\{\int_\Omega c_0 dx,\ m_*\right\} \quad \text{for all } t \in (0, T_{\max}). \tag{2.4}$$

Proof. Integrate the first equation of (1.4) to get

$$\frac{d}{dt}\int_\Omega n dx = r\int_\Omega n dx - \mu \int_\Omega n^k dx \leq r \int_\Omega n dx - \frac{\mu}{|\Omega|^{k-1}}\Big(\int_\Omega n dx\Big)^k \quad \text{for all } t \in (0, T_{\max}) \tag{2.5}$$

by the Hölder's inequality and $\nabla \cdot u = 0$. Then we obtain (2.3) by the Bernoulli inequality.

Integrate the second equation of (1.4) to get

$$\frac{d}{dt}\int_\Omega c dx = -\int_\Omega c dx + \int_\Omega n dx \quad \text{for all } t \in (0, T_{\max}).$$

We obtain (2.4), again by the Bernoulli inequality. □

To obtain a positive uniform-in-time lower bound of c, we need to construct estimates of negative exponents to make use of reverse Hölder's inequality.

Lemma 2.3 *With $k > 1$, we have for $p > 0$ and $q > q_+ := \frac{p+1}{2}\left(\sqrt{1 + p\chi^2} - 1\right)$ that*

$$\frac{d}{dt}\int_\Omega n^{-p} c^{-q} dx \leq (q - rp) \int_\Omega n^{-p} c^{-q} dx + \mu p \int_\Omega n^{-p-1+k} c^{-q} dx - q \int_\Omega n^{-p+1} c^{-q-1} dx \tag{2.6}$$

for all $t \in (0, T_{\max})$.

Proof. With $p, q > 0$ to be determined, a direct computation with (1.4) shows

$$\begin{aligned} &\frac{d}{dt}\int_\Omega n^{-p} c^{-q} dx \\ &= -p \int_\Omega n^{-p-1} c^{-q} [\Delta n - \chi \nabla \cdot (\frac{n}{c} \nabla c) + rn - \mu n^k - u \cdot \nabla n] dx \\ &\quad - q \int_\Omega n^{-p} c^{-q-1} (\Delta c - c + n - u \cdot \nabla c) dx \end{aligned}$$

$$
\begin{aligned}
&= -p(p+1)\int_\Omega n^{-p-2}c^{-q}|\nabla n|^2dx + [p(p+1)\chi - 2pq]\int_\Omega n^{-p-1}c^{-q-1}\nabla n\cdot\nabla c dx \\
&\quad + [pq\chi - q(q+1)]\int_\Omega n^{-p}c^{-q-2}|\nabla c|^2dx + (q-rp)\int_\Omega n^{-p}c^{-q}dx + \mu p\int_\Omega n^{-p-1+k}c^{-q}dx \\
&\quad - q\int_\Omega n^{-p+1}c^{-q-1}dx - \int_\Omega u\cdot\nabla(n^{-p}c^{-q})dx \\
&\le \left\{\frac{p[(p+1)\chi-2q]^2}{4(p+1)} + pq\chi - q(q+1)\right\}\int_\Omega n^{-p}c^{-q-2}|\nabla c|^2dx + (q-rp)\int_\Omega n^{-p}c^{-q}dx \\
&\quad + \mu p\int_\Omega n^{-p-1+k}c^{-q}dx - q\int_\Omega n^{-p+1}c^{-q-1}dx,\ \ t\in(0,T)
\end{aligned}
$$

by the Young's inequality and $\nabla\cdot u = 0$. Let $f(q;p,\chi) := \frac{p[(p+1)\chi-2q]^2}{4(p+1)} + pq\chi - q(q+1)$. It is easy to see that $f(q;p,\chi) < 0$ is equivalent to $-4q^2 - 4(p+1)q + p(p+1)^2\chi^2 < 0$. Since $\Delta_q = 16(p+1)^2(1+p\chi^2) > 0$, we get $q > \frac{p+1}{2}(\sqrt{1+p\chi^2}-1) = q_+$.
□

The following two lemmas are cornerstone of our work.

Lemma 2.4 *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that the initial and boundary conditions satisfy* (1.5) *and* $(1.4)_4$*, and suppose that n_0, c_0 and u_0 comply with* (1.6)*. For any χ satisfying*

$$
\begin{cases}
\chi < 2\sqrt{r}, & 0<\chi\le 2,\ k\in(1,2],\\
\max\left\{\chi-1, \dfrac{\chi^2}{4}\left(1-p_0^2\right)\right\} < r < \dfrac{\chi^2}{4}, & \chi>2,\ k\in(1,2],\ p_0 := \frac{4(k-1)}{4+(2-k)k\chi^2},\\
0<\chi<\min\left\{\dfrac{\sqrt{4r(r+1)k+r^2}-r}{k}, \dfrac{2}{\sqrt{k(k-1)(k-2)}}\right\}, & k\in(2,+\infty),
\end{cases}
$$

there exists $\eta > 0$ such that

$$
c(x,t)\ge\eta \quad \textit{for all } x\in\Omega,\ t\in(0,T_{\max}). \tag{2.7}
$$

Proof. To improve readability, we divide it into two steps.
Let $\beta_0 := \frac{1}{2}\inf_{x\in\Omega} c_0(x) > 0$. Thanks to Lemma 2.1 and the sign-preserving property of limit, there exists $t_0\in(0,T_{\max})$, such that $c(x,t)\ge\beta_0$ for all $x\in\Omega, t\in(0,t_0]$ and $n(x,t_0)\ge\gamma_0$ for all $x\in\Omega$ with some constants $\gamma_0>0$. So we only need to prove (2.7) for all $t\in(t_0,T_{\max})$. If $1-p+(k-2)(q+1)\in(0,1)$, we can see from Young's inequality and (2.3) that

$$
\mu p\int_\Omega n^{-p-1+k}c^{-q}dx \le q\int_\Omega n^{-p+1}c^{-q-1}dx + \left(\frac{\mu p}{q}\right)^{q+1}\int_\Omega n^{1-p+(k-2)(q+1)} \le q\int_\Omega n^{-p+1}c^{-q-1}dx + C_1. \tag{2.8}
$$

Substituting (2.8) into (2.6), we have

$$
\frac{d}{dt}\int_\Omega n^{-p}c^{-q}dx \le (q-rp)\int_\Omega n^{-p}c^{-q}dx + C_1,\ \ t\in(t_0,T_{\max}) \tag{2.9}
$$

for some $C_1>0$. If $q-rp<0$, we can obtain the boundedness of $\int_\Omega n^{-p}c^{-q}dx$ by using Gronwall's inequality.

Next, we will prove that there exist p,q such that $p>0, q>q_+(p) = \frac{p+1}{2}(\sqrt{1+p\chi^2}-1)$, $g_1(p,q_+) := 1-p+(k-2)(q+1)\in(0,1)$ and $g_2(p,q_+) := q_+ - rp < 0$. By the continuity of g_1 and g_2, there exist p^* and $q^* > q_+$ such that $g_1(p^*,q^*)>0$ and $g_2(p^*,q^*)<0$.

Step 1. At first we consider the case $k\in(1,2]$. Since $1-p+(k-2)(q+1)<1$, we only need to find p and q_+ such that $g_1(p,q_+)>0$. In fact, thanks to $\sqrt{1+s}-1<\frac{s}{2}$ for $s>0$, we just need to check the following inequality

$$
g_1(p,q_+) > -p+k-1+\frac{1}{4}(k-2)(p+1)p\chi^2 \ge -p+k-1+\frac{1}{4}(k-2)kp\chi^2 > 0
$$

for all $0<p<p_0<k-1$ and find some $q>q_+(p)$ satisfying $g_1(p,q)>0$.

Notice that $g_2(p,q_+) = q_+ - rp = \frac{p+1}{2}(\sqrt{1+p\chi^2}-1) - rp < 0$ is equivalent to $\chi^2p^2 + 2(\chi^2-2r^2-2r)p + \chi^2 - 4r < 0$. Taking $\Delta_p = 16r^2((1+r)^2-\chi^2) > 0$, we can get $r > \max\{\chi-1, 0\}$ such that $p\in(p_1,p_2)$, where

$$
p_{1,2} := \frac{2r^2+2r-\chi^2}{\chi^2} \mp \frac{2r\sqrt{(1+r)^2-\chi^2}}{\chi^2}.
$$

If $r \geq \frac{\chi^2}{4}$, we have $p_1 \leq 0 < p_2$ by the Vieta Theorem. Taking $0 < p^* < \min\{p_0, p_2\}$ and $q^* > q_+$, we gather that $g_1(p^*, q^*) = 1 - p^* + (k-2)(q^*+1) \in (0,1)$ and $g_2(p^*, q^*) = q^* - rp^* < 0$. If $\max\{\chi - 1, \frac{\chi^2}{4}(1-p_0^2)\} < r < \frac{\chi^2}{4}$, owing to Vieta Theorem again, there exist p^*, q^* such that $g_1(p^*, q^*) \in (0,1)$ and $g_2(p^*, q^*) < 0$, where $p^* \in (p_1, \min\{p_0, p_2\})$, $q^* > q_+$, and $0 < p_1 < \sqrt{p_1 p_2} = \sqrt{\frac{\chi^2 - 4r}{\chi^2}} < p_0$.

Using Gronwall's inequality for (2.9), one has

$$\int_\Omega n^{-p^*} c^{-q^*} dx \leq \frac{C_1}{rp^* - q^*} + e^{-(rp^*-q^*)(t-t_0)} \Big(\int_\Omega n(x,t_0)^{-p^*} c(x,t_0)^{-q^*} dx - \frac{C_1}{rp^* - q^*} \Big) \leq C_2 \quad \text{for all } t \in (t_0, T_{\max}) \tag{2.10}$$

with some $C_2 > 0$.

If $\alpha_3 := \frac{p^*}{1+q^*} \in (0, p^*)$, then we have

$$\int_\Omega n^{-\alpha_3} dx \leq \Big(\int_\Omega n^{-p^*} c^{-q^*} dx \Big)^{\frac{\alpha_3}{p^*}} \Big(\int_\Omega c dx \Big)^{\frac{p^* - \alpha_3}{p^*}} \leq C_3 \quad \text{for all } t \in (t_0, T_{\max}) \tag{2.11}$$

by the Hölder's inequality and (2.4) with some $C_3 > 0$. Again using the Hölder's inequality, we have

$$\int_\Omega n dx \geq |\Omega|^{\frac{\alpha_3+1}{\alpha_3}} \Big(\int_\Omega n^{-\alpha_3} dx \Big)^{-\frac{1}{\alpha_3}} \geq C_3^{-\frac{1}{\alpha_3}} |\Omega|^{\frac{\alpha_3+1}{\alpha_3}} =: \eta_0 > 0. \tag{2.12}$$

By the pointwise lower bound estimate for the Neumann heat semigroup $\{e^{t\Delta}\}_{t \geq 0}$ ([13], Lemma 3.1), we obtain from (2.12) that

$$\begin{aligned} c(\cdot, t) &= e^{t(\Delta - 1)} c_0 + \int_0^t e^{(t-s)(\Delta-1)} (n(\cdot, s) + u(\cdot, s) \cdot \nabla c(\cdot, s)) ds \\ &\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-((t-s) + \frac{(\operatorname{diam}\Omega)^2}{4(t-s)})} \cdot \Big(\int_\Omega n(x,s)\, dx \Big) ds + \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-((t-s) + \frac{(\operatorname{diam}\Omega)^2}{4(t-s)})} \cdot \Big(\int_\Omega u(x,s) \cdot \nabla c(x,s)\, dx \Big) ds \\ &\geq \eta_0 \int_0^t \frac{1}{(4\pi l)^{\frac{n}{2}}} e^{-(l + \frac{(\operatorname{diam}\Omega)^2}{4l})} dl \geq \eta_0 \int_0^{t_0} \frac{1}{(4\pi l)^{\frac{n}{2}}} e^{-(l + \frac{(\operatorname{diam}\Omega)^2}{4l})} dl := \eta_1 > 0 \quad \text{for all } t \in (t_0, T_{\max}), \end{aligned}$$

where $\operatorname{diam}\Omega := \max_{x,y \in \bar\Omega} |x - y|$.

Step 2. We consider the case $k \in (2, +\infty)$. If $p \in (\frac{4(k-2)}{4 - k(k-2)\chi^2}, k-1)$, $\chi^2 < \frac{4}{k(k-1)(k-2)}$, $q > q_+(p)$ by the definition of g_1, we have

$$g_1(p,q) = 1 - p + (k-2)(q+1) > 1 - p + (k-2) > 0$$

and

$$\begin{aligned} g_1(p, q_+) - 1 &= -p + k - 2 + \frac{1}{2}(k-2)(p+1)(\sqrt{1 + p\chi^2} - 1) \\ &< -p + k - 2 + \frac{1}{4} p\chi^2 (k-2)(p+1) \\ &< -p + k - 2 + \frac{1}{4} p\chi^2 k(k-2) < 0, \end{aligned}$$

So there exist p^* and q^* such that $g_1(p^*, q^*) \in (0,1)$.

Next, we check the following inequalities

$$\begin{cases} p_1 = \dfrac{2r^2 + 2r - \chi^2}{\chi^2} - \dfrac{2r\sqrt{(1+r)^2 - \chi^2}}{\chi^2} < k - 1, \\ p_2 = \dfrac{2r^2 + 2r - \chi^2}{\chi^2} + \dfrac{2r\sqrt{(1+r)^2 - \chi^2}}{\chi^2} > k - 1 > \dfrac{4(k-2)}{4 - k(k-2)\chi^2} \end{cases}$$

to obtain that $(\frac{4(k-2)}{4 - k(k-2)\chi^2}, k-1) \cap (p_1, p_2) \neq \emptyset$.

A simple calculation shows that, for all $k > 2, r > 0$

$$0 < \chi < \min\left\{2\sqrt{\frac{(k-1)r^2 + rk}{k}}, \frac{\sqrt{(4k+1)r^2 + 4kr} - r}{k}, \frac{2}{\sqrt{k(k-1)(k-2)}}\right\},$$

which yields

$$0 < \chi < \min\left\{\frac{\sqrt{(4k+1)r^2 + 4kr} - r}{k}, \frac{2}{\sqrt{k(k-1)(k-2)}}\right\},$$

because $\frac{\sqrt{(4k+1)r^2+4kr}-r}{k} < 2\sqrt{\frac{(k-1)r^2+rk}{k}}$. For such $p^* \in \left(\max\left\{\frac{4(k-2)}{4-k(k-2)\chi^2}, p_1\right\}, k-1\right)$ and $q^* > q_+(p^*)$, by using a similar method in step 1, we can obtain $\eta_2 > 0$ such that $c(x,t) \geq \eta_2$. We complete the proof of Lemma 2.4 by taking $\eta = \min\{\eta_1, \eta_2\}$. □

Lemma 2.5 *If $0 < \chi < 1$, $p \in (1, \frac{1}{\chi^2})$ and $s \in I_p$, where*

$$I_p := \left(\frac{p-1}{2}\left(1 - \sqrt{1-p\chi^2}\right), \frac{p-1}{2}\left(1 + \sqrt{1-p\chi^2}\right)\right) := (s_1, s_2),$$

then there exists $C_4 > 0$ independent of μ such that

$$\int_\Omega n(x,t)^p c(x,t)^{-s} dx \leq \frac{C_4}{\mu^{\frac{p}{k-1}}} \quad for\ all\ t \in (0, T_{\max}). \tag{2.13}$$

Furthermore, there exists $C(T) > 0$ such that

$$\int_0^T \int_\Omega |\nabla(n^{\frac{p}{2}} c^{-\frac{s}{2}})|^2 dxdt \leq C(T) \quad for\ any\ T \in (0, T_{\max}). \tag{2.14}$$

Proof. Multiplying the first equation of (1.4) by $pn^{p-1}c^{-s}$ and the second equation of (1.4) by $-sn^p c^{-s-1}$, summing up and then integrating by parts, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega n^p c^{-s} dx = &- p(p-1)\int_\Omega n^{p-2}c^{-s}|\nabla n|^2 dx + [2ps + \chi p(p-1)]\int_\Omega n^{p-1}c^{-s-1}\nabla n \cdot \nabla c dx \\ &- [\chi ps + s(s+1)]\int_\Omega n^p c^{-s-2}|\nabla c|^2 dx + (s+pr)\int_\Omega n^p c^{-s} dx \\ &- \mu p\int_\Omega n^{p-1+k}c^{-s}dx - s\int_\Omega n^{p+1}c^{-s-1}dx - \int_\Omega u \cdot \nabla(n^p c^{-s})dx \\ := &J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7.\end{aligned} \tag{2.15}$$

The condition $s \in I_p$ implies that

$$(s - s_1)(s - s_2) = s^2 - (p-1)s + \frac{p(p-1)^2\chi^2}{4} < 0,$$

that is

$$[2ps + \chi p(p-1)]^2 < 4p(p-1)[\chi ps + s(s+1)].$$

Therefore, it will be a positive constant ε_1 small enough such that

$$2ps + \chi p(p-1) \leq 2\sqrt{p(p-1) - \varepsilon_1}\sqrt{\chi ps + s(s+1) - \varepsilon_1}.$$

Thus, one has

$$J_2 \leq 2\sqrt{p(p-1) - \varepsilon_1}\sqrt{\chi ps + s(s+1) - \varepsilon_1}\int_\Omega n^{p-1}c^{-s-1}\nabla n \cdot \nabla c dx.$$

Using Young's inequality for above inequality, we obtain

$$\begin{aligned} J_2 &\leq [p(p-1)-\varepsilon_1]\int_\Omega n^{p-2}c^{-s}|\nabla n|^2dx + [\chi ps + s(s+1)) - \varepsilon_1]\int_\Omega n^p c^{-s-2}|\nabla c|^2 dx \\ &= -J_1 - \varepsilon_1 \int_\Omega n^{p-2}c^{-s}|\nabla n|^2dx - J_3 - \varepsilon_1\int_\Omega n^p c^{-s-2}|\nabla c|^2dx \\ &= -(J_1+J_3) - \varepsilon_1\Big(\int_\Omega n^{p-2}c^{-s}|\nabla n|^2dx + \int_\Omega n^p c^{-s-2}|\nabla c|^2dx\Big). \end{aligned} \tag{2.16}$$

Applying the inequality $|\nabla(n^{\frac{p}{2}}c^{-\frac{s}{2}})|^2 \leq \frac{p^2}{2}n^{p-2}c^{-s}|\nabla n|^2 + \frac{s^2}{2}n^pc^{-s-2}|\nabla c|^2$, we have

$$\varepsilon|\nabla(n^{\frac{p}{2}}c^{-\frac{s}{2}})|^2 \leq \frac{p^2\varepsilon}{2}n^{p-2}c^{-s}|\nabla n|^2 + \frac{s^2\varepsilon}{2}n^pc^{-s-2}|\nabla c|^2 \leq \varepsilon_1(n^{p-2}c^{-s}|\nabla n|^2 + n^pc^{-s-2}|\nabla c|^2), \tag{2.17}$$

where $\varepsilon = \min\{\frac{2\varepsilon_1}{p^2}, \frac{2\varepsilon_1}{s^2}\} > 0$.

Since $J_5 \leq 0,\ J_6 \leq 0,\ J_7 = 0$, we substitute (2.16) and (2.17) into (2.15) to obtain

$$\frac{d}{dt}\int_\Omega n^pc^{-s}dx + \varepsilon\int_\Omega|\nabla(n^{\frac{p}{2}}c^{-\frac{s}{2}})|^2dx \leq (s+pr)\int_\Omega n^pc^{-s}dx - \mu p\int_\Omega n^{p-1+k}c^{-s}dx, \tag{2.18}$$

Using Young's inequality and (2.7), we obtain

$$\begin{aligned} (s+pr+1)\int_\Omega n^pc^{-s}dx &\leq \mu p\int_\Omega n^{p-1+k}c^{-s}dx + (s+pr+1)^{\frac{p+k-1}{k-1}}(\mu p)^{-\frac{p}{k-1}}\int_\Omega c^{-s}dx \\ &\leq \mu p\int_\Omega n^{p-1+k}c^{-s}dx + (s+pr+1)^{\frac{p+k-1}{k-1}}(\mu p)^{-\frac{p}{k-1}}\eta^{-s}|\Omega|. \end{aligned} \tag{2.19}$$

Adding (2.19) to (2.18), one has

$$\frac{d}{dt}\int_\Omega n^pc^{-s}dx + \varepsilon\int_\Omega|\nabla(n^{\frac{p}{2}}c^{\frac{s}{2}})|^2dx \leq -\int_\Omega n^pc^{-s}dx + (s+pr+1)^{\frac{p+k-1}{k-1}}(\mu p)^{-\frac{p}{k-1}}\eta^{-s}|\Omega|. \tag{2.20}$$

Using Gronwall's inequality and integrating (2.20) from 0 to t, we have

$$\int_\Omega n^pc^{-s}dx \leq e^{-t}\int_\Omega n_0^p(x)c_0^{-s}(x)dx + (s+pr+1)^{\frac{p+k-1}{k-1}}(\mu p)^{-\frac{p}{k-1}}\eta^{-s}\,|\Omega|(1-e^{-t}).$$

Combining this with (2.20), it immediately yields (2.13) and (2.14). □

Lemma 2.6 *For all $q \geq 1$,*

$$\frac{1}{q}\frac{d}{dt}\int_\Omega c^q(x,t)dx = -\frac{4(q-1)}{q^2}\int_\Omega|\nabla c^{\frac{q}{2}}(x,t)|^2dx - \int_\Omega c^q(x,t)dx + \int_\Omega n(x,t)c^{q-1}(x,t)dx \tag{2.21}$$

holds on $(0, T_{\max})$.

Proof. Multiplying c^{q-1} the second equation of (1.4) by c^{q-1}, and then integrating by parts, we conclude (2.21).

Lemma 2.7 *For all $0 < \chi < \sqrt{\frac{2}{N}}$, $q \in (1,\infty)$ and any $T \in (0, T_{\max})$, there exists a constant $C_5 > 0$ such that*

$$\|c(x,t)\|_{L^q(\Omega)} \leq C_5 \quad for\ all\ t \in (0,T). \tag{2.22}$$

Proof. Without loss of generality, we can assume that $q \geq 2$. Then we take $p \in (\frac{N}{2}, \min\{\frac{1}{\chi^2}, 2\})$, $s := \frac{p-1}{2} \in I_p$ and $q > p - s$. Using Hölder's inequality and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \int_\Omega n(x,t)c^{q-1}(x,t)dx &\leq (\int_\Omega n^p(x,t)c^{-s}(x,t)dx)^{\frac{1}{p}}(\int_\Omega c^{\frac{pq-p+s}{p-1}}(x,t)dx)^{\frac{p-1}{p}} \\ &\leq (\frac{C_4}{\mu^{\frac{p}{k-1}}})^{\frac{1}{p}}\left\|c^{\frac{q}{2}}(x,t)\right\|^{\frac{2(pq-p+s)}{pq}}_{L^{\frac{2(pq-p+s)}{q(p-1)}}(\Omega)} \end{aligned} \tag{2.23}$$

and

$$\left\|c^{\frac{q}{2}}(x,t)\right\|_{L^{\frac{2(pq-p+s)}{q(p-1)}}(\Omega)} \leq C_{GN}\|\nabla c^{\frac{q}{2}}(x,t)\|^{\tilde{\theta}}_{L^2(\Omega)}\|c^{\frac{q}{2}}(x,t)\|^{1-\tilde{\theta}}_{L^2(\Omega)} + C_{GN}\|c^{\frac{q}{2}}(x,t)\|_{L^{\frac{2}{q}}(\Omega)}$$

for all $t \in (0,T)$, where $\tilde{\theta} = \frac{q-(p-s)}{\frac{2}{N}[pq-(p-s)]} \in (0,1)$, $q > p-s$. By Young's inequality, one has

$$\begin{aligned}
\|\nabla c^{\frac{q}{2}}(x,t)\|^{\frac{2(pq-p+s)\tilde{\theta}}{pq}}_{L^2(\Omega)}\|c^{\frac{q}{2}}(x,t)\|^{\frac{2(pq-p+s)(1-\tilde{\theta})}{pq}}_{L^2(\Omega)} &\leq \frac{2(q-1)}{q^2 C_{GN}^{\frac{2(pq-p+s)}{pq}}(\frac{C_4}{\mu^{\frac{p}{k-1}}})^{\frac{1}{p}}}\|\nabla c^{\frac{q}{2}}(x,t)\|^2_{L^2(\Omega)} + C_6\|c^{\frac{q}{2}}(x,t)\|^{\frac{2[q(2p-N)+(N-2)(p-s)]}{2pq-N(q-p+s)}}_{L^2(\Omega)} \\
&\leq \frac{2(q-1)}{q^2 C_{GN}^{\frac{2(pq-p+s)}{pq}}(\frac{C_4}{\mu^{\frac{p}{k-1}}})^{\frac{1}{p}}}\|\nabla c^{\frac{q}{2}}(x,t)\|^2_{L^2(\Omega)} + \frac{1}{2C_{GN}^{\frac{2(pq-p+s)}{pq}}(\frac{C_4}{\mu^{\frac{p}{k-1}}})^{\frac{1}{p}}}\|c^{\frac{q}{2}}(x,t)\|^2_{L^2(\Omega)} + C_7,
\end{aligned} \tag{2.24}$$

because the power $0 < \frac{2[q(2p-N)+(N-2)(p-s)]}{2pq-N(q-p+s)} = \frac{2[2pq-N(q-p+s)-2(p-s)]}{2pq-N(q-p+s)} < 2$.

Substituting (2.23)-(2.24) into (2.21), we have

$$\frac{1}{q}\frac{d}{dt}\int_\Omega c^q(x,t)dx + \frac{2(q-1)}{q^2}\int_\Omega |\nabla c^{\frac{q}{2}}(x,t)|^2 dx \leq -\frac{1}{2}\int_\Omega c^q(x,t)dx + C_8,$$

which means that there exists C_5 such that

$$\|c(\cdot,t)\|_{L^q(\Omega)} \leq C_5 \quad \text{for all } t > 0.$$

Lemma 2.8 *Let $0 < \chi < \sqrt{\frac{2}{N}}$. For any $p \in [1, \frac{1}{\chi^2})$ and any finite $T \in (0, T_{\max}]$, there exists a constant $C_9 > 0$ such that*

$$\|n(x,t)\|_{L^p(\Omega)} \leq C_9 \quad for\ all\ t \in (0,T). \tag{2.25}$$

Proof. Since we can use the interpolation inequality to get the boundedness of $\|n\|_{L^p(\Omega)}$ in $p \in (1, \frac{N}{2}]$, we can assume that $p \in (\frac{N}{2}, \frac{1}{\chi^2})$ for simplicity. Let $p_0 \in (p, \frac{1}{\chi^2})$ and $s_0 := \frac{p_0-1}{2}$. Then we can see from Hölder's inequality, Lemma 2.7 and (2.13) that

$$\begin{aligned}
\int_\Omega n^p(x,t)dx &= \int_\Omega (n^{p_0}(x,t)c^{-s_0}(x,t))^{\frac{p}{p_0}} c^{\frac{ps_0}{p_0}}(x,t)dx \\
&= \left\|(n^{p_0}(x,t)c^{-s_0}(x,t))^{\frac{p}{p_0}}\right\|_{L^{\frac{p_0}{p}}(\Omega)}\left\|c^{\frac{ps_0}{p_0}}(x,t)\right\|_{L^{\frac{p_0}{p_0-p}}(\Omega)} \\
&\leq \left\|n^{p_0}(x,t)c^{-s_0}(x,t)\right\|^{\frac{p}{p_0}}_{L^1(\Omega)}\left\|c(x,t)\right\|^{\frac{ps_0}{p_0}}_{L^{\frac{ps_0}{p_0-p}}(\Omega)} \\
&\leq (\frac{C_4}{\mu^{\frac{p}{k-1}}})^{\frac{p}{p_0}} C_5^{\frac{ps_0}{p_0}}.
\end{aligned}$$

Thus we complete the proof. □

3 Boundedness of u

Having obtained boundedness of $\|n\|_{L^p(\Omega)}$, we can use the standard semigroup technique in [3, 31, 51, 47, 50, 55] to obtain the estimates for the fluid velocity.

Lemma 3.1 *If $0 < \chi < \sqrt{\frac{2}{N}}$, then for any finite $T \in (0, T_{\max}]$ there is $C(T) > 0$ such that*

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq C(T) \quad for\ all\ t \in (0,T) \tag{3.1}$$

and

$$\|\nabla u(\cdot,t)\|^2_{L^2(0,T;L^2(\Omega))} \leq C(T). \tag{3.2}$$

Proof. Multiplying the third equation (1.4) by u and integrating by parts, then using Hölder's inequality, Sobolev's embedding and Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} &= \int_\Omega nu\cdot\nabla\phi dx \\ &\le \|\nabla\phi\|_{L^\infty(\Omega)}\|n\|_{L^p(\Omega)}\|u\|_{L^{\frac{p}{p-1}}(\Omega)} \\ &\le C_{10}\|\nabla\phi\|_{L^\infty(\Omega)}\|n\|_{L^p(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}C_9^2C_{10}^2\|\nabla\phi\|^2_{L^\infty(\Omega)}.\end{aligned} \tag{3.3}$$

Here, we take $p \in (\frac{2N}{N+2}, \frac{1}{\chi^2}) \subset [1, \frac{1}{\chi^2})$, thanks to $W_0^{1,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$ and $\frac{p}{p-1} < \frac{2N}{N-2}$. Integrating the two sides of (3.3), we obtain (3.1), furthermore, we get (3.2). □

We have obtained the estimate for n and u in (2.25) and Lemma 3.1, respectively. These are sufficient to prove the boundedness of u even in the case of $N = 3$. Arguments appearing in the proof of the lemmas below have been previously used in [3, 47, 55].

The goal of this section will be to obtain the boundedness of the norm of $n(\cdot,t)$ in $L^\infty(\Omega)$ to finish the proof of Theorem 1.1. We need some spatio-temporal integrability of n and Au, the estimates of $\|\nabla u\|_{L^2(\Omega)}$, $\|A^{\alpha_0}u\|_{L^2(\Omega)}$, $\|u\|_{L^\infty(\Omega)}$, and $\|\nabla c\|_{L^q(\Omega)}$, respectively. We shall use the following lemmas.

3.1 The case $\lambda = 0$

Lemma 3.2 *If $0 < \chi < \sqrt{\frac{2}{N}}$, then for any finite $T \in (0, T_{\max}]$ and any $\alpha_0 \in (\frac{N}{4}, \alpha] \subset (\frac{N}{4}, 1)$ satisfying $\alpha_0 < 1 - \frac{N}{2}\chi^2 + \frac{N}{4}$, there exists $C(T) > 0$ such that*

$$\|A^{\alpha_0}u(\cdot,t)\|_{L^2(\Omega)} = \Big\|A^{\alpha_0}(e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}(n(\cdot,s)\nabla\phi)ds)\Big\|_{L^2(\Omega)} \le C(T) \tag{3.4}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(T) \tag{3.5}$$

for all $t \in (0,T)$.

Proof. We pick $p \in (\frac{N}{2}, \min\{2, \frac{1}{\chi^2}\})$ and $\delta_1 \in (0,1)$ sufficiently small such that $\alpha_0 + \delta_1 + \frac{N}{2}(\frac{1}{p_0} - \frac{1}{2}) < 1$ holds. We then fix $p_0 > p$ satisfying $\frac{N}{p} - \frac{N}{p_0} < 2\delta_1$ and notice that $\alpha_0 + \delta_1 + \frac{N}{2}(\frac{1}{p_0} - \frac{1}{2}) < 1$. Since $u_0 \in D(A^{\alpha_0})$ by $\alpha_0 \le \alpha$, applying the operator A^{α_0} acting on the Variation-of-constant formula for u, we have

$$\begin{aligned}\|A^{\alpha_0}u(\cdot,t)\|_{L^2(\Omega)} &= \Big\|A^{\alpha_0}\Big(e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}(n(\cdot,s)\nabla\phi)ds\Big)\Big\|_{L^2(\Omega)} \\ &\le \Big\|A^{\alpha_0}e^{-tA}u_0\Big\|_{L^2(\Omega)} + \int_0^t \Big\|A^{\alpha_0}e^{-(t-s)A}\mathcal{P}(n(\cdot,s)\nabla\phi)\Big\|_{L^2(\Omega)}ds \\ &= \Big\|e^{-tA}A^{\alpha_0}u_0\Big\|_{L^2(\Omega)} + \int_0^t \Big\|A^{\alpha_0+\delta_1}e^{-(t-s)A}A^{-\delta_1}\mathcal{P}(n(\cdot,s)\nabla\phi)\Big\|_{L^2(\Omega)}ds \\ &\le \Big\|A^{\alpha_0}u_0\Big\|_{L^2(\Omega)} + K_1\int_0^t (t-s)^{-\alpha_0-\delta_1-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{2})}e^{-\lambda_1(t-s)}\Big\|A^{-\delta_1}\mathcal{P}(n(\cdot,s)\nabla\phi)\Big\|_{L^{p_0}(\Omega)}ds \\ &\le C_{11} + K_1\int_0^t (t-s)^{-\alpha_0-\delta_1-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{2})}e^{-\lambda_1(t-s)}K_2\Big\|n(\cdot,s)\nabla\phi\Big\|_{L^p(\Omega)}ds \\ &\le C_{11} + K_1K_2\|\nabla\phi\|_{L^\infty(\Omega)}\int_0^t (t-s)^{-\alpha_0-\delta_1-\frac{N}{2}(\frac{1}{p_0}-\frac{1}{2})}e^{-\lambda_1(t-s)}\|n(\cdot,s)\|_{L^p(\Omega)}ds \le K_3,\end{aligned}$$

where λ_1, K_1, K_2, K_3 are positive constants. Using the conditions $\alpha_0 + \delta_1 + \frac{N}{2}(\frac{1}{p_0} - \frac{1}{2}) < 1$, $\frac{N}{p} - \frac{N}{p_0} < 2\delta_1$, and $p < \frac{1}{\chi^2}$, we can obtain

$$\begin{aligned}1 &> \alpha_0 + \delta_1 + \frac{N}{2}(\frac{1}{p_0} - \frac{1}{2}) = \alpha_0 + \frac{1}{2}(2\delta_1 + \frac{N}{p_0} - \frac{N}{2}) \\ &> \alpha_0 + \frac{1}{2}(\frac{N}{p} - \frac{N}{p_0} + \frac{N}{p_0} - \frac{N}{2}) > \alpha_0 + \frac{N}{2}\chi^2 - \frac{N}{4}.\end{aligned} \tag{3.6}$$

Next, we use the embedding $D(A^{\alpha_0}) \hookrightarrow C^{\gamma_1}(\Omega)$ to obtain (3.5) for arbitrary $\gamma_1 \in (0, 2\alpha_0 - \frac{N}{2})$.

3.2 The case $\lambda = 1$ and $N = 2$

Lemma 3.3 *If $0 < \chi < \sqrt{\frac{2}{N}}$, then for any finite $T \in (0, T_{\max}]$, there exists $C(T) > 0$ such that*

$$\int_0^T \int_\Omega n^2(x,t)dxdt \leq C(T).$$

Proof. Let $p \in (1, \frac{1}{\chi^2})$ and $s := \frac{p-1}{2} \in I_p$. First we recall from (2.13)-(2.14) that there exist $C_{12} > 0$ and $C_{13} > 0$ such that

$$\int_\Omega n^p(\cdot,t)c^{-s}(\cdot,t)dx \leq C_{12} \qquad \text{and} \qquad \int_0^T \int_\Omega |\nabla(n^{\frac{p}{2}} c^{-\frac{s}{2}})|^2 dxdt \leq C_{13}$$

hold. In virtue of Gagliardo-Nirenberg inequality there exist $C_{14} > 0$ such that

$$\begin{aligned}\int_0^T \|n(\cdot,t)c(\cdot,t)^{-\frac{s}{p}}\|^{2p}_{L^{2p}(\Omega)}dt &= \int_0^T \|n(\cdot,t)^{\frac{p}{2}}c(\cdot,t)^{-\frac{s}{2}}\|^4_{L^4(\Omega)}dt \\ &\leq C_{14}\int_0^T \|\nabla(n(\cdot,t)^{\frac{p}{2}}c(\cdot,t)^{-\frac{s}{2}})\|^2_{L^2(\Omega)}\|n(\cdot,t)^{\frac{p}{2}}c(\cdot,t)^{-\frac{s}{2}}\|^2_{L^2(\Omega)}dt \\ &\quad + C_{14}\int_0^T \|n(\cdot,t)^{\frac{p}{2}}c(\cdot,t)^{-\frac{s}{2}}\|^2_{L^2(\Omega)}dt \\ &\leq C_{12}C_{13}C_{14} + C_{12}C_{14}T,\end{aligned}$$

which means that

$$\int_0^T \|n(\cdot,t)^{2p}c(\cdot,t)^{-(p-1)}\|_{L^1(\Omega)}dt \leq C(T) \tag{3.7}$$

holds. Thanks to Young's inequality, we can estimate

$$\begin{aligned}\int_0^T\int_\Omega n^2(x,t)dxdt &= \int_0^T\int_\Omega \left(n^2(x,t)c^{-\frac{p-1}{p}}(x,t)\right)c^{\frac{p-1}{p}}(x,t)dxdt \\ &\leq \int_0^T\int_\Omega n^{2p}(x,t)c^{-(p-1)}(x,t)dxdt + \int_0^T\int_\Omega c(x,t)dxdt.\end{aligned}$$

Therefore, we get (2.4) and (3.7) to finish the proof of Lemma 3.3. □

Then we use the methods in [3, 5, 12, 14, 46, 50] to lead the following estimates.

Lemma 3.4 *For any $0 < \chi < \sqrt{\frac{2}{N}}$, $T \in (0, T_{\max}]$, there exists $C(T) > 0$ such that*

$$\int_\Omega |\nabla u(x,t)|^2 dx \leq C(T) \quad for\ all\ t \in (0,T) \tag{3.8}$$

and

$$\int_0^T\int_\Omega |Au(x,t)|^2dxdt \leq C(T). \tag{3.9}$$

Proof. Multiplying the third equation in (1.4) by Au, integrating by parts and using the Young's inequality, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2dx + \int_\Omega |Au|^2dx &= \int_\Omega (n\nabla\phi)Audx - \int_\Omega (u\cdot\nabla u)Au \\ &\leq \frac{1}{2}\int_\Omega |Au|^2dx + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^2dx + \int_\Omega |u|^2|\nabla u|^2dx.\end{aligned} \tag{3.10}$$

For the last term, we use the Gagliardo-Nirenberg inequality, Young's inequality and (3.1) to obtain

$$\begin{aligned}\int_\Omega |u|^2|\nabla u|^2 dx &\le \|u\|^2_{L^\infty(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\\ &\le C_{GN}\|u\|_{W^{2,2}(\Omega)}\|u\|_{L^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\\ &\le C_{15}C_{GN}\|u\|_{W^{2,2}(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\\ &\le \frac{1}{4}\|Au\|^2_{L^2(\Omega)} + C_{16}\|\nabla u\|^4_{L^2(\Omega)}.\end{aligned} \tag{3.11}$$

Thus, substituting (3.11) into (3.10) and taking $C_{17} = 2\max\{\|\nabla\phi\|_{L^\infty(\Omega)}, C_{16}\}$, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |Au|^2 dx &= \int_\Omega (n\nabla\phi)Au dx - \int_\Omega (u\cdot\nabla u)Au\\ &\le C_{17}\Big(\int_\Omega n^2 dx + (\int_\Omega |\nabla u|^2 dx)^2\Big).\end{aligned} \tag{3.12}$$

We define $y(t) := \int_\Omega |\nabla u|^2 dx$ on $(0,T)$, which satisfies

$$y'(t) \le C_{17}(\int_\Omega n^2 dx + y^2(t)). \tag{3.13}$$

Using the Variation-of-constant formula, we obtain

$$y(t) \le y(0)e^{C_{17}\int_0^t\int_\Omega |\nabla u(\cdot,s)|^2 ds} + C_{17}\int_0^t e^{C_{17}\int_s^t\int_\Omega |\nabla u(\cdot,\sigma)|^2 d\sigma}\Big(\int_\Omega n^2(x,s)dx\Big)ds \le C_{18}, \tag{3.14}$$

Noticing that $\|\nabla u\|^2_{L^2(0,T;\,L^2(\Omega))} \le C$ in (3.2). That is (3.8). Integrating the two sides of inequality (3.12) about the time t and applying the boundedness of $y(t)$ in (3.14), we obtain (3.9). □

Lemma 3.5 *For any $0 < \chi < \sqrt{\frac{2}{N}}$, $T \in (0, T_{\max}]$ and any $\alpha_0 \in (\frac{N}{4}, \alpha] \subset (\frac{N}{4}, 1)$, there exists $C(T) > 0$ such that*

$$\|A^{\alpha_0}u(\cdot,t)\|_{L^2(\Omega)} \le C(T) \tag{3.15}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(T) \tag{3.16}$$

for all $t \in (0,T)$.

Proof. We fix $\alpha_0 \in (\frac{N}{4}, 1)$ and take p such that $\frac{p\alpha_0}{p-1} \in (0,1)$. Applying the operator A^{α_0} on the Variation-of constant formula of u and Hölder's inequality, due to (3.4), we have

$$\begin{aligned}\|A^{\alpha_0}u(\cdot,t)\|_{L^2(\Omega)} &= \Big\|A^{\alpha_0}\Big(e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}(n(\cdot,s)\nabla\phi - u(\cdot,s)\cdot\nabla u(\cdot,s))ds\Big)\Big\|_{L^2(\Omega)}\\ &\le C_0 + C_{19}\int_0^t (t-s)^{-\alpha_0}\|u(\cdot,s)\cdot\nabla u(\cdot,s)\|_{L^2(\Omega)}ds\\ &\le C_0 + C_{19}\Big(\int_0^t (t-s)^{-\frac{p\alpha_0}{p-1}}ds\Big)^{\frac{p-1}{p}}\Big(\int_0^t \|u(\cdot,s)\cdot\nabla u(\cdot,s)\|^p_{L^2(\Omega)}ds\Big)^{\frac{1}{p}} \quad \text{for all } t \in (0,T).\end{aligned}$$

In order to estimate the last term of above inequality, using the Hölder's inequality, Sobolev's embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, $p \in (\frac{1}{1-\alpha_0}, +\infty)$, Poincaré's inequality, we can find positive constants C_{20}, C_{21}, C_{GN} such that

$$\begin{aligned}\Big(\int_0^t \|u(\cdot,s)\cdot\nabla u(\cdot,s)\|^p_{L^2(\Omega)}ds\Big) &\le \int_0^T \|u(\cdot,s)\|^p_{L^p(\Omega)}\|\nabla u(\cdot,s)\|^p_{L^{\frac{2p}{p-2}}(\Omega)}ds\\ &\le C_{20}\int_0^T \|u(\cdot,s)\|^p_{W^{1,2}(\Omega)}\|\nabla u(\cdot,s)\|^p_{L^{\frac{2p}{p-2}}(\Omega)}ds\\ &\le C_{20}C_{21}\int_0^T \|\nabla u(\cdot,s)\|^p_{L^2(\Omega)}\|\nabla u(\cdot,s)\|^p_{L^{\frac{2p}{p-2}}(\Omega)}ds\end{aligned}$$

$$\le C_{20}C_{21}C_{GN}\int_0^T \|\nabla u(\cdot,s)\|^p_{L^2(\Omega)}\|\nabla u(\cdot,s)\|^{p-2}_{L^2(\Omega)}\|\Delta u(\cdot,s)\|^2_{L^2(\Omega)}ds$$
$$\le C_{20}C_{21}C_{GN}\sup_{t\in(0,T)}\|\nabla u(\cdot,s)\|^{2p-2}_{L^2(\Omega)}\int_0^T \|\Delta u(\cdot,s)\|^2_{L^2(\Omega)}ds$$
$$\le C_{20}C_{21}C_{GN}\sup_{t\in(0,T)}\|\nabla u(\cdot,s)\|^{2p-2}_{L^2(\Omega)}\int_0^T \|Au(x,s)\|^2_{L^2(\Omega)}ds.$$

In virtue of (3.8) and (3.9), we obtain (3.15). Using the embedding of $D(A^{\alpha_0}) \hookrightarrow L^\infty(\Omega)$, we get (3.16). □

4 Boundedness of n

Next, we will give the following estimates of $\|\nabla c\|_{L^q(\Omega)}$ and $\|n\|_{L^\infty(\Omega)}$ through obtained results.

Lemma 4.1 *Assume that* $0 < \chi < \sqrt{\frac{2}{N}}$ *and* $\lambda = 0, N = \{2,3\}$ *or* $\lambda = 1$, $N = 2$ *hold. For any* $1 \le p \le q < \infty$ *satisfying* $q \in [1, \frac{1}{\chi^2-\frac{1}{N}}) \cap [1,\vartheta]$ *and* $\frac{1}{2}+\frac{N}{2}(\frac{1}{p}-\frac{1}{q}) < 1$, *there exists* $C(T) > 0$ *such that*

$$\|\nabla c(\cdot,t)\|_{L^q(\Omega)} \le C(q,T) \quad for\ all\ T\in(0,T_{\max})\ and\ t\in(0,T). \tag{4.1}$$

Proof. Applying the Variation-of-constant formula of c, one has

$$c(\cdot,t) = e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-s)(\Delta-1)}(n(\cdot,s)+u(\cdot,s)\cdot\nabla c(\cdot,s))ds \quad \text{for all } t\in(0,T_{\max}). \tag{4.2}$$

Using standard semigroup estimates for the Neumann heat semigroup of ([46]) Lemma 1.3) provide us positive constants $C_{22} > 0$ and $C_{23} > 0$ such that for any $\vartheta > N$, $t \in (0, T_{\max})$ fulfilling

$$\|\nabla e^{t(\Delta-1)}c_0\|_{L^q(\Omega)} \le C_{22}\|\nabla c_0\|_{L^\vartheta(\Omega)} \tag{4.3}$$

and

$$\begin{aligned}\int_0^t \|\nabla e^{(t-s)(\Delta-1)}n(\cdot,s)\|_{L^q(\Omega)}ds &\le C_{23}\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})})e^{-\lambda_1(t-s)}\|n(\cdot,s)\|_{L^p(\Omega)}ds\\ &\le C_{23}\sup_{s\in(0,T)}\|n(\cdot,s)\|_{L^p(\Omega)}\int_0^\infty (1+\sigma^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})})e^{-\lambda_1\sigma}d\sigma.\end{aligned} \tag{4.4}$$

Because $0 < \frac{1}{2}+\frac{N}{2}(\frac{1}{p}-\frac{1}{q}) < 1$, the last integral of above inequality is finite.

For any positive constants κ_1 and δ_2, if $\frac{1}{2}+\frac{N}{2}(\frac{1}{\iota}-\frac{1}{q}) < \kappa_1$ and $\delta_2 < \frac{1}{2}-\kappa_1$, we can use the following embedding in ([12], Theorem 1.6.1) and ([14], Lemma 2.1) to obtain

$$\|w\|_{W^{1,q}(\Omega)} \le C_{24}\|(-\Delta+1)^{\kappa_1}w\|_{L^\iota(\Omega)} \quad \text{for any } w\in D((-\Delta+1)^{\kappa_1}),\ \iota > q \tag{4.5}$$

and

$$\|(-\Delta+1)^{\kappa_1}e^{-\tau(-\Delta+1)}\nabla\cdot w\|_{L^\iota(\Omega)} \le C_{25}\tau^{-\kappa_1-\frac{1}{2}-\delta_2}e^{-\lambda_1\tau}\|w\|_{L^\iota(\Omega)} \quad \text{for any } \tau>0 \text{ and } w\in L^\iota(\Omega). \tag{4.6}$$

Applying (4.5) and (4.6), one has

$$\begin{aligned}\int_0^t \|\nabla e^{(t-s)(\Delta-1)}u(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^q(\Omega)}ds &\le \int_0^t \|e^{(t-s)(\Delta-1)}\nabla\cdot(u(\cdot,s)c(\cdot,s))\|_{W^{1,q}(\Omega)}ds\\ &\le C_{24}\int_0^t \|(-\Delta+1)^{\kappa_1}e^{(t-s)(\Delta-1)}\nabla\cdot(u(\cdot,s)c(\cdot,s))\|_{L^\iota(\Omega)}ds\\ &\le C_{24}C_{25}\int_0^t (t-s)^{-\kappa_1-\frac{1}{2}-\delta_2}\|c(\cdot,s)u(\cdot,s)\|_{L^\iota(\Omega)}ds\\ &\le C_{24}C_{25}\int_0^t (t-s)^{-\kappa_1-\frac{1}{2}-\delta_2}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|c(\cdot,s)\|_{L^\iota(\Omega)}ds \quad \text{for all } t\in(0,T_{\max}).\end{aligned} \tag{4.7}$$

Since by (2.22), (3.5) or (3.16), and $\frac{1}{2} < \kappa_1+\frac{1}{2}+\delta_2 < 1$, we can conclude that (4.7) is bounded. Combining (4.2)-(4.4) with (4.7), we establish the asserted inequality (4.1). □

Lemma 4.2 *Suppose that χ satisfies* (1.7). *If $\lambda = 1$, additionally suppose that $N = 2$. Then for any finite $T \in (0, T_{\max}]$ there exists a constant $C(T) > 0$ satisfying*

$$\|n\|_{L^\infty(\Omega)} \le C(T) \quad for\ all\ t \in (0, T).$$

Proof. We define $M(T') := \sup_{t\in(0,T')} \|n(\cdot,t)\|_{L^\infty(\Omega)}$ for all $T' \in (0,T)$ and let $t_0 = (t-1)_+$. The Variation-of-constant formula and the nonnegative of $n(\cdot,t)$ imply that

$$\begin{aligned} n(\cdot,t) &= e^{(t-t_0)\Delta}n(\cdot,t_0) - \int_{t_0}^t e^{(t-s)\Delta}\left\{\nabla\cdot\Big(\chi\frac{n(\cdot,s)}{c(\cdot,s)}\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s)\Big) - rn(\cdot,s) + \mu n^k(\cdot,s)\right\}ds \\ &\le e^{(t-t_0)\Delta}n(\cdot,t_0) - \int_{t_0}^t e^{(t-s)\Delta}\left\{\nabla\cdot\Big(\chi\frac{n(\cdot,s)}{c(\cdot,s)}\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s)\Big) - rn(\cdot,s)\right\}ds. \end{aligned}$$

If $t_0 = 0$ (i.e. $t \le 1$), then we have the pointwise estimate to the first term on the right hand side

$$\|e^{(t-t_0)\Delta}n(\cdot,t_0)\|_{L^\infty(\Omega)} = \|e^{t\Delta}n_0\|_{L^\infty(\Omega)} \le \|n_0\|_{L^\infty(\Omega)}.$$

Otherwise, if $t_0 > 0$ (i.e. $t - t_0 = 1$), we can obtain from the semigroup estimate ([46], Lemma 1.3 (i)) and (2.3) that of the above inequality

$$\|e^{(t-t_0)\Delta}n(\cdot,t_0)\|_{L^\infty(\Omega)} \le (1 + (t-t_0)^{-\frac{N}{2}})\|n(\cdot,t_0)\|_{L^1(\Omega)} \le C_{26}.$$

Again using the semigroup estimate in([46], Lemma 1.3 (iv)), Hölder's inequality, interpolation inequality, (2.3), (2.7), (3.16) and (4.1), one has

$$\begin{aligned} \|n(\cdot,t)\|_{L^\infty(\Omega)} &\le C_{27} + \int_0^1 \chi\Big(1 + (t-s)^{-\frac{1}{2}-\frac{N}{2\varsigma}}\Big)\Big\|\frac{n(\cdot,s)}{c(\cdot,s)}\nabla c(\cdot,s) + n(\cdot,s)u(\cdot,s)\Big\|_{L^\varsigma(\Omega)} ds + \int_0^1 (1 + (t-s)^{-\frac{N}{2l}})\|n\|_{L^\varsigma(\Omega)}ds \\ &\le C_{27} + C_{28}\eta^{-1}\int_0^1 (\|n(\cdot,s)\nabla c(\cdot,s)\|_{L^\varsigma(\Omega)} + \eta\|n(\cdot,s)u(\cdot,s)\|_{L^\varsigma(\Omega)})\,ds + m_*^{\frac{1}{\varsigma}}(M(T'))^{\frac{\varsigma-1}{\varsigma}}\int_0^1 (1 + (t-s)^{-\frac{N}{2\varsigma}})ds \\ &\le C_{27} + C_{28}\eta^{-1}\int_0^1 (\|n(\cdot,s)\|_{L^{\frac{\varsigma q}{q-\varsigma}}(\Omega)}\|\nabla c(\cdot,s)\|_{L^q(\Omega)} + \eta\|u(\cdot,s)\|_{L^\infty(\Omega)}\|n(\cdot,s)\|_{L^\varsigma(\Omega)})ds + C_{29}m_*^{\frac{1}{\varsigma}}(M(T'))^{\frac{\varsigma-1}{\varsigma}} \\ &\le C_{27} + C_{28}\eta^{-1}\int_0^1 (\|n(\cdot,s)\|_{L^1(\Omega)}^{\frac{q-\varsigma}{\varsigma q}}\|n(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{q-\varsigma}{q\varsigma}}\|\nabla c(\cdot,s)\|_{L^q(\Omega)} + \eta\|u(\cdot,s)\|_{L^\infty(\Omega)}\|n(\cdot,s)\|_{L^1(\Omega)}^{\frac{1}{\varsigma}}\|n(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{\varsigma-1}{\varsigma}})ds \\ &\quad + C_{29}m_*^{\frac{1}{\varsigma}}(M(T'))^{\frac{\varsigma-1}{\varsigma}} \\ &\le C_{27} + C_{30}\Big((M(T'))^{1-\frac{q-\varsigma}{\varsigma q}} + (M(T'))^{\frac{\varsigma-1}{\varsigma}}\Big), \end{aligned}$$

for all $q > \varsigma > N$, where $C_{27} = \max\{\|n_0\|_{L^\infty(\Omega)}, C_{26}\} > 0, C_{28} > 0, C_{29} > 0$, and $C_{30} > 0$. The conditions $q > \varsigma > N \ge 2$ implies that $0 < 1 - \frac{q-\varsigma}{\varsigma q} < 1$ and $0 < \frac{\varsigma-1}{\varsigma} < 1$. Thus, we have

$$M(T') \le C_{27} + C_{30}\Big((M(T'))^{1-\frac{q-\varsigma}{\varsigma q}} + (M(T'))^{\frac{\varsigma-1}{\varsigma}}\Big),$$

which implies $M(T')$ is finite by means of Young's inequality. This completes the proof. □

Proof of Theorem 1.1. We take $\alpha_0 \in (\frac{N}{4}, \alpha]$ satisfying $\alpha_0 < 1 - \frac{N}{2}\chi^2 + \frac{N}{4}$ and $u_0 \in D(A^{\alpha_0})$. Using Lemma 2.1, we obtain the solution that either global exists or satisfies

$$\lim_{t\to T_{\max}} (\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\vartheta}} + \|A^{\alpha_0}u(\cdot,t)\|_{L^2(\Omega)}) = \infty. \tag{4.8}$$

If $T_{\max}$ were finite, we could apply Lemma 4.2, Lemma 2.7 and invoking Lemma 4.1, and (3.4) or (3.15) with $T = T_{\max}$ to see that $\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\vartheta}} + \|A^{\alpha_0}u(\cdot,t)\|_{L^2(\Omega)} < \infty$ on $(0, T_{\max})$. That would give rise to a contradiction to (4.8). Therefore, we get $T_{\max} = \infty$. Thus we conclude the proof of (1.8). In order to complete the proof of (1.9), we give the following several lemmas.

Lemma 4.3 *Assume that the conditions in Theorem 1.1 hold. Let (n, c, u) be the global classical solution of* (1.4). *There are $C_{31} > 0$ and $\theta \in (0,1)$ such that for every $t > 0$*

$$\|A^{\alpha_0}u(\cdot,t)\|_{L^2(\Omega)} \le C_{31} \tag{4.9}$$

and

$$\|u(\cdot,t)\|_{C^{\theta,\frac{\theta}{2}}(\bar{\Omega}\times(0,\infty))} \le C_{31}. \tag{4.10}$$

Proof. Thanks to (3.4) and (3.15), we obtain (4.9) immediately. Now by a straightforward adaptation of a well-known reasoning [9], in quite a similar method it is furthermore possible to find $\theta_1 \in (0,1)$ and $b_1 = b_1(T) > 0$ satisfying

$$\|A^{\alpha_0}u(\cdot,t) - A^{\alpha_0}u(\cdot,t_0)\|_{L^2(\Omega)} \le b_1|t-t_0|^{\theta_1} \quad \text{for all } t \in (0,T) \text{ and } t_0 \in (0,T),$$

which finally implies (4.10) due to the fact that $D(A_2^{\alpha}) \hookrightarrow C^{\theta_2}(\bar{\Omega}; \mathrm{R}^N)$ in [12] for any $\theta_2 \in (0, 2\alpha_0 - \frac{N}{2})$. □

Lemma 4.4 *Assume that the conditions in Theorem 1.1 are satisfied. Let (n,c,u) be the global classical solution of* (1.4). *Then there exist $C_{32} > 0$ and $\theta \in (0,1)$ such that*

$$\|u(\cdot,t)\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times(0,+\infty))} \le C_{32}.$$

Proof. Due to the estimates provided by Lemma 4.2 and Lemma 4.3, this follows upon a straightforward application of well-known Schauder theory for the linear inhomogeneous Stokes evolution equation in [32].

Lemma 4.5 *Assume that the conditions in Theorem 1.1 are satisfied. Let (n,c,u) be the global classical solution of* (1.4). *There are $C_{33} > 0$ and $\theta \in (0,1)$ such that for every $t > 0$*

$$\|n(\cdot,t)\|_{C^{\theta,\frac{\theta}{2}}(\Omega\times(0,\infty))} \le C_{33} \tag{4.11}$$

and

$$\|c(\cdot,t)\|_{C^{2+\theta,1+\frac{\theta}{2}}(\Omega\times(0,\infty))} \le C_{33} \tag{4.12}$$

as well as

$$\|n(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_{33}. \tag{4.13}$$

Proof. We rewrite the first equation in (1.4) in the form

$$n_t = \nabla\cdot a(x,t,\xi) + b(x,t), \quad x \in \Omega,\ t \in (0,\infty),\ \xi = \nabla n \in \mathbb{R}^N,$$

with

$$a(x,t,\xi) = \xi - \chi\frac{n(x,t)}{c(x,t)}\nabla c(x,t) - n(x,t)u(x,t),\ (x,t,\xi) \in \Omega\times(0,\infty)\times\mathbb{R}^N$$

and

$$b(x,t) = rn - \mu n^k,\ (x,t,\xi) \in \Omega\times(0,\infty)\times\mathbb{R}^N.$$

Using the Young's inequality and (2.7), (3.5) or (3.16) and Lemma 4.2, there exist some $c_{18} > 0, c_{19} > 0, c_{20} > 0$ fulfilling

$$a(x,t,\xi)\cdot\xi \ge \frac{|\xi|^2}{2} - c_{18}|\nabla c|^2 - c_{18} \quad \text{for all } (x,t,\xi) \in \Omega\times(0,\infty)\times\mathbb{R}^N$$

and

$$|a(x,t,\xi)| \le |\xi| + c_{19}|\nabla c| + c_{19} \quad \text{for all } (x,t,\xi) \in \Omega\times(0,\infty)\times\mathbb{R}^N$$

as well as

$$|b(x,t)| \le c_{20} \quad \text{for all } (x,t) \in \Omega\times(0,\infty).$$

Since Lemma 4.1 provided a boundedness for $|\nabla c|^2$ in $L^8((0,T); L^2(\Omega))$, with the exponents fulfilling $\frac{1}{8} + \frac{N}{2\times 2} = \frac{1+2N}{8} \le \frac{7}{8} < 1$ for $N = \{2,3\}$, the estimate (4.11) directly results on applying the standard result on Hölder regularity in scalar parabolic equations (see [30], Theorem 1.3). Thereupon (4.12) are immediate consequences of Lemma 4.3 and standard parabolic Schauder theory in [16]. Finally, with the aforementioned regularity properties of n and c at hand, we can obtain form ([16] Theorem IV. 5.3) that (4.13) holds. □

The stated boundedness of the classical solution in (1.9) comes from Lemma 4.3-4.5. This completes the proof of Theorem 1.1. □

5 Asymptotic behavior

In this section, we first consider the asymptotic behavior of $k \geq 2$, and then give the asymptotic behavior of $1 < k < 2$ by suitable energy functional for (1.4). The following ideas originate from [34] and [45], respectively. To show the solution (n, c, u) exponentially stabilizes to the constant stationary solution $((\frac{r}{\mu})^{\frac{1}{k-1}}, (\frac{r}{\mu})^{\frac{1}{k-1}}, \mathbf{0})$, we use the following scale transformation.

Let $U(x,t) = \frac{\mu}{r} n(x,t)$ and $V(x,t) = c(x,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}$. Then we can transform (1.4) into the following model:

$$\begin{cases} U_t + u \cdot \nabla U = \Delta U - \chi \nabla \cdot (\dfrac{U}{c} \nabla V) + rU\Big(1 - (\dfrac{r}{\mu})^{k-2} U^{k-1}\Big), & x \in \Omega,\ t > 0, \\ V_t + u \cdot \nabla V = \Delta V - V + (\dfrac{r}{\mu})^{\frac{1}{k-1}} \Big((\dfrac{r}{\mu})^{\frac{k-2}{k-1}} U - 1\Big), & x \in \Omega,\ t > 0, \\ u_t + \lambda (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \phi, & x \in \Omega,\ t > 0, \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ U(x,0) := U_0(x) = \dfrac{\mu}{r} n_0(x),\ V(x,0) := V_0(x) = c_0(x) - (\dfrac{r}{\mu})^{\frac{1}{k-1}}, & x \in \Omega, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0. \end{cases} \tag{5.1}$$

Lemma 5.1 *Assume that $s \geq 0$, $k \geq 2$ and in addition r, μ are given in* (1.4). *Then*

$$\left(1 - \left(\frac{r}{\mu}\right)^{k-2} s^{k-1}\right)\left(\left(\frac{r}{\mu}\right)^{\frac{k-2}{k-1}} s - 1\right) + \left(\left(\frac{r}{\mu}\right)^{\frac{k-2}{k-1}} s - 1\right)^2 \leq 0. \tag{5.2}$$

Proof. It is obvious that if $\left(\frac{r}{\mu}\right)^{\frac{k-2}{k-1}} s = 1$, then (5.2) holds. Now we assume that $\left(\frac{r}{\mu}\right)^{\frac{k-2}{k-1}} s > 1$. One has

$$\left(1 - \left(\frac{r}{\mu}\right)^{k-2} s^{k-1}\right) = 1 - \left(\left(\frac{r}{\mu}\right)^{\frac{k-2}{k-1}} s\right)^{k-1} \leq 1 - \left(\left(\frac{r}{\mu}\right)^{\frac{k-2}{k-1}} s\right) < 0. \tag{5.3}$$

Multiplying (5.3) by $\left(\frac{r}{\mu}\right)^{\frac{k-2}{k-1}} s - 1$, we obtain (5.2). For the case of $\left(\frac{r}{\mu}\right)^{\frac{k-2}{k-1}} s < 1$, proceeding similarly, we can also obtain (5.2). □

Lemma 5.2 *Let (n, c) be a global classical solution of* (1.4). *Assume that $\frac{\chi^2}{8\eta^2} < L < \mu^{\frac{2}{k-1}} r^{\frac{k-3}{k-1}}$ holds for $k \geq 2$. Then for all $t > 0$ the function*

$$\mathcal{F}(t) := \int_\Omega \Big((\frac{r}{\mu})^{\frac{k-2}{k-1}} U - 1 - \ln (\frac{r}{\mu})^{\frac{k-2}{k-1}} U + LV^2\Big) dx$$

satisfies

$$\mathcal{F}'(t) \leq -\mathcal{D}(t), \tag{5.4}$$

with

$$\mathcal{D}(t) := \mathcal{D}_0 \int_\Omega \left(\Big((\frac{r}{\mu})^{\frac{k-2}{k-1}} U - 1\Big)^2 + LV^2\right) dx \tag{5.5}$$

and

$$\mathcal{D}_0 := \min\left\{1,\ r - L\Big(\frac{r}{\mu}\Big)^{\frac{2}{k-1}}\right\} > 0.$$

Proof. The strong maximum principle along with the assumption $U_0 \not\equiv 0$ yields $U > 0$ in $\bar{\Omega} \times (0, \infty)$. Multiplying the second equation in (5.1) by V and using the Young's inequality, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_\Omega V^2 dx + \int_\Omega |\nabla V|^2 dx &= -\int_\Omega V^2 dx + (\frac{r}{\mu})^{\frac{1}{k-1}} \int_\Omega V((\frac{r}{\mu})^{\frac{k-2}{k-1}} U - 1) dx \\ &\leq -\frac{1}{2} \int_\Omega V^2 dx + \frac{1}{2} (\frac{r}{\mu})^{\frac{2}{k-1}} \int_\Omega ((\frac{r}{\mu})^{\frac{k-2}{k-1}} U - 1)^2 dx \quad \text{for all } t > 0. \end{aligned} \tag{5.6}$$

Multiplying the first equation in (5.1) by $(\frac{r}{\mu})^{\frac{k-2}{k-1}} - \frac{1}{U}$, integrating by parts, and using the Young's inequality and (5.2), one has

$$\frac{d}{dt}\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}U)dx+\int_\Omega\frac{|\nabla U|^2}{U^2}dx=\chi\int_\Omega\frac{1}{Uc}\nabla U\cdot\nabla Vdx+r\int_\Omega\Big(1-(\frac{r}{\mu})^{k-2}U^{k-1}\Big)\Big((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1\Big)dx$$
$$\le\int_\Omega\frac{|\nabla U|^2}{U^2}dx+\frac{\chi^2}{4}\int_\Omega\frac{|\nabla V|^2}{c^2}dx-r\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2dx$$
$$\le\int_\Omega\frac{|\nabla U|^2}{U^2}dx+\frac{\chi^2}{4\eta^2}\int_\Omega|\nabla V|^2dx-r\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2dx$$

for all $t>0$, because $c(x,t)\ge\eta>0$ and (5.2). That is

$$\frac{d}{dt}\int_\Omega\Big((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}U\Big)dx\le\frac{\chi^2}{4\eta^2}\int_\Omega|\nabla V|^2dx-r\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2dx\quad\text{for all } t>0. \tag{5.7}$$

We multiply (5.6) by $2L$, where $L\in(\frac{\chi^2}{8\eta^2},\ r^{\frac{k-3}{k-1}}\mu^{\frac{2}{k-1}})$ and then add it into (5.7) to obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}U+LV^2\Big)dx+(2L-\frac{\chi^2}{4\eta^2})\int_\Omega|\nabla V|^2dx\\&+(r-L(\frac{r}{\mu})^{\frac{2}{k-1}})\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2dx+\int_\Omega LV^2dx\le0\quad\text{for all } t>0.\end{aligned} \tag{5.8}$$

Taking $\mathcal{D}_0=\min\{1,\ r-L(\frac{r}{\mu})^{\frac{2}{k-1}}\}$ together with the definition of $\mathcal{F}$ and $\mathcal{D}$ implies that (5.4) holds. □

Lemma 5.3 *Let $\varphi(s):=(\frac{r}{\mu})^{\frac{k-2}{k-1}}s-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}s$. Then there exists a positive constant $\delta_0<\frac{1}{10}$ satisfying*

$$\varphi(s)\ge0\quad for\ all\ s>0 \tag{5.9}$$

and

$$\frac{1}{6}((\frac{r}{\mu})^{\frac{k-2}{k-1}}s-1)^2<\varphi(s)<\frac{5}{6}((\frac{r}{\mu})^{\frac{k-2}{k-1}}s-1)^2\quad for\ all\ 0<\Big|s-(\frac{\mu}{r})^{\frac{k-2}{k-1}}\Big|<\delta_0. \tag{5.10}$$

Proof. Since $\varphi'(s)=(\frac{r}{\mu})^{\frac{k-2}{k-1}}-\frac{1}{s}$ and $\varphi''(s)=\frac{1}{s^2}>0$ for all $s>0$, we have $\varphi(s)\ge\varphi((\frac{\mu}{r})^{\frac{k-2}{k-1}})=0$. That is (5.9). Because

$$\lim_{s\to(\frac{\mu}{r})^{\frac{k-2}{k-1}}}\frac{(\frac{r}{\mu})^{\frac{k-2}{k-1}}s-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}s}{((\frac{r}{\mu})^{\frac{k-2}{k-1}}s-1)^2}=\frac{1}{2},$$

we can find a positive constant $\delta_0<\frac{1}{10}$ such that

$$\left|\frac{(\frac{r}{\mu})^{\frac{k-2}{k-1}}s-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}s}{((\frac{r}{\mu})^{\frac{k-2}{k-1}}s-1)^2}-\frac{1}{2}\right|<\frac{1}{3}\quad\text{for all } 0<|s-(\frac{\mu}{r})^{\frac{k-2}{k-1}}|<\delta_0.$$

We complete the proof of (5.10). □

Corollary 5.1 *Under the assumption of Lemma 5.2, we have*

$$\int_1^\infty\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2dxdt<\infty\quad and\quad\int_1^\infty\int_\Omega V^2dxdt<\infty. \tag{5.11}$$

Proof. Integrating the two side of (5.4) from 1 to ∞, we have

$$\mathcal{F}(t)+\int_1^\infty\mathcal{D}(s)ds\le\mathcal{F}(1)\qquad\text{for all } t>1.$$

Since $\mathcal{F}(t)$ is nonnegative by Lemma 5.3, this entails that $\int_1^\infty\mathcal{D}(s)ds\le\mathcal{F}(1)<\infty$, which according to the definition of $\mathcal{D}(t)$ in (5.5) directly implies (5.11). □

Lemma 5.4 *Assume the hypothesis of Theorem 1.2 holds. Then if (n, c) is a nonnegative global classical solution of* (1.4)*, we have*

$$\lim_{t\to\infty} \|U(\cdot,t) - (\frac{\mu}{r})^{\frac{k-2}{k-1}}\|_{L^\infty(\Omega)} = 0 \tag{5.12}$$

and

$$\lim_{t\to\infty} \|V(\cdot,t)\|_{L^\infty(\Omega)} = 0. \tag{5.13}$$

Proof. Suppose to the contrary that (5.12) was false. There would exist $c_{21} > 0$ and sequences $(t_j)_{j\in\mathbb{N}} \subset (1, +\infty)$ and $(x_j)_{j\in\mathbb{N}} \subset \Omega$ such that

$$\left|U(x_j, t_j) - (\frac{\mu}{r})^{\frac{k-2}{k-1}}\right| \geq c_{21} \text{ for all } j \in \mathbb{N}.$$

Also note that Lemma 4.5 implies that the function $|U(x,t) - 1|$ is uniformly continuous in $\Omega \times (1, \infty)$. So, we can find $r_1 > 0$ and $\tau > 0$ fulfilling

$$|U(x,t) - (\frac{\mu}{r})^{\frac{k-2}{k-1}}| \geq \frac{c_{21}}{2} \text{ for all } x \in B_{r_1}(x_j) \cap \Omega \text{ and } t \in (t_j, t_{j+\tau}). \tag{5.14}$$

Now since the smoothness of $\partial\Omega$ ensures the existence of $c_{22} > 0$ such that

$$|B_{r_1}(x) \cap \Omega| \geq c_{22} \text{ for all } x \in B_{r_1}(x_j) \cap \Omega, \tag{5.15}$$

from (5.14) and (5.15) we infer that

$$\begin{aligned}\int_{t_j}^{t_j+\tau}\int_\Omega \left|U(x,t) - (\frac{\mu}{r})^{\frac{k-2}{k-1}}\right|^2 dxdt &\geq \int_{t_j}^{t_j+\tau}\int_{B_{r_1}(x_j)\cap\Omega} \left|U(x,t) - (\frac{\mu}{r})^{\frac{k-2}{k-1}}\right|^2 dxdt \\ &\geq \frac{c_{21}^2}{4}\int_{t_j}^{t_j+\tau} |B_{r_1}(x_j)\cap\Omega| dt \\ &\geq \frac{c_{21}^2 c_{22}\tau}{4} \text{ for all } j \in \mathbb{N}.\end{aligned} \tag{5.16}$$

But on the other hand, from Corollary 5.1 we know that since $t_j \to \infty$ as $j \to \infty$ we must have

$$\int_{t_j}^{t_j+\tau}\int_\Omega |U(x,t) - (\frac{\mu}{r})^{\frac{k-2}{k-1}}|^2 dxdt \leq \int_{t_j}^{\infty}\int_\Omega |U(x,t) - (\frac{\mu}{r})^{\frac{k-2}{k-1}}|^2 dxdt \to 0 \qquad \text{as } j \to \infty,$$

which is a contradiction to (5.16). We show that (5.12) actually was true. Similarly, we can obtain (5.13). □

Proof of Theorem 1.2. First, we prove the convergence of $n(\cdot, t)$ and $c(\cdot, t)$. In light of (5.12), then there exists $t_0 > 0$ such that for the above $\delta_0 > 0$

$$\|U(\cdot,t) - (\frac{\mu}{r})^{\frac{k-2}{k-1}}\|_{L^\infty(\Omega)} < \delta_0 \qquad \text{for all } t > t_0.$$

Thanks to (5.10), we obtain

$$\frac{1}{6}((\frac{r}{\mu})^{\frac{k-2}{k-1}}U(x,t) - 1)^2 < \varphi(U) < \frac{5}{6}((\frac{r}{\mu})^{\frac{k-2}{k-1}}U(x,t) - 1)^2 \leq ((\frac{r}{\mu})^{\frac{k-2}{k-1}}U(x,t) - 1)^2 \quad \text{for all } x \in \Omega,\ t > t_0,$$

which on account of the definition of $\mathcal{F}$ yields to

$$\frac{1}{6}\int_\Omega \left((\frac{r}{\mu})^{\frac{k-2}{k-1}}U(\cdot,t) - 1\right)^2 dx + L\int_\Omega V^2(\cdot,t)dx < \mathcal{F}(t) \leq \int_\Omega \left((\frac{r}{\mu})^{\frac{k-2}{k-1}}U(x,t) - 1\right)^2 dx + L\int_\Omega V^2(\cdot,t)dx. \tag{5.17}$$

In virtue of (5.4), we have

$$\mathcal{F}'(t) \leq -\mathcal{D}_0\int_\Omega \left(((\frac{r}{\mu})^{\frac{k-2}{k-1}}U - 1)^2 + LV^2\right)dx \leq -\mathcal{D}_0\mathcal{F}(t),$$

from which we obtain

$$\mathcal{F}(t) \leq \mathcal{F}(t_0)e^{-\mathcal{D}_0(t-t_0)}. \tag{5.18}$$

Combining (5.17) with (5.18), one has

$$\frac{1}{6}\int_\Omega \left((\frac{r}{\mu})^{\frac{k-2}{k-1}}U(\cdot,t) - 1\right)^2 dx + L\int_\Omega V^2(\cdot,t)dx \leq \mathcal{F}(t_0)e^{-\mathcal{D}_0(t-t_0)},$$

which implies that there exists a constant $C_{34} > 0$ such that

$$\|(\frac{r}{\mu})^{\frac{k-2}{k-1}}U(\cdot,t)-1\|_{L^2(\Omega)} \leq C_{34}e^{-\frac{\mathcal{D}_0}{2}(t-t_0)}$$

and

$$\|V(\cdot,t)\|_{L^2(\Omega)} \leq C_{34}e^{-\frac{\mathcal{D}_0}{2}(t-t_0)}$$

for all $t > t_0$.
Due to the relationship of translation about (n, c) and (U, V), there exists a constant $C_{35} > 0$ such that

$$\|n(\cdot,t)-(\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^2(\Omega)} \leq C_{35}e^{-\frac{\mathcal{D}_0}{2}(t-t_0)} \tag{5.19}$$

and

$$\|c(\cdot,t)-(\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^2(\Omega)} \leq C_{35}e^{-\frac{\mathcal{D}_0}{2}(t-t_0)}$$

for all $t > t_0$.
Using the Gagliardo-Nirenberg inequality and (4.13),we can find two positive constants C_{36} and C_{37} such that

$$\begin{aligned}\Big\|n(\cdot,t)-(\frac{r}{\mu})^{\frac{1}{k-1}}\Big\|_{L^\infty(\Omega)} &\leq C_{36}\Big\|\nabla n(\cdot,t)\Big\|^{\frac{N}{N+2}}_{L^\infty(\Omega)}\Big\|n(\cdot,t)-(\frac{r}{\mu})^{\frac{1}{k-1}}\Big\|^{\frac{2}{N+2}}_{L^2(\Omega)}+C_{36}\Big\|n(\cdot,t)-(\frac{r}{\mu})^{\frac{1}{k-1}}\Big\|_{L^2(\Omega)}\\ &\leq C_{37}e^{-\frac{\mathcal{D}_0}{N+2}t}\end{aligned}$$

and

$$\|c(\cdot,t)-(\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \leq C_{37}e^{-\frac{\mathcal{D}_0}{N+2}t}$$

for all $t > t_0$.
Taking $\gamma = \frac{\mathcal{D}_0}{N+2}$, we obtain (1.11) and (1.12).
Next, we give the proof of convergence for u. Multiplying the third of (1.4) by u and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx+\int_\Omega|\nabla u|^2dx=\int_\Omega n\nabla\phi\cdot udx=\int_\Omega(n(x,t)-(\frac{r}{\mu})^{\frac{1}{k-1}})\nabla\phi\cdot udx. \tag{5.20}$$

By using Hölder's inequality, Young's inequality and Poincaré's inequality, we can find $\kappa_1 > 0$ such that

$$\int_\Omega\Big|(n(x,t)-(\frac{r}{\mu})^{\frac{1}{k-1}})\nabla\phi\cdot u\Big|dx \leq \kappa_1\int_\Omega(n(x,t)-(\frac{r}{\mu})^{\frac{1}{k-1}})^2dx+\frac{1}{2}\int_\Omega|\nabla u|^2dx \tag{5.21}$$

holds for all $t \in (0, T_{\max})$, where we used the boundedness of $\|\nabla\phi\|_{L^\infty(\Omega)}$.
Thus substituting (5.21) into (5.20), we have

$$\frac{d}{dt}\int_\Omega|u|^2dx+\int_\Omega|\nabla u|^2dx \leq 2\kappa_1\int_\Omega(n(x,t)-(\frac{r}{\mu})^{\frac{1}{k-1}})^2dx. \tag{5.22}$$

Due to $U(x,t) = \frac{\mu}{r}n(x,t)$, we can obtain

$$\frac{d}{dt}\int_\Omega|u|^2dx+\int_\Omega|\nabla u|^2dx \leq 2\kappa_1(\frac{r}{\mu})^{\frac{2}{k-1}}\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2dx. \tag{5.23}$$

Substituting (5.23) into (5.8), we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}U+LV^2+|u|^2\Big)dx+(2L-\frac{\chi^2}{4\eta^2})\int_\Omega|\nabla V|^2dx+\int_\Omega|\nabla u|^2dx\\ &+(r-(L+2\kappa_1)(\frac{r}{\mu})^{\frac{2}{k-1}})\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2dx+\int_\Omega LV^2dx \leq 0 \quad \text{for all } t > 0.\end{aligned}$$

As once again using the Poincaré's inequality, there is a constant $\kappa_2 > 0$ such that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}U+LV^2+|u|^2\Big)dx+(2L-\frac{\chi^2}{4\eta^2})\int_\Omega|\nabla V|^2dx\\&+(r-(L+2\kappa_1)(\frac{r}{\mu})^{\frac{2}{k-1}})\int_\Omega((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2dx+\int_\Omega LV^2dx+\kappa_2\int_\Omega u^2dx\le 0\quad\text{for all } t>0.\end{aligned}\tag{5.24}$$

Let $\mathcal{G}(t) := \int_\Omega\Big((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1-\ln(\frac{r}{\mu})^{\frac{k-2}{k-1}}U+LV^2+|u|^2\Big)dx$ and let $\mathcal{H}_0=\min\{r-(L+2\kappa_1)(\frac{r}{\mu})^{\frac{2}{k-1}},\ 1,\ \kappa_2\}>0$. We also require $\frac{\chi^2}{8\eta^2}<L<\mu^{\frac{2}{k-1}}r^{\frac{k-3}{k-1}}-2\kappa_1$. Thanks to (5.24), we have

$$\frac{d}{dt}\mathcal{G}(t)\le-\mathcal{H}(t),$$

where

$$\mathcal{H}(t) := \mathcal{H}_0\int_\Omega\Big(((\frac{r}{\mu})^{\frac{k-2}{k-1}}U-1)^2+LV^2+u^2\Big)dx.$$

By means of quite a similarly argument, we have

$$\frac{d}{dt}\mathcal{G}(t)\le-\mathcal{H}_0\mathcal{G}(t)$$

and hence there exists a constant $C_{38}>0$ such that

$$\|u(\cdot,t)\|_{L^2(\Omega)}\le C_{38}e^{-\frac{\mathcal{H}_0}{2}(t-t_0)}\qquad\text{for all } t>t_0.\tag{5.25}$$

We also recall from the Gagliardo-Nirenberg inequality, (5.25), and Lemma 4.4 that there are some constants $C_{39}>0$ and $C_{40}>0$ fulfilling

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)}&\le C_{39}\big\|u(\cdot,t)\big\|_{L^2(\Omega)}^{\frac{2}{N+2}}\big\|\nabla u(\cdot,t)\big\|_{L^\infty(\Omega)}^{\frac{N}{N+2}}+C_{39}\|u(\cdot,t)\|_{L^2(\Omega)}\\&\le C_{40}e^{-\frac{\mathcal{H}_0}{N+2}t}\qquad\text{for all } t>t_0.\end{aligned}$$

Taking $\gamma_\star=\frac{\mathcal{H}_0}{N+2}$, we obtain (1.14) and (1.15) as well as (1.16). □

Now let us consider the asymptotic behavior of $1<k<2$ to prove Theorem 1.3. The following ideas come from the proof of Theorem 1.2 and the energy construction of asymptotic behavior in [45].

Let $n_*=(\frac{r}{\mu})^{\frac{1}{k-1}}$. We define the following functions

$$\mathcal{F}_{n_*,B}(n,c) := \int_\Omega\psi_{n_*}(n)dx+\frac{B}{2}\int_\Omega(c-n_*)^2dx,\tag{5.26}$$

where

$$\psi_{n_*}(s) := s-n_*-n_*\ln\frac{s}{n_*}\tag{5.27}$$

and B is a fixed positive constant.

From the discussion of Lemma 5.3, we can see that

$$\psi_{n_*}(s)\ge 0\quad\text{for all } s>0.\tag{5.28}$$

For the convenience of proving the Theorem 1.3, we first prove the following several lemmas.

Lemma 5.5 *Suppose that* $\chi>0, 1<k<2$ *and*

$$\mu\ge\max\left\{\left(\frac{\chi^2M_0^{2-k}}{4(k-1)\eta^2}\right)^{\frac{k-1}{k}}r^{\frac{1}{k}},\ \left(\frac{\chi}{2\eta\sqrt{k-1}}\right)^{k-1}r^{\frac{3-k}{2}}\right\}$$

hold. Here $\eta>0$ *come from* (2.7) *and* $M_0=\sup_t\|n(\cdot,t)\|_{L^\infty(\Omega)}$ *be defined in Theorem 1.3. Then, we have the following energy inequality*

$$\frac{d}{dt}\mathcal{F}_{n_*,B}(n,c)+\mathcal{A}(n,c)\le 0,\tag{5.29}$$

where

$$\mathcal{A}(n,c) = \frac{n_*}{2}\int_\Omega \frac{|\nabla n|^2}{n^2}dx + \Big(B - \frac{\chi^2 n_*}{2\eta^2}\Big)\int_\Omega |\nabla c|^2 dx + \frac{B}{4}\int_\Omega (c-n_*)^2 dx + \frac{B}{6}\int_\Omega (n-n_*)^2 dx. \tag{5.30}$$

Proof. We use the first two equation in (1.4) and the fact that $\nabla \cdot u = 0$ to compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_{n_*,B}(n,c) &= \int_\Omega n_t dx - n_*\int_\Omega \frac{n_t}{n}dx + B\int_\Omega (c-n_*)c_t dx \\
&= r\int_\Omega n dx - \mu\int_\Omega n^k dx - n_*\int_\Omega \frac{1}{n}\cdot\Big\{\Delta n - \chi\nabla\cdot(\frac{n}{c}\nabla c) + rn - \mu n^k - u\cdot\nabla n\Big\} \\
&\quad + B\int_\Omega (c-n_*)\cdot\Big\{\Delta c - c + n - u\cdot\nabla c\Big\}dx \\
&= r\int_\Omega n dx - \mu\int_\Omega n^k dx - n_*\int_\Omega \frac{|\nabla n|^2}{n^2}dx + \chi n_*\int_\Omega \frac{\nabla n\cdot\nabla c}{nc}dx - rn_*|\Omega| + \mu n_*\int_\Omega n^{k-1}dx \\
&\quad - B\int_\Omega |\nabla c|^2 dx - B\int_\Omega (c-n_*)^2 dx + B\int_\Omega (c-n_*)(n-n_*)dx \\
&:= \tilde{J}_1 + \cdots + \tilde{J}_9.
\end{aligned} \tag{5.31}$$

We deduce from the Young's inequality and (2.7) that

$$\tilde{J}_4 \le \frac{n_*}{2}\int_\Omega \frac{|\nabla n|^2}{n^2}dx + \frac{\chi^2 n_*}{2\eta^2}\int_\Omega |\nabla c|^2 dx. \tag{5.32}$$

For $\tilde{J}_9$, using Young's inequality again, we have

$$\tilde{J}_9 \le \frac{3B}{4}\int_\Omega (c-n_*)^2 dx + \frac{B}{3}\int_\Omega (n-n_*)^2 dx. \tag{5.33}$$

We control $\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_5 + \tilde{J}_6$ as follows

$$\begin{aligned}
\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_5 + \tilde{J}_6 + \frac{B}{2}\int_\Omega (n-n_*)^2 dx &= \int_\Omega \Big\{rn - \mu n^k - rn_* + \mu n_* n^{k-1} + \frac{B}{2}(n-n_*)^2\Big\}dx \\
&= \int_\Omega (n-n_*)\cdot\Big\{\mu(\frac{r}{\mu} - n^{k-1}) + \frac{B}{2}(n-n_*)\Big\}dx.
\end{aligned} \tag{5.34}$$

We now substitute (5.32)-(5.34) into (5.31) to obtain that

$$\begin{aligned}
&\frac{d}{dt}\mathcal{F}_{n_*,B}(n,c) + \frac{n_*}{2}\int_\Omega \frac{|\nabla n|^2}{n^2}dx + \Big(B - \frac{\chi^2 n_*}{2\eta^2}\Big)\int_\Omega |\nabla c|^2 dx + \frac{B}{4}\int_\Omega (c-n_*)^2 dx \\
&\quad + \frac{B}{6}\int_\Omega (n-n_*)^2 dx \le \int_\Omega (n-n_*)\cdot\Big\{\mu(\frac{r}{\mu} - n^{k-1}) + \frac{B}{2}(n-n_*)\Big\}dx \quad \text{for all } B \ge \frac{\chi^2}{2\eta^2}n_*.
\end{aligned}$$

Let $\omega(s) := (s-n_*)h(s)$, where $h(s) = \mu(\frac{r}{\mu} - s^{k-1}) + \frac{B}{2}(s-n_*)$. A derivation with respect to s gives

$$h'(s) = -\mu(k-1)s^{k-2} + \frac{B}{2} \quad \text{for all } B \ge \frac{\chi^2}{2\eta^2}(\frac{r}{\mu})^{\frac{1}{k-1}}.$$

Notice that $h'(s) = 0$ is equivalent to $s = \left(\frac{2\mu(k-1)}{B}\right)^{\frac{1}{2-k}}$. We take two times derivatives for $\omega(s)$ to obtain

$$\begin{aligned}
\omega'(s) &= \mu(\frac{r}{\mu} - s^{k-1}) - \mu(k-1)s^{k-2}(s-n_*) + B(s-n_*) \\
\omega''(s) &= -2\mu(k-1)s^{k-2} - \mu(k-1)(k-2)s^{k-3}(s-n_*) + B.
\end{aligned}$$

A direct computation shows that $\omega(n_*) = \omega'(n_*) = 0$ and $\omega''(n_*) = B - 2(k-1)\mu^{\frac{1}{k-1}}r^{\frac{k-2}{k-1}}$. We take $B \le 2(k-1)\mu^{\frac{1}{k-1}}r^{\frac{k-2}{k-1}}$ to ensure $\omega''(n_*) \le 0$. Therefore, we have $\left(\frac{2\mu(k-1)}{B}\right)^{\frac{1}{2-k}} \ge \left(\frac{2\mu(k-1)}{2(k-1)\mu^{\frac{1}{k-1}}r^{\frac{k-2}{k-1}}}\right)^{\frac{1}{2-k}} = n_*$, $h(0) = r - \frac{B}{2}n_* \ge r - (k-1)\mu^{\frac{1}{k-1}}r^{\frac{k-2}{k-1}}(\frac{r}{\mu})^{\frac{1}{k-1}} = (2-k)r > 0$, $h(n_*) = 0$, and $\omega(0) = -rn_* + \frac{B}{2}n_*^2 = -n_*(r - \frac{B}{2}n_*) < 0$. We use the method of analyzing graph to capture a point which is called arrest point. Thus arrest point is $\left((\frac{2\mu(k-1)}{B})^{\frac{1}{2-k}}, \omega((\frac{2\mu(k-1)}{B})^{\frac{1}{2-k}})\right)$. If we truncate the right part about arrest point, then the function is not positive. That is if $M_0 \le \left(\frac{2\mu(k-1)}{B}\right)^{\frac{1}{2-k}}$, we have

$$\omega(n) \le 0 \quad \text{for all } \ B \le \frac{2(k-1)\mu}{M_0^{2-k}},$$

which completes the proof of Lemma 5.5. □

Corollary 5.2 *Under the assumptions of Lemma 5.5, we have*

$$\int_1^\infty \int_\Omega (c-n_*)^2 dxdt < \infty \quad \textit{and} \quad \int_1^\infty \int_\Omega (n-n_*)^2 dxdt < \infty.$$

Proof. Integrating both side of (5.29) from 1 to ∞ about the time, we have

$$\mathcal{F}_{n_*,B}(n,c) + \int_1^\infty \mathcal{A}(n,c)dt \le \mathcal{F}_{n_*,B}(n(\cdot,1), c(\cdot,1)).$$

Since $\mathcal{F}_{n_*,B}(n,c)$ is nonnegative by (5.26) and (5.28), this entails that $\int_1^\infty \mathcal{A}(n,c)dt \le \mathcal{F}_{n_*,B}(n(\cdot,1),c(\cdot,1)) < \infty$, which according to the definition of $\mathcal{A}(n,c)$ in (5.30) directly implies Corollary 5.2. □

Lemma 5.6 *Assume the hypothesis of Theorem 1.3 holds. Then if* (n,c) *is a nonnegative global bounded classical solution of* (1.4)*, we have*

$$\lim_{t\to\infty} \|n(\cdot,t) - n_*\|_{L^\infty(\Omega)} = 0 \tag{5.35}$$

and

$$\lim_{t\to\infty} \|c(\cdot,t) - n_*\|_{L^\infty(\Omega)} = 0.$$

Proof. Thanks to the boundedness of $\int_1^\infty \int_\Omega (c-n_*)^2 dxdt$ and $\int_1^\infty \int_\Omega (n-n_*)^2 dxdt$, which is obtained in Corollary 5.2. The following processes are similar to the proof of Lemma 5.4. Thus we omit the details. □

Lemma 5.7 *Let* $\psi_{n_*}(s)$ *be defined by* (5.27)*. Then there exists a positive constant* $\delta_3 < \frac{1}{10}$ *satisfying*

$$\frac{1}{6n_*}(s-n_*)^2 < \psi_{n_*}(s) < \frac{5}{6n_*}(s-n_*)^2 \quad for\ all \quad 0 < \left|s-n_*\right| < \delta_3. \tag{5.36}$$

Proof Because

$$\lim_{s\to n_*} \frac{s-n_*-n_*\ln\frac{s}{n_*}}{(s-n_*)^2} = \frac{1}{2n_*},$$

we can find a positive constant $\delta_3 < \frac{1}{10}$ such that

$$\left|\frac{s-n_*-n_*\ln\frac{s}{n_*}}{(s-n_*)^2} - \frac{1}{2n_*}\right| < \frac{1}{3n_*} \quad \text{for all} \quad 0 < |s-n_*| < \delta_3.$$

We thereupon readily arrive at (5.36). □

Proof of Theorem 1.3. The proof is similar to that of Theorem 1.2. Here we only give the key steps. In virtue of (5.35), there exists $t_1 > 0$ large enough such that

$$\frac{1}{6n_*}(n-n_*)^2 < \psi_{n_*}(n) < \frac{5}{6n_*}(n-n_*)^2 \quad \text{for all} \quad t > t_1,\ x \in \Omega. \tag{5.37}$$

Due to (5.29) and (5.37), there exists $\mathcal{A}_0 = \min\{\frac{1}{2}, \frac{Bn_*}{5}\}$ such that

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_{n_*,B}(n,c) &\le -\mathcal{A}(n,c) \\ &\le -\frac{B}{4}\int_\Omega (c-n_*)^2dx - \frac{B}{6}\int_\Omega (n-n_*)^2dx \\ &\le -\mathcal{A}_0\Big\{\int_\Omega \psi_{n_*}(n)dx + \frac{B}{2}\int_\Omega (c-n_*)^2dx\Big\} \\ &= -\mathcal{A}_0\mathcal{F}_{n_*,B}(n,c) \quad \text{for all} \quad t>t_1,\ x\in\Omega. \end{aligned} \tag{5.38}$$

Using the Gronwall's inequality, we have

$$\mathcal{F}_{n_*,B}(n,c) \le C_{41}e^{-\mathcal{A}_0 t}, \tag{5.39}$$

where $C_{41} = \mathcal{F}_{n_*,B}(n_0,c_0) > 0$.
Using the Gagliardo-Nirenberg inequality and (5.39), we can find two positive constants C_{42} and C_{43} such that

$$\Big\|n(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\Big\|_{L^\infty(\Omega)} \le C_{42}\big\|\nabla n(\cdot,t)\big\|_{L^\infty(\Omega)}^{\frac{N}{N+2}}\Big\|n(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\Big\|_{L^2(\Omega)}^{\frac{2}{N+2}} + C_{42}\Big\|n(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\Big\|_{L^2(\Omega)} \le C_{43}e^{-\frac{\mathcal{A}_0}{N+2}t}$$

and

$$\|c(\cdot,t) - (\frac{r}{\mu})^{\frac{1}{k-1}}\|_{L^\infty(\Omega)} \le C_{43}e^{-\frac{\mathcal{A}_0}{N+2}t}$$

for all $t>t_1$. Taking $\tilde{\gamma} = \frac{\mathcal{A}_0}{N+2}$, we obtain (1.18) and (1.19).
Thanks to (5.22), we obtain

$$\frac{d}{dt}\int_\Omega |u|^2dx + \int_\Omega |\nabla u|^2dx \le 2\kappa_1\int_\Omega (n(x,t)-n_*)^2dx. \tag{5.40}$$

Adding (5.40) into (5.38) and then using the Poincaré's inequality and (5.37), we have

$$\begin{aligned} \frac{d}{dt}\Big(\mathcal{F}_{n_*,B}(n,c) + \int_\Omega u^2dx\Big) + \kappa_2\int_\Omega u^2dx &\le \frac{d}{dt}\Big(\mathcal{F}_{n_*,B}(n,c) + \int_\Omega u^2dx\Big) + \int_\Omega |\nabla u|^2dx \\ &\le -\mathcal{A}_0\mathcal{F}_{n_*,B}(n,c) + 2\kappa_1\int_\Omega (n(x,t)-n_*)^2dx \\ &\le -\mathcal{A}_0\mathcal{F}_{n_*,B}(n,c) + 12\kappa_1 n_*\int_\Omega \psi_{n_*}(n)dx \\ &\le -\mathcal{A}_1\mathcal{F}_{n_*,B}(n,c), \end{aligned}$$

where $0<\mathcal{A}_1 \le \mathcal{A}_0 - 12\kappa_1 n_* \le \min\{\frac{1}{2}, \frac{Bn_*}{5}\} - 12\kappa_1 n_*$ and κ_2 is given by (5.24).
Moving the second term on the left hand side to the right hand side, we have

$$\begin{aligned} \frac{d}{dt}\Big(\mathcal{F}_{n_*,B}(n,c) + \int_\Omega u^2dx\Big) &\le -\mathcal{A}_1\mathcal{F}_{n_*,B}(n,c) - \kappa_2\int_\Omega u^2dx \\ &\le -\mathcal{A}_2\Big(\mathcal{F}_{n_*,B}(n,c) + \int_\Omega u^2dx\Big), \end{aligned}$$

where $0<\mathcal{A}_2 = \min\{\mathcal{A}_1, \kappa_2\} \le \min\Big\{\min\{\frac{1}{2}, \frac{Bn_*}{5}\} - 12\kappa_1 n_*,\ \kappa_2\Big\} = \min\Big\{\min\{\frac{1}{2}, \frac{B}{5}(\frac{r}{\mu})^{\frac{1}{k-1}}\} - 12\kappa_1(\frac{r}{\mu})^{\frac{1}{k-1}},\ \kappa_2\Big\}$.
Using the Gronwall's inequality, we can find $C_{44}>0$ such that

$$\mathcal{F}_{n_*,B}(n,c) + \int_\Omega u^2dx \le C_{44}e^{-\mathcal{A}_2 t}, \tag{5.41}$$

where $\mu > (24\kappa_1)^{k-1}r$ and $B > 60\kappa_1$.
Again using the Gagliardo-Nirenberg inequality, (5.26), (5.36), and (5.41), we can find two positive constants C_{45} and C_{46} such that

$$\begin{aligned} \big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} &\le C_{45}\big\|\nabla u(\cdot,t)\big\|_{L^\infty(\Omega)}^{\frac{N}{N+2}}\big\|u(\cdot,t)\big\|_{L^2(\Omega)}^{\frac{2}{N+2}} + C_{45}\big\|u(\cdot,t)\big\|_{L^2(\Omega)} \\ &\le C_{46}e^{-\frac{\mathcal{A}_2}{N+2}t} \qquad \text{for all } t>t_1. \end{aligned}$$

Taking $\tilde{\gamma}_\star = \frac{\mathcal{A}_2}{N+2}$, we complete the proof of Theorem 1.3. □

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