J. Math. Anal. Appl. 489 (2020) 124152

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Optimal control of an eddy current problem with a dipole source

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ARTICLE INFO

Article history:
Received 18 March 2020
Available online 20 April 2020
Submitted by D. Donatelli

Keywords:
Optimal control
Pointwise control
Time-harmonic Maxwell equations
Fundamental solution formulation
Optimality conditions

ABSTRACT

This paper is concerned with the analysis of a class of optimal control problems governed by a time-harmonic eddy current system with a dipole source, which is taken as the control variable. A mathematical model is set up for the state equation where the dipole source takes the form of a Dirac mass located in the interior of the conducting domain. A non-standard approach featuring the fundamental solution of a curl curl −Id operator is proposed to address the well-posedness of the state problem, leading to a split structure of the state field as the sum of a singular part and a regular part. The aim of the control is the best approximation of desired electric and magnetic fields via a suitable L^2-quadratic tracking cost functional. Here, special attention is devoted to establishing an adjoint calculus which is consistent with the form of the state variable and in this way first order optimality conditions are eventually derived.

1. Introduction

The aim of devising optimal control procedures for Maxwell's equations and eddy current systems is not new in itself, considering the important role of electromagnetic fields in various modern technologies. Once suitable mathematical tools became available in the literature[1] many researchers have started focusing their attention on this kind of problems, most of the times considering distributed controls in the form of a current density in the interior of a conducting domain, or in the form of a voltage excitation on the boundary (i.e., via electric ports): we refer to Tröltzsch and Valli [22], [21] and to Yousept [25] for linear time-harmonic eddy current problems, and to Tröltzsch and Valli [23] or Nicaise et al. [15] for the time-dependent case. We also mention the work of Bommer and Yousept [6] featuring the full Maxwell system as well as the one of Yousept [26] where a quasi-linear case is investigated.

At the same time, there are several applied contexts in which one is interested in finding an optimal way to *place* sensors or actuators; along with this, if it is not possible - or not enough efficient - to distribute

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[1] The very first exhaustive characterization of the traces of $\boldsymbol{H}(\mathrm{curl})$ functions in rather general domains came out in the early 2000s with [7] by Buffa et al.

https://doi.org/10.1016/j.jmaa.2020.124152

control devices all over the domain, the problem of identifying which sub-regions are actually important in order to achieve the minimization of the objective functional arises. Following the archetype work of Stadler [18], it became clear that the addition of a non-smooth L^1-regularization term in the cost functional entails sparse properties of the optimal solutions, namely that they have small support with respect to the Lebesgue measure. These techniques have already been applied, though not extensively, in the context electromagnetic PDEs, see for instance Tröltzsch and Valli [22] and the author [9].

In more recent times, the lack of reflexivity, compactness and differentiability (regularity) properties of the L^1-spaces and norms led to the study of optimal control problems in measure spaces like $\mathcal{M}(\Omega)$, the space of regular Borel measures, or $L^2(I, \mathcal{M}(\Omega))$, which both exhibit better functional properties as well as similar sparsity features of optimal solutions; see Casas et al. [8], Clason and Kunish [10] and Trautmann et al. [19], where all these issues are widely discussed.

Let us focus our attention on controls of the form:

$$u = \sum_{i=1}^{N} u_i \delta_{x_i}, \quad x_i \in \Omega, \tag{1}$$

where u_i is, say, either a complex number or a time-dependent intensity $t \mapsto u_i(t)$. These are typical examples of singular elements in $\mathcal{M}(\Omega)$ (respectively in $L^2(I; \mathcal{M}(\Omega))$ in the time-dependent framework) that are usually of interest for modeling phenomena related to geology or acoustics: we refer to Pieper et al. [17], where an inverse problem from point-wise measurements (state observations) is analyzed; nevertheless, a work by Alonso Rodríguez et al. [3] concerning inverse problems for eddy current equations suggests that sources (controls) of type (1) can be meaningful also for electromagnetic problems: a weighted Dirac mass $\mathbf{p}\delta_{\mathbf{x}_0}$ represents a dipole source of intensity $\mathbf{p} \in \mathbb{R}^3$ concentrated at $\mathbf{x} = \mathbf{x}_0$.

In principle, this would lead to consider controls that can be a priori expressed as a linear combination of deltas with *unknown positions* and *unknown intensities*, an assumption which, in turn, yields a non-convex optimization problem. A common idea to overcome this difficulty is precisely to lift the problem to a more general one with controls lying in a suitable space of measures, and *then* discuss if and under what conditions the solution has the desired structure (1).

However, the latter step is far from being reliable, often providing just some necessary conditions and/or information on the support of the optimal measure. For what concerns electromagnetic state equations, the situation is even more complicated since the analysis of PDEs with measure-valued sources usually requires some structural regularity of the differential operator, while Maxwell's equations naturally exhibit singular solutions in most instances; see e.g. Costabel et al. [12]. For these reasons, we decided to work with a fixed number of deltas (i.e., one, without loss of generality) in a fixed location, say $\mathbf{x} = \mathbf{x}_0$. A similar approach has been carried out rather recently by Allendes et al. [1], but there the state equation takes the form of a Poisson problem and the focus is shifted on the a posteriori error analysis for a FEM approximation.

Despite the adopted simplifications, several mathematical difficulties are here present: the most important, as mentioned, is that our state equation is an eddy current system with a Dirac distribution as source. We propose an approach that seems new in this context; the resolution of the problem is split into three steps, the first one being the determination of a fundamental solution to deal with the singularity at $\mathbf{x}_0$ (this idea has been already used to tackling some inverse problems; see for instance Wolters et al. [24] and Alonso Rodríguez et al. [3]). After that, the specific structure of the eddy current problem leads to a state variable that is composed by two terms, a vector one and (the gradient of) a scalar one. The control analysis inherits these issues and thus two adjoint states, corresponding to two different *parts* of the state variable, need to be defined in a non-standard way. Moreover, the underlying complex structure of the spaces involved in the analysis of time-harmonic Maxwell's equations entails that some attention is required to discuss the differentiability of the objective functional.

It is worth to note that this kind of approach, based on the determination of a fundamental solution, could be used also for tackling the control problem associated with more canonical operators, as the Laplace operator or other elliptic operators.

Now we briefly summarize the content of this paper. In next section we introduce our notation and our basic geometrical assumptions. Section 3 is devoted to the mathematical analysis of the state equation: here, its solution is built up starting from the fundamental solution of a curl curl operator. In section 4, we present the optimal control problem and eventually derive first order optimality conditions.

To our best knowledge, this article represents the first contribution towards the optimal control of electromagnetic fields in the presence of spike sources.

2. Preliminaries

Geometrical assumptions. The computational domain Ω is a bounded simply connected open set in $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega =: \Gamma$. A non-empty open, connected subset $\Omega_C \subset \Omega$ denotes the conducting region and consequently $\Omega_I := \Omega \setminus \overline{\Omega_C}$ is the insulator, which is also assumed to be connected for simplicity; Ω_C is strictly contained in Ω in such a way that $\Gamma \cap \partial\Omega_C = \emptyset$ and it is assumed to be simply connected, implying that Ω_I is also simply connected. The set $\Gamma_C := \partial\Omega_I \cap \partial\Omega_C$ is the interface between the conductor and the insulator. We finally set $\Gamma_I := \partial\Omega_I = \Gamma \cap \Gamma_C$ and denote by $\mathbf{n}, \mathbf{n}_C$ and $\mathbf{n}_I$ respectively the unit outward normal vectors on Γ, Γ_C and Γ_I. From now on, for the sake of clarity we use the notation $\mathbf{H}_I := \mathbf{H}|_{\Omega_I}, \boldsymbol{\sigma}_C := \boldsymbol{\sigma}|_{\Omega_C}$ (and similar for other fields) to explicitly underline to which subdomain a certain vector or matrix valued map is restricted.

Notation. Throughout this paper, we shall work with functional spaces on the field of complex numbers – unless otherwise specified - and we shall use a bold typeface to denote a three-dimensional vector map, or a vector space of three-dimensional vector functions. For instance, we set:

$$\begin{aligned} &\boldsymbol{L}^2(\Omega) := \{\mathbf{u} : \Omega \to \mathbb{C}^3 \mid |\mathbf{u}| \in L^2_{\mathbb{R}}(\Omega)\} \\ &H^1(\Omega) := \{u : \Omega \to \mathbb{C} \mid |u| \in L^2_{\mathbb{R}}(\Omega), |\nabla u| \in L^2_{\mathbb{R}}(\Omega)\}; \end{aligned} \tag{2}$$

the spaces $\boldsymbol{H}(\mathrm{curl};\Omega), \boldsymbol{H}(\mathrm{div};\Omega)$ are thus defined as

$$\begin{aligned} \boldsymbol{H}(\mathrm{curl};\Omega) &:= \{\mathbf{u} : \Omega \to \mathbb{C}^3 \mid \mathbf{u} \in \boldsymbol{L}^2(\Omega),\ \mathrm{curl}\,\mathbf{u} \in \boldsymbol{L}^2(\Omega)\}, \\ \boldsymbol{H}(\mathrm{div};\Omega) &:= \{\mathbf{u} : \Omega \to \mathbb{C}^3 \mid \mathbf{u} \in \boldsymbol{L}^2(\Omega),\ \mathrm{div}\,\mathbf{u} \in L^2_{\mathbb{C}}(\Omega)\}. \end{aligned}$$

The corresponding trace spaces are defined, e.g., in Monk [14, Section 3.5] or in Alonso Rodríguez and Valli [2, Appendix A.1].

The matrix-valued coefficients $\boldsymbol{\mu} \in L^\infty(\Omega;\mathbb{R}^{3\times3}), \boldsymbol{\sigma} \in L^\infty(\Omega_C;\mathbb{R}^{3\times3})$ and $\boldsymbol{\epsilon} \in L^\infty(\Omega_I;\mathbb{R}^{3\times3})$ are all assumed to be symmetric and uniformly positive definite; moreover they satisfy an *homogeneity condition* which is below introduced and motivated, see (4).

3. Analysis of the state equation: the eddy current problem with a dipole source

Let us consider an **E**-based formulation for the eddy current problem with a dipole source in the form of a Dirac mass, namely:

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}) + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{p}\delta_{\mathbf{x}_0} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\epsilon}_I\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_I)\times\mathbf{n} = \mathbf{0} & \text{on } \Gamma \\ \boldsymbol{\epsilon}_I\mathbf{E}_I\cdot\mathbf{n} = 0 & \text{on } \Gamma, \end{cases} \tag{3}$$

where $\mathbf{p} \in \mathbb{R}^3, \omega > 0, \mathbf{x}_0 \in \Omega_C$ and $\delta_{\mathbf{x}_0}$ stands for the Dirac distribution centered at $\mathbf{x}_0$. Equations $(3)_{3,4}$ correspond to the choice of the so called *magnetic boundary condition(s)*, see Alonso Rodríguez and Valli [2, Section 1.3], reinterpreted after eliminating the magnetic field from the eddy current system.

We also point out that (3) is somehow already a simplified model, since our geometrical assumptions entail that a couple of equations related to the topology of Ω_I can be a priori dropped, see again Alonso Rodríguez and Valli [2, p. 22].

Prior to the control analysis, we need to address the well-posedness of problem (3). The first existence and uniqueness result for (3) is due to Alonso Rodríguez et al. [3] in the context of inverse problems; in this sense, Theorem 3 is not new. However, here we are focused towards the analysis of a corresponding optimal control problem and thus we need to keep track of the dependence of the solution (that is, the state variable) on the control $\mathbf{p}$, see Remarks 1, 2 and Corollary 2, and we also need to introduce and study some operators that will be involved in the adjoint calculus (for instance the sesquilinear forms (25)). None of these issues are addressed in [3]. Moreover we discuss in Remark 3 how to treat another significant boundary condition.

From now on we shall assume that physical parameters $\boldsymbol{\mu}, \boldsymbol{\sigma}$ satisfy a *local homogeneity condition*: there exists a ball $B_r(\mathbf{x}_0)$ centered at $\mathbf{x}_0$ and two real positive constants μ_0, σ_0 for which:

$$\boldsymbol{\mu}(\mathbf{x}) = \mu_0 \operatorname{Id}_{\mathbb{R}^3} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{x}) = \sigma_0 \operatorname{Id}_{\mathbb{R}^3} \qquad \forall \mathbf{x} \in B_r(\mathbf{x}_0). \tag{4}$$

The latter assumption is not that much restrictive in most instances because the location of the point source, i.e. $\mathbf{x}_0$, is more or less free to choose and it seems reasonable to opt for a point that does not lie on an interface region separating different materials. On the other hand, it is pivotal for giving a meaning to our fundamental solution-based approach: $\boldsymbol{\mu}$ being constant in a neighborhood of $\mathbf{x}_0$ entails that locally we are dealing with the $\operatorname{curl}\operatorname{curl} - \operatorname{Id}$ operator - up to constants -, whose fundamental solution is known in the literature. The following result is adapted from Ammari et al. [5]:

Proposition 1. *Let $z = \sqrt{-i\omega\mu_0\sigma_0}$ with $\operatorname{Re} z < 0$ and $\mathbf{q} = -i\omega\mathbf{p}$; the distributional solution $\mathbf{K} = \mathbf{K}(\cdot;\mathbf{x}_0)$ of the equation*

$$\operatorname{curl}\operatorname{curl}\mathbf{K} - z^2\mathbf{K} = \mathbf{q}\delta_{\mathbf{x}_0} \tag{5}$$

is given by

$$\mathbf{K} = \mathbf{K}(\mathbf{x};\mathbf{x}_0) = \mathbf{q}\Phi_{\mathbf{x}_0}(\mathbf{x}) + \frac{1}{z^2}(\mathbf{q}\cdot\nabla)\nabla\Phi_{\mathbf{x}_0}(\mathbf{x}), \tag{6}$$

where

$$\Phi_{\mathbf{x}_0}(\mathbf{x}) = \frac{\exp(iz|\mathbf{x}-\mathbf{x}_0|)}{4\pi|\mathbf{x}-\mathbf{x_0}|} \tag{7}$$

is the fundamental solution - up to translation in $\mathbf{x}_0$ - of the Helmholtz operator

$$-\Delta - z^2 \operatorname{Id}.$$

Remark 1 *(Dependence of* $\mathbf{K}$ *on the intensity* $\mathbf{q}$*).* Since $\mathbf{q}$ is constant, we have

$$\begin{aligned}\mathbf{K} &= \mathbf{q}\Phi + \frac{1}{z^2}(\mathbf{q}\cdot\nabla)\nabla\Phi \\ &= \mathrm{Id}(\mathbf{q}\Phi) + \frac{1}{z^2}(\nabla^2\Phi)\mathbf{q} \\ &= [\mathrm{Id}\,\Phi + \nabla^2\Phi]\mathbf{q} =: N\mathbf{q},\end{aligned} \tag{8}$$

where $N = N(\mathbf{x}_0, \Phi_{\mathbf{x}_0})$ is then a symmetric matrix with entries in $H^{-2}(\Omega)$, since it inherits the singularity of $\Phi_{\mathbf{x}_0}(\cdot)$ at $\mathbf{x} = \mathbf{x}_0$.

If $\mathbf{x} \in B_r(\mathbf{x}_0)$, equation $(3)_1$ reads

$$\mu_0^{-1}\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}(\mathbf{x}) + i\omega\sigma_0\mathbf{E}(\mathbf{x}) = -i\omega\mathbf{p}\delta_{\mathbf{x}_0}(\mathbf{x}),$$

thus Proposition 1 applies and we are suggested to look for the solution of (3) in the form

$$\mathbf{E} = \mathbf{K} + \mathbf{M}, \tag{9}$$

where $\mathbf{M}$ has to read the behavior outside the ball $B_r(\mathbf{x}_0)$ through a modified source on the RHS. More precisely, $\mathbf{M}$ is formally the solution to

$$\begin{cases} \mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{M}) + i\omega\boldsymbol{\sigma}\mathbf{M} = \mathbf{J} & \text{in } \Omega \\ \mathrm{div}(\boldsymbol{\epsilon}\mathbf{M}) = -\,\mathrm{div}(\boldsymbol{\epsilon}\mathbf{K}) & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{M})\times\mathbf{n} = -(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{K})\times\mathbf{n} & \text{on } \Gamma \\ \boldsymbol{\epsilon}\mathbf{M}\cdot\mathbf{n} = -\boldsymbol{\epsilon}\mathbf{K}\cdot\mathbf{n} & \text{on } \Gamma, \end{cases} \tag{10}$$

where

$$\mathbf{J} = \begin{cases} \mathbf{0} & \text{in } B_r(\mathbf{x}_0) \\ -\,\mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{K}) - i\omega\boldsymbol{\sigma}\mathbf{K} & \text{in } \Omega\setminus B_r(\mathbf{x}_0); \end{cases} \tag{11}$$

later on we shall see the correct weak formulation of this formal problem.

Focusing now on (10), we first aim at homogenizing it by finding a vector field – in the form of a gradient – which has both the same divergence in Ω_I and the same normal component on Γ of $\boldsymbol{\epsilon}\mathbf{Q}$.

Let

$$W := \{w \in H^1(\Omega_I) : w = 0 \text{ on } \Gamma_C\}; \tag{12}$$

$\eta_I \in W$ is defined as the solution of the weak boundary value problem

$$b[\eta_I, \xi] := \int_{\Omega_I} \boldsymbol{\epsilon}_I \nabla\eta_I \cdot \nabla\overline{\xi} = -\int_{\Omega_I} \boldsymbol{\epsilon}_I \mathbf{K}\cdot\nabla\overline{\xi} \qquad \forall\xi \in W, \tag{13}$$

which is clearly well-posed since $\mathbf{K}|_{\Omega_I} \in \boldsymbol{L}^2(\Omega_I)^3$. It is straightforward to see that η_I is the weak solution of the strong, mixed boundary value problem

$$\begin{cases} -\operatorname{div}(\boldsymbol{\epsilon}_I \nabla \eta_I) = \operatorname{div}(\boldsymbol{\epsilon}_I \mathbf{K}) & \text{in } \Omega_I \\ \eta_I = 0 \quad \text{on } \Gamma_C \\ \boldsymbol{\epsilon}_I \nabla \eta_I \cdot \mathbf{n} = -\boldsymbol{\epsilon}_I \mathbf{K} \cdot \mathbf{n} & \text{on } \Gamma. \end{cases} \tag{14}$$

We then extend η_I by zero outside Ω_I and define

$$\eta := \begin{cases} \eta_I & \text{in } \Omega_I \\ 0 & \text{in } \Omega_C \end{cases} \in H^1(\Omega). \tag{15}$$

Remark 2 *(Dependence of η on $\mathbf{p}$).* Since $\mathbf{q} = -i\omega\mu_0\mathbf{p}$, the dependence with respect to $\mathbf{p}$ is given by

$$\mathbf{K} = -i\omega\mu_0 N\mathbf{p} =: A\mathbf{p}, \tag{16}$$

where $A = A(\mathbf{x}_0, \Phi_{\mathbf{x}_0})$ is defined as $A = -i\omega\mu_0 N$. Since linearity is preserved by extensions to zero, the mapping $\mathbb{R}^3 \ni \mathbf{p} \mapsto \eta(\mathbf{p}) \in H^1(\Omega)$ is linear; in particular, the same is true for $\mathbf{p} \mapsto (\nabla\eta)(\mathbf{p})$.

Back to problem (10), we can now split its solution as

$$\mathbf{M} = \mathbf{Q} + \nabla\eta, \tag{17}$$

where $\mathbf{Q} \in \boldsymbol{H}(\operatorname{curl};\Omega)$ has now to satisfy

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{Q}) + i\omega\boldsymbol{\sigma}\mathbf{Q} = \mathbf{J} \quad \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\epsilon}\mathbf{Q}) = 0 \quad \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{Q}) \times \mathbf{n} = -(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{K}) \times \mathbf{n} \quad \text{on } \Gamma \\ \boldsymbol{\epsilon}\mathbf{Q} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \end{cases} \tag{18}$$

$\mathbf{J}$ being still defined as in (11). Notice that the singularity at $\mathbf{x} = \mathbf{x}_0$ of the initial problem (3) is directly *read* by the fundamental solution $\mathbf{K}$ via (9), hence we are left with a boundary value problem where $\mathbf{K}$ appears as a datum, but only in subsets of the domain where it is smooth.

In order to set up a weak formulation of the eddy current problem (18), we introduce the linear space

$$\mathbf{V} := \{\mathbf{v} \in \boldsymbol{H}(\operatorname{curl};\Omega) : \operatorname{div}(\boldsymbol{\epsilon}\mathbf{v}_I) = 0 \text{ in } \Omega_I,\ \boldsymbol{\epsilon}\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}, \tag{19}$$

which turns out to be a Hilbert space if endowed with the (semi)weighted inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}} := \int_\Omega \boldsymbol{\epsilon}\mathbf{u} \cdot \overline{\mathbf{v}} + \int_\Omega \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \overline{\mathbf{v}}. \tag{20}$$

Notice that the linear space $\mathbf{V}$ turns out to be suitable thanks to the preliminary homogenization by means of $\nabla\eta$.

Multiplying equation $(18)_1$ by (the complex conjugate of) a test function $\mathbf{v} \in \mathbf{V}$, integrating in Ω and then by parts we obtain:

$$\begin{aligned} \int_\Omega \mathbf{J} \cdot \overline{\mathbf{v}} &= \int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{Q} \cdot \operatorname{curl} \overline{\mathbf{v}} - \int_\Gamma [(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{Q}) \times \mathbf{n}] \cdot \overline{\mathbf{v}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{Q} \cdot \overline{\mathbf{v}} \\ &= \int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{Q} \cdot \operatorname{curl} \overline{\mathbf{v}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{Q} \cdot \overline{\mathbf{v}} + \int_\Gamma [(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{K}) \times \mathbf{n}] \cdot \overline{\mathbf{v}}. \end{aligned} \tag{21}$$

It is important to point out that the boundary integrals shall be generally understood as duality pairings between $(\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{K}\times\mathbf{n})\in \boldsymbol{H}^{-1/2}(\mathrm{div}_\tau;\Gamma)$ and $\mathbf{n}\times\overline{\mathbf{v}}\times\mathbf{n}\in\boldsymbol{H}^{-1/2}(\mathrm{curl}_\tau;\Gamma)$. Let us rigorously see what is $\int\limits_\Omega \mathbf{J}\cdot\overline{\mathbf{v}}$ for the datum $\mathbf{J}$ defined in (11). Let $B^c_{\mathbf{x}_0} := \Omega\setminus B_r(\mathbf{x}_0)$; by the homogeneity condition (4) we can write:

$$
\begin{aligned}
&\int\limits_\Omega \mathbf{J}\cdot\overline{\mathbf{v}}\\
&\quad=\int\limits_{B^c_{\mathbf{x}_0}}[-\operatorname{curl}(\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{K})-i\omega\boldsymbol{\sigma}\mathbf{K}]\cdot\overline{\mathbf{v}}\\
&\quad=\int\limits_\Omega[-\operatorname{curl}(\boldsymbol{\mu}^{-1}-\mu_0^{-1})\operatorname{curl}\mathbf{K})\cdot\overline{\mathbf{v}}-i\omega(\boldsymbol{\sigma}-\sigma_0)\mathbf{K}\cdot\overline{\mathbf{v}}]\\
&\quad=\int\limits_\Omega[-(\boldsymbol{\mu}^{-1}-\mu_0^{-1})\operatorname{curl}\mathbf{K}\cdot\operatorname{curl}\overline{\mathbf{v}}-i\omega(\boldsymbol{\sigma}-\sigma_0)\mathbf{K}\cdot\overline{\mathbf{v}}]\\
&\qquad-\int\limits_\Gamma \mathbf{n}\times[(\boldsymbol{\mu}^{-1}-\mu_0^{-1})\operatorname{curl}\mathbf{K}]\cdot\overline{\mathbf{v}}\\
&\quad=\int\limits_\Omega[-(\boldsymbol{\mu}^{-1}-\mu_0^{-1})\operatorname{curl}\mathbf{K}\cdot\operatorname{curl}\overline{\mathbf{v}}-i\omega(\boldsymbol{\sigma}-\sigma_0)\mathbf{K}\cdot\overline{\mathbf{v}}]+\int\limits_\Gamma(\mathbf{n}\times\mu_0^{-1}\operatorname{curl}\mathbf{K})\cdot\overline{\mathbf{v}}\\
&\qquad-\int\limits_\Gamma(\mathbf{n}\times\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{K})\cdot\overline{\mathbf{v}},
\end{aligned}
$$

where with a slight abuse of notation μ_0^{-1} has been used in place of $\mu_0^{-1}\,\mathrm{Id}_{\mathbb{R}^3}$. Combining the above computation with (21) and $(18)_3$, we conclude that the weak formulation of (18) reads as follows:

Problem 1. Let $\mathbf{K}$ be defined in (6). To find $\mathbf{Q}\in\mathbf{V}$ such that

$$
\begin{aligned}
a^+[\mathbf{Q},\mathbf{v}] &:= \int\limits_\Omega \boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{Q}\cdot\operatorname{curl}\overline{\mathbf{v}}+i\omega\int\limits_{\Omega_C}\boldsymbol{\sigma}\mathbf{Q}_C\cdot\overline{\mathbf{v}}\\
&=\int\limits_{B^c_{\mathbf{x}_0}}[-(\boldsymbol{\mu}^{-1}-\mu_0^{-1})\operatorname{curl}\mathbf{K}\cdot\operatorname{curl}\overline{\mathbf{v}}-i\omega(\boldsymbol{\sigma}-\sigma_0)\mathbf{K}\cdot\overline{\mathbf{v}}]\\
&\quad+\int\limits_\Gamma(\mathbf{n}\times\mu_0^{-1}\operatorname{curl}\mathbf{K})\cdot\overline{\mathbf{v}},
\end{aligned}
\tag{22}
$$

for all $\mathbf{v}\in\mathbf{V}$.

The following Poincaré-type inequality (see Alonso Rodríguez and Valli [2, Lemma 2.1], Fernandes and Gilardi [13]) will be pivotal to prove the well-posedness of Problem 1.

Lemma 1. *There is a constant $C_0>0$ such that*

$$
\begin{aligned}
\|\mathbf{w}_I\|_{0,\Omega_I}\le C_0(&\|\operatorname{curl}\mathbf{w}_I\|_{0,\Omega_I}+\|\operatorname{div}(\boldsymbol{\epsilon_I}\mathbf{w}_I)\|_{0,\Omega_I}+\|\mathbf{w}_I\times\mathbf{n}_I\|_{-1/2,\mathrm{div}_\tau,\Gamma_C}\\
&+\|\boldsymbol{\epsilon}\mathbf{w}_I\cdot\mathbf{n}\|_{-1/2,\Gamma})
\end{aligned}
\tag{23}
$$

for all $\mathbf{w}_I \in \boldsymbol{H}(\mathrm{curl};\Omega_I) \cap \boldsymbol{H}_{\boldsymbol{\epsilon}_I}(\mathrm{div};\Omega_I)$ *with* $\mathbf{w}_I \perp^{\boldsymbol{\epsilon}_I} \mathcal{H}_{\boldsymbol{\epsilon}_I}(\Gamma_C, \Gamma; \Omega_I)$*, where*

$$\begin{aligned} \mathcal{H}_{\boldsymbol{\epsilon}_I}(\Gamma_C, \Gamma; \Omega_I) = \{\mathbf{q}_I \in \boldsymbol{L}^2(\Omega_I) : \mathrm{curl}\,\mathbf{q}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\epsilon}_I \mathbf{q}_I) = 0, \\ \mathbf{q}_I \times \mathbf{n}_I = \mathbf{0} \text{ on } \Gamma_C,\ \boldsymbol{\epsilon}_I \mathbf{q}_I \cdot \mathbf{n}_I = 0 \text{ on } \Gamma\}, \end{aligned} \tag{24}$$

and $\perp^{\boldsymbol{\epsilon}_I}$ *denotes the orthogonality with respect to the* $\boldsymbol{\epsilon}_I$*-weighted* $\boldsymbol{L}^2(\Omega_I)$ *inner product, that is* $(\boldsymbol{\epsilon}_I \cdot, \cdot)_{\boldsymbol{L}^2(\Omega_I)}$.

We shall briefly explain why the above lemma actually applies to functions in $\mathbf{V}$. Indeed first of all $\mathbf{V} \subset \boldsymbol{H}(\mathrm{curl};\Omega_I) \cap \boldsymbol{H}_{\boldsymbol{\epsilon}_I}(\mathrm{div};\Omega_I)$ due to the divergence-free constraint. Moreover, the space $\mathcal{H}_{\boldsymbol{\epsilon}_I}(\Gamma_C, \Gamma; \Omega_I)$ has dimension equal to $p_{\Gamma_C} + n_\Gamma$, where the former denotes the number of connected components of Γ_C minus one, while the latter denotes the number of Γ-independent non-bounding cycles[2] in Ω_I, and both these numbers vanish under the hypothesis that Γ_C is connected and Ω is simply connected.[3]

A consequence of the previous lemma is the following:

Corollary 1. *The sesquilinear forms*

$$\begin{aligned} a^+[\mathbf{w}, \mathbf{v}] &= \int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{w} \cdot \mathrm{curl}\,\overline{\mathbf{v}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{w} \cdot \overline{\mathbf{v}}, \\ a^-[\mathbf{w}, \mathbf{v}] &:= \int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{w} \cdot \mathrm{curl}\,\overline{\mathbf{v}} - i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{w} \cdot \overline{\mathbf{v}} \end{aligned} \tag{25}$$

are (strongly) coercive in $\mathbf{V} \times \mathbf{V}$.

Proof. For all $\mathbf{v} \in \mathbf{V}$, we have:

$$\begin{aligned} |a^+[\mathbf{v}, \mathbf{v}]|^2 &= \left(\int_\Omega \boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{v} \cdot \mathrm{curl}\,\overline{\mathbf{v}} \right)^2 + \omega^2 \left(\int_{\Omega_C} \boldsymbol{\sigma}\mathbf{v}_C \cdot \overline{\mathbf{v}}_C \right)^2 \\ &\geq \left\{ \mu_{\min}^{-2} \|\,\mathrm{curl}\,\mathbf{v}\|_{0,\Omega}^4 + \omega^2 \sigma_{\min}^{-2} \|\mathbf{v}_C\|_{0,\Omega_C}^4) \right\} \\ &\geq C(\|\,\mathrm{curl}\,\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{v}_C\|_{0,\Omega_C}^2)^2. \end{aligned}$$

By Lemma 1 together with the continuity of the tangential trace, we also have:

$$\begin{aligned} \|\mathbf{v}\|_{0,\Omega_I}^2 &\leq C_0(\|\,\mathrm{curl}\,\mathbf{v}_I\|_{0,\Omega_I} + \|\mathbf{v}_I \times \mathbf{n}_I\|_{-1/2,\mathrm{div}_\tau,\Gamma_C})^2 \\ &= C_0(\|\,\mathrm{curl}\,\mathbf{v}_I\|_{0,\Omega_I} + \|\mathbf{v}_C \times \mathbf{n}_C\|_{-1/2,\mathrm{div}_\tau,\Gamma_C})^2 \\ &\leq C_1(\|\,\mathrm{curl}\,\mathbf{v}_I\|_{0,\Omega_I}^2 + \|\mathbf{v}_C\|_{0,\Omega_C}^2 + \|\,\mathrm{curl}\,\mathbf{v}_C\|_{0,\Omega_C}^2). \end{aligned}$$

Therefore

$$\begin{aligned} |a^+[\mathbf{v}, \mathbf{v}]|^2 &\geq C_2(\|\,\mathrm{curl}\,\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{v}_C\|_{0,\Omega_C}^2 + \|\mathbf{v}_I\|_{0,\Omega_I}^2)^2 \\ &= C_2\|\mathbf{v}\|_{\mathbf{V}}^4, \end{aligned}$$

[2] More precisely, we say that a family $\mathcal{C}$ of disjoint cycles of Ω_I is formed by Γ-independent, non-bounding cycles if, for each non trivial subfamily $\mathcal{C}^* \subset \mathcal{C}$, the union of the cycles in $\mathcal{C}^*$ cannot be equal to $S \setminus \gamma$, where S denotes a surface contained in Ω_I and γ a union of cycles contained in Γ.

[3] The fact that the computational domain Ω is simply connected is sufficient to make n_Γ equal to zero. However, this may also happen when the topology of Ω is non-trivial. For a detailed discussion and examples we refer to Alonso Rodríguez and Valli [2, Section 1.4].

C, C_0, C_1, C_2 being positive real constants, which do not depend on $\mathbf{v}$. Since $a^+[\cdot,\cdot]$ and $a^-[\cdot,\cdot]$ have the same magnitude, the proof is complete. $\square$

For the sake of completeness, we briefly discuss how to proceed when Γ_C is *not* assumed to be connected[4]; in this case $p_{\Gamma_C} \geq 1$, then it is known (Alonso Rodríguez and Valli [2, Appendix A.4]) that $\mathcal{H}_{\boldsymbol{\epsilon}_I}(\Gamma_C, \Gamma; \Omega_I)$ is spanned by $\{\nabla w_i\}_{i=1\ldots p_{\Gamma_C}}$, $w_i \in H^1(\Omega_I)$ being the solution of the mixed problem:

$$\begin{cases} \operatorname{div}(\boldsymbol{\epsilon}_I \nabla w_i) = 0 & \text{in } \Omega_I \\ \boldsymbol{\epsilon}_I \nabla w_i \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ w_i = 0 & \text{on } \Gamma_C \setminus \Gamma_i \\ w_i = 1 & \text{on } \Gamma_i, \end{cases} \tag{26}$$

where $(\Gamma_i)_{i=1\ldots p_{\Gamma_C}}$ denotes the i-th connected component. Fix any $j \in \{1 \ldots p_{\Gamma_C}\}$; for each $\mathbf{v} \in \mathbf{V}$, we have:

$$\begin{aligned} \int_{\Omega_I} \boldsymbol{\epsilon}_I \mathbf{v}_I \cdot \nabla w_j &= -\int_{\Omega_I} w_j \operatorname{div}(\boldsymbol{\epsilon}_I \mathbf{v}_I) + \int_{\partial\Omega_I} w_j \boldsymbol{\epsilon}_I \mathbf{v}_I \cdot \mathbf{n} \\ &= \int_{\Gamma} w_j \boldsymbol{\epsilon}_I \mathbf{v}_I \cdot \mathbf{n} + \sum_{i=1}^{p_{\Gamma_C}} \int_{\Gamma_i} w_j \boldsymbol{\epsilon}_I \mathbf{v}_I \cdot \mathbf{n} \\ &= \int_{\Gamma_j} \boldsymbol{\epsilon}_I \mathbf{v}_I \cdot \mathbf{n}, \end{aligned}$$

since $\boldsymbol{\epsilon}_I \mathbf{v}_I \cdot \mathbf{n}$ vanishes identically on the external boundary Γ. Hence we see that it suffices to require the functions of $\mathbf{V}$ to satisfy the additional constraints

$$\int_{\Gamma_i} \boldsymbol{\epsilon}_I \mathbf{v}_I \cdot \mathbf{n} = 0 \qquad \forall i = 1 \ldots p_{\Gamma_C}$$

concerning the fluxes through each connected component of the boundary of the conductor, Γ_C.

With this adjustment, $\mathbf{V}$ is yet again a Hilbert space endowed with the $\boldsymbol{H}(\operatorname{curl};\Omega)$ inner product (20) and its elements satisfy the orthogonality hypothesis of Lemma 1, which, in turn, implies that Corollary 1 and the following lemma still hold.

Lemma 2 *(Existence for* $\mathbf{Q}$*). Problem 1 has a unique solution* $\mathbf{Q} \in \mathbf{V}$*.*

Proof. The mapping $L : \boldsymbol{H}(\operatorname{curl};\Omega) \mapsto \mathbb{C}$ defined via

$$\begin{aligned} &L(\mathbf{v}) \\ &:= \int_{B^c_{\mathbf{x}_0}} [-(\boldsymbol{\mu}^{-1} - \mu_0^{-1}) \operatorname{curl} \mathbf{K} \cdot \operatorname{curl} \mathbf{v} - i\omega(\boldsymbol{\sigma} - \sigma_0)\mathbf{K} \cdot \mathbf{v}] + \int_{\Gamma} (\mathbf{n} \times \mu_0^{-1} \operatorname{curl} \mathbf{K}) \cdot \mathbf{v} \end{aligned} \tag{27}$$

[4] Since Ω_I is assumed to be connected, this can only happen if Ω_C itself is a non-connected conductor, that is $\Omega_C = \coprod_i \Omega_C^{(i)}$ with $\Omega_C^{(i)}$ connected for each i. The presence of more conductors in a device is a situation that often arises in engineering applications.

(that is, the complex conjugate of the right hand side of (22)) is linear and continuous on $\mathbf{V}$ owing again to the continuity of the tangential trace; moreover by Corollary 1 the sesquilinear form $a[\cdot,\cdot]$ is coercive on $\mathbf{V}\times\mathbf{V}$, hence the Lax-Milgram lemma applies ensuring the existence of a unique weak solution to (22). □

Summarizing the whole discussion on the state equation, we end up with:

Theorem 3 *(Well-posedness for the state equation). Assuming that condition (4) is satisfied, there exists a solution $\mathbf{E}\in \boldsymbol{H}^{-2}(\Omega)$ to (3), which can be written as:*

$$\mathbf{E} = \mathbf{Q} + \nabla\eta + \mathbf{K}, \tag{28}$$

where $\mathbf{Q}$ is the solution of (18), η is the solution of (13) and $\mathbf{K}$ is the fundamental solution defined in (6). Moreover, it is unique among all solutions $\widehat{\mathbf{E}}$ such that $(\widehat{\mathbf{E}}-\mathbf{K})\in \boldsymbol{H}(\mathrm{curl};\Omega)$.

Proof. Uniqueness is the only assertion yet to be proved. Assume that $\widehat{\mathbf{E}}$ is another solution for which $(\widehat{\mathbf{E}}-\mathbf{K})\in \boldsymbol{H}(\mathrm{curl};\Omega)$, we can write it as $\widehat{\mathbf{E}} = \mathbf{K}+(\widehat{\mathbf{E}}-\mathbf{K})$ and it is easy to see that the addendum $\widehat{\mathbf{E}}-\mathbf{K}$ is a solution to (10), a problem for which one has uniqueness in $\boldsymbol{H}(\mathrm{curl};\Omega)$. Hence we conclude $\widehat{\mathbf{E}}-\mathbf{K} = \mathbf{E}-\mathbf{K}$ and $\mathbf{E} = \widehat{\mathbf{E}}$. □

Corollary 2 *(Linearity in $\mathbf{p}$). The solution mapping $S:\mathbb{R}^3\to \boldsymbol{H}^{-2}(\Omega)$ acting as*

$$\mathbf{p}\mapsto S\mathbf{p} := \mathbf{E}(\mathbf{p}),\ \text{ with } (\mathbf{E}(\mathbf{p})-\mathbf{K}_{\mathbf{p}})\in \boldsymbol{H}(\mathrm{curl};\Omega) \tag{29}$$

is linear (with respect to real numbers).

Proof. We see that each term on the RHS of (28) is linear in $\mathbf{p}$. Indeed Remark 2 is enough for $\mathbf{K},\nabla\eta$; for what concerns $\mathbf{Q}$, it suffices to observe that the mapping $\mathbf{V}\ni\mathbf{v}\mapsto L_{\mathbf{p}}(\mathbf{v})$ defined via (27) depends linearly on $\mathbf{p}$, that is $L_{\alpha\mathbf{p}_1+\beta\mathbf{p}_2} = \alpha L_{\mathbf{p}_1}+\beta L_{\mathbf{p}_2}$ as elements of $\mathcal{L}(\mathbf{V};\mathbb{C})$, with $\alpha,\beta\in\mathbb{R}$ and $\mathbf{p}_1,\mathbf{p}_2\in\mathbb{R}^3$. □

Remark 3 *(Other boundary conditions).* The boundary conditions $(3)_{3,4}$ are not the most commonly seen for an $\mathbf{E}$-based eddy current system. Let us briefly state what changes if the so called *electric boundary condition*

$$\mathbf{E}_I\times\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma \tag{30}$$

is considered in place of $(3)_{3,4}$. Again we look for a solution in the form $\mathbf{E} = \mathbf{K}+\mathbf{M}+\nabla\eta$ (see (9) together with (17)), exception made for the fact that now $\eta|_{\partial\Omega_I} = 0$ instead of $(14)_3$ and we are left with the following formal problem:

$$\begin{cases} \mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{Q}) + i\omega\boldsymbol{\sigma}\mathbf{Q} = \mathbf{J} \quad \text{in } \Omega \\ \mathrm{div}(\boldsymbol{\epsilon}\mathbf{Q}) = 0 \quad \text{in } \Omega_I \\ \mathbf{Q}\times\mathbf{n} = -\mathbf{K}\times\mathbf{n} =: \mathbf{G} \quad \text{on } \Gamma. \end{cases} \tag{31}$$

We set

$$\mathbf{V}_0 := \{\mathbf{u}\in \boldsymbol{H}(\mathrm{curl};\Omega) : \mathrm{div}(\boldsymbol{\epsilon}_I\mathbf{u}_I) = 0 \text{ in } \Omega_I,\ \mathbf{u}_I\times\mathbf{n} = \mathbf{0} \text{ on } \Gamma\};$$

since the bilinear form $a^+[\cdot,\cdot]$ is coercive in $\mathbf{V}_0$ (Lemma 1, and thus Corollary 1, applies to functions of $\mathbf{V}_0$ too), the resolution procedure becomes standard if we are able to find a suitable[5] lifting $\widetilde{\mathbf{G}}$ of $\mathbf{G}$, that is $\widetilde{\mathbf{G}} \in \mathbf{V}_0$ and $\widetilde{\mathbf{G}} \times \mathbf{n} = \mathbf{G}$ on Γ.

Let us consider the following curl – div system for $\widetilde{\mathbf{G}}_I \in \boldsymbol{H}(\mathrm{curl};\Omega_I)$:

$$\begin{cases} \mathrm{curl}\,\widetilde{\mathbf{G}}_I = \boldsymbol{\Psi} & \text{in } \Omega_I \\ \mathrm{div}(\boldsymbol{\epsilon}_I \widetilde{\mathbf{G}}_I) = 0 & \text{in } \Omega_I \\ \widetilde{\mathbf{G}}_I \times \mathbf{n} = \mathbf{G} & \text{on } \Gamma \\ \widetilde{\mathbf{G}}_I \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma_C \\ \displaystyle\int_\Gamma \widetilde{\mathbf{G}}_I \cdot \mathbf{n} = 0, \end{cases} \tag{32}$$

where $\boldsymbol{\Psi} = \nabla\phi$ and $\phi \in H^1(\Omega_I)$ satisfies

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega_I \\ \nabla\phi \cdot \mathbf{n} = 0 & \text{on } \Gamma_C \\ \nabla\phi \cdot \mathbf{n} = \mathrm{div}_\tau \mathbf{G} & \text{on } \Gamma \\ \displaystyle\int_{\Omega_I} \phi = 0. \end{cases} \tag{33}$$

In this way, we see that all compatibility conditions for the solvability of the curl – div system (we refer to Alonso Rodríguez et al. in [4, Chap. 1, Sec. 2.1]) are satisfied. In particular, they are also sufficient for existence and uniqueness.

Indeed the Neumann problem (33) is well-posed since $\int_\Gamma \mathrm{div}_\tau \mathbf{G} = -\int_\Gamma \mathbf{G} \cdot (\nabla_\tau 1) = 0$, while for (32) we have $\mathrm{div}\,\boldsymbol{\Psi} = \mathrm{div}\,\nabla\phi = 0$ in Ω_I and $\mathrm{div}_\tau \mathbf{G} = \nabla\phi \cdot \mathbf{n} = \boldsymbol{\Psi} \cdot \mathbf{n}$ on Γ by construction. Moreover the space of harmonic fields

$$\mathcal{H}(m;\Omega_I) := \{\boldsymbol{\rho} \in \boldsymbol{L}^2(\Omega_I) : \mathrm{curl}\,\boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega_I, \mathrm{div}(\boldsymbol{\rho}) = 0 \text{ in } \Omega_I, \boldsymbol{\rho} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_I\}$$

is trivial since Ω_I is simply connected (see Alonso Rodríguez and Valli [2, Appendix A.4]). Hence (32) has a unique solution and eventually we can define

$$\widetilde{\mathbf{G}} := \begin{cases} \widetilde{\mathbf{G}}_I & \text{in } \Omega_I \\ \mathbf{0} & \text{in } \Omega_C \end{cases} \in \mathbf{V}_0 \subset \boldsymbol{H}(\mathrm{curl};\Omega),$$

which is the desired lifting.

4. The control problem

Let us now discuss the optimal control problem; our analysis will be driven by the following task: suppose we want to approach two given desired electric field (state functions) $\mathbf{E}_d, \mathbf{H}_d \in \boldsymbol{L}^2(\Omega)$ controlling the dipole intensity $\mathbf{p} \in \mathbb{R}^3$ (its location has already been fixed in $\mathbf{x}_0$, see (3)); since the solution $\mathbf{E}$ to (3) does not

[5] Note that $\widetilde{\mathbf{G}} \in \boldsymbol{H}(\mathrm{curl};\Omega)$ would not be enough: if, say, $\mathbf{Q}_0$ solves the problem with homogeneous boundary datum $(31)_3$ and $\widetilde{\mathbf{G}} \in \boldsymbol{H}(\mathrm{curl};\Omega)$, then $\mathbf{Q} = \mathbf{Q}_0 + \widetilde{\mathbf{G}}$ does not need to satisfy the divergence-free constraint $(31)_2$, although it satisfies the boundary condition $(31)_3$.

belong to $\boldsymbol{L}^2(\Omega)$ due to the singularity at $\mathbf{x} = \mathbf{x}_0$ of the fundamental solution $\mathbf{K}$, we shall optimize the distance between the solution and the desired fields with respect to $\boldsymbol{L}^2(B_{\mathbf{x}_0^c})$, where $B^c_{\mathbf{x}_0} = \Omega \setminus \overline{B_r(\mathbf{x_0})}$ (the radius r has already been chosen prior to the homogeneity assumption (4)). In other words, although the eddy current state equation is driven by a (Dirac) dipole source concentrated at $\mathbf{x}_0$, the optimization problem disregards the behavior of the state variable around (close to) the point $\mathbf{x}_0$. This may seem to be unreasonable at first sight, however, our resolution approach guarantees a priori the presence of a singularity of the same kind of $\mathbf{K}$ at $\mathbf{x} = \mathbf{x}_0$ and therefore we precisely focus the attention on the state variable away from that point. In other words, we shall not be interested in a specific *shape* at the actuators, we aim at given fields in the complement of the actuators instead. This kind of approach is often seen in optimal control problems for PDEs where a control domain Ω_{ctr} and a disjoint state observation domain Ω_o are considered, see e.g. Clason and Kunisch [11] or Pieper and Vexler [16]. In this sense, here we are doing something similar taking $\Omega_o := B^c_{\mathbf{x}_0}$ and $\Omega_{ctr} := \{\mathbf{x}_0\}$.

Summing up, we are then led to the following regularized problem:

$$\min_{\mathbf{p}\in\mathcal{P}_{ad}} F(\mathbf{E},\mathbf{p}) := \frac{\nu_E}{2}\int\limits_{B^c_{\mathbf{x}_0}} |\mathbf{E}-\mathbf{E}_d|^2 + \frac{\nu_H}{2}\int\limits_{B^c_{\mathbf{x}_0}} |\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}-\mathbf{H}_d|^2 + \frac{\nu}{2}|\mathbf{p}|^2_{\mathbb{R}^3}, \tag{34}$$

subject to

$$\boxed{\begin{aligned} \operatorname{curl}(\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}) + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{p}\delta_{\mathbf{x}_0} \qquad & \text{in } \Omega && (35)\\ \operatorname{div}(\boldsymbol{\epsilon}_I\mathbf{E}_I) = 0 \qquad & \text{in } \Omega_I && (36)\\ (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_I)\times\mathbf{n} = \mathbf{0} \qquad & \text{on } \Gamma && (37)\\ \boldsymbol{\epsilon}_I\mathbf{E}_I\cdot\mathbf{n} = 0 \qquad & \text{on } \Gamma, && (38) \end{aligned}}$$

where

$$\mathcal{P}_{ad} := \{\mathbf{p}\in\mathbb{R}^3 : |(\mathbf{p})_i| \le p_{max},\ i = 1\ldots 3\},$$

$0 < p_{max}$ being a bound for the maximal component-wise dipole intensity.

The fact that $\mathbf{K}$ is smooth far from $\mathbf{x}_0$ together with the assumption that $\mathbf{E}_d, \mathbf{H}_d \in \boldsymbol{L}^2(\Omega)$ ensure that both $(\mathbf{E}-\mathbf{E}_d)$ and $(\mu^{-1}\operatorname{curl}\mathbf{E}-\mathbf{H}_d)$ lie in $\boldsymbol{L}^2(B^c_{\mathbf{x}_0})$, making F well-defined on $\boldsymbol{H}^{-2}(\Omega)\times\mathcal{P}_{ad}$.

Before proceeding further, we define the following reduced cost functional by composition with the control-to-state mapping (29):

$$\begin{aligned} F(\mathbf{p}) &:= \frac{\nu_E}{2}\|S\mathbf{p}-\mathbf{E}_d\|^2_{0,B^c_{\mathbf{x}_0}} + \frac{\nu_H}{2}\|\boldsymbol{\mu}^{-1}\operatorname{curl}(S\mathbf{p})-\mathbf{H}_d\|^2_{0,B^c_{\mathbf{x}_0}} + \frac{\nu}{2}|\mathbf{p}|^2_{\mathbb{R}^3}\\ &= \frac{\nu_E}{2}\|\mathbf{E}_{\mathbf{p}}-\mathbf{E}_d\|^2_{0,B^c_{\mathbf{x}_0}} + \frac{\nu_H}{2}\|\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_{\mathbf{p}}-\mathbf{H}_d\|^2_{0,B^c_{\mathbf{x}_0}} + \frac{\nu}{2}|\mathbf{p}|^2_{\mathbb{R}^3}; \end{aligned} \tag{39}$$

if $\nu > 0$, thanks to the continuity of S we obtain at once that F is weakly lower semi-continuous and strictly convex. This together with the fact that $\mathcal{P}_{ad}$ is compact entails by standard arguments (see Tröltzsch [20, Section 2.5]) the existence and uniqueness of an optimal control $\mathbf{p}^*\in\mathcal{P}_{ad}$ such that

$$F(\mathbf{p}^*) = \min_{\mathbf{p}\in\mathcal{P}_{ad}} F(\mathbf{p});$$

with this optimal control an optimal state $\mathbf{E}^* = S\mathbf{p}^* \in \boldsymbol{H}^{-2}(\Omega)$ is associated. If $\nu = 0$, we still have existence but uniqueness is no longer guaranteed.

5. Necessary and sufficient conditions for optimality

By theorem (3), we know that to each control $\mathbf{p} \in \mathcal{P}_{ad}$ there corresponds a unique state

$$\mathbf{E}_\mathbf{p} = \mathbf{Q}_\mathbf{p} + \nabla\eta_\mathbf{p} + \mathbf{K}_\mathbf{p}; \tag{40}$$

prior to deriving and discussing necessary (and sufficient) conditions for optimality, we need to further clarify Corollary 2 on the dependence of $\mathbf{E}$ on $\mathbf{p}$, in particular the one of $\mathbf{Q}, \eta$ on $\mathbf{p}$.

We shall verify that the whole RHS of (13) depends linearly (at least w.r.t real numbers) on the control $\mathbf{p}$: this will be pivotal for deriving optimality conditions with an effective notation. We then perform a similar computation for the RHS of problem (22). For (13) we have:

$$-\int\limits_{\Omega_I} \boldsymbol{\epsilon}_I \mathbf{K}_\mathbf{p} \cdot \nabla\overline{\xi} = -\int\limits_{\Omega_I} \boldsymbol{\epsilon}_I A\mathbf{p} \cdot \nabla\overline{\xi} = -\int\limits_{\Omega_I} \mathbf{p} \cdot A^T(\boldsymbol{\epsilon}_I \nabla\overline{\xi}) = \mathbf{p} \cdot \left(\int\limits_{\Omega_I} -A^T(\boldsymbol{\epsilon}_I \nabla\overline{\xi}) \right), \tag{41}$$

and we thus define

$$\widetilde{\mathcal{G}}(\xi) := \int\limits_{\Omega_I} -A^T(\boldsymbol{\epsilon}_I \nabla\overline{\xi}), \qquad \xi \in W. \tag{42}$$

Instead for (22) we obtain

$$\begin{aligned}
&\int\limits_{B^c_{\mathbf{x}_0}} [-(\boldsymbol{\mu}^{-1} - \mu_0^{-1}) \operatorname{curl} \mathbf{K}_\mathbf{p} \cdot \operatorname{curl} \overline{\mathbf{v}} - i\omega(\boldsymbol{\sigma} - \sigma_0)\mathbf{K}_\mathbf{p} \cdot \overline{\mathbf{v}}] \\
&\quad + \int\limits_{\Gamma} (\mathbf{n} \times \mu_0^{-1} \operatorname{curl} \mathbf{K}_\mathbf{p}) \cdot \overline{\mathbf{v}} \\
&\quad = \int\limits_{B^c_{\mathbf{x}_0}} [-(\boldsymbol{\mu}^{-1} - \mu_0^{-1}) \operatorname{curl}(A\mathbf{p}) \cdot \operatorname{curl} \overline{\mathbf{v}} - i\omega(\boldsymbol{\sigma} - \sigma_0)A\mathbf{p} \cdot \overline{\mathbf{v}}] \\
&\quad + \int\limits_{\Gamma} (\mathbf{n} \times \mu_0^{-1} \operatorname{curl}(A\mathbf{p})) \cdot \overline{\mathbf{v}} \\
&\quad = \int\limits_{B^c_{\mathbf{x}_0}} [-(\boldsymbol{\mu}^{-1} - \mu_0^{-1}) \sum_{j=1}^{3} \operatorname{curl} A^{(j)} p_j \cdot \operatorname{curl} \overline{\mathbf{v}} - i\omega \mathbf{p} \cdot A^T(\boldsymbol{\sigma} - \sigma_0)\overline{\mathbf{v}}] \\
&\quad + \int\limits_{\Gamma} \mathbf{n} \times \mu_0^{-1} \sum_{j=1}^{3} \operatorname{curl} A^{(j)} p_j \cdot \overline{\mathbf{v}} \\
&\quad = \sum_{j=1}^{3} p_j \left(-\int\limits_{B^c_{\mathbf{x}_0}} [(\boldsymbol{\mu}^{-1} - \mu_0^{-1}) \operatorname{curl} A^{(j)} \cdot \operatorname{curl} \overline{\mathbf{v}} - \int\limits_{B^c_{\mathbf{x}_0}} i\omega \mathbf{p} \cdot A^{(j)}[(\boldsymbol{\sigma} - \sigma_0)\overline{\mathbf{v}}] \right. \\
&\quad \left. + \int\limits_{\Gamma} [\mathbf{n} \times \mu_0^{-1} \operatorname{curl} A^{(j)}] \cdot \overline{\mathbf{v}} \right),
\end{aligned} \tag{43}$$

and we define the vector $\mathcal{G}(\mathbf{v})$ component-wise via

$$(\mathcal{G}(\mathbf{v}))_j := -\int\limits_{B^c_{\mathbf{x}_0}} [(\boldsymbol{\mu}^{-1} - \mu_0^{-1}) \operatorname{curl} A^{(j)} \cdot \operatorname{curl} \overline{\mathbf{v}} - \int\limits_{B^c_{\mathbf{x}_0}} i\omega \mathbf{p} \cdot A^{(j)} [(\boldsymbol{\sigma} - \sigma_0)\overline{\mathbf{v}}] + \int\limits_{\Gamma} [\mathbf{n} \times \mu_0^{-1} \operatorname{curl} A^{(j)}] \cdot \overline{\mathbf{v}}. \tag{44}$$

Exploiting this notation, (13), (22) now respectively read:

$$b[\eta, \xi] = \widetilde{\mathcal{G}}(\xi) \cdot \mathbf{p} \qquad \forall \xi \in W, \tag{45}$$

and

$$a^+[\mathbf{Q}, \mathbf{v}] = \mathcal{G}(\mathbf{v}) \cdot \mathbf{p} \qquad \forall \mathbf{v} \in \mathbf{V}. \tag{46}$$

The squared norm $|\cdot|^2 : \mathbb{C} \to \mathbb{R}$ is *nowhere* complex differentiable,[6] exception made for the origin; however, since our controls lie in a real vector space, this has no consequences concerning Fréchet differentiability or the existence of the Gateaux derivative. We compute the directional (Gateaux) derivatives at each point $\widehat{\mathbf{p}} \in \mathbb{R}^3$:

$$\begin{aligned} &\frac{F(\widehat{\mathbf{p}} + t\mathbf{p}) - F(\widehat{\mathbf{p}})}{t} \\ &= \nu_E t \int\limits_{B^c_{\mathbf{x}_0}} |\mathbf{E}_{\mathbf{p}}|^2 + \nu_E \operatorname{Re} \int\limits_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d) \cdot \overline{\mathbf{E}}_{\mathbf{p}} + t\nu_H \int\limits_{B^c_{\mathbf{x}_0}} |\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\mathbf{p}}|^2 \\ &+ \nu_H \operatorname{Re} \left\{ \int\limits_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \operatorname{curl} \overline{\mathbf{E}}_{\mathbf{p}} \right\} + t\nu |\mathbf{p}|^2 + \nu \widehat{\mathbf{p}} \cdot \mathbf{p}, \end{aligned}$$

therefore

$$\begin{aligned} &\lim_{t \to 0^+} \frac{F(\widehat{\mathbf{p}} + t\mathbf{p}) - F(\widehat{\mathbf{p}})}{t} \\ &= \nu_E \operatorname{Re} \int\limits_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d) \cdot \overline{\mathbf{E}}_{\mathbf{p}} + \nu_H \operatorname{Re} \left\{ \int\limits_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \operatorname{curl} \overline{\mathbf{E}}_{\mathbf{p}} \right\} \\ &+ \nu \widehat{\mathbf{p}} \cdot \mathbf{p} \end{aligned}$$

for each chosen direction $\mathbf{p}$.

Hence it follows that the directional derivative of the cost functional F in the direction $\mathbf{p} \in \mathbb{R}^3$ at an arbitrary fixed control $\widehat{\mathbf{p}}$ with associated state $\mathbf{E} = \mathbf{E}_{\widehat{\mathbf{p}}}$ is given by:

[6] Indeed if $z_0 \neq 0$,

$$\lim_{z \to z_0} \frac{|z|^2 - |z_0|^2}{z - z_0} = \lim_{z \to z_0} \frac{|z| + |z_0|}{z - z_0} (|z| - |z_0|),$$

and the latter limit vanishes if we move along the circle $\{z : |z| = |z_0|\}$ and is equal to $2\overline{z}_0$ if we move on the ray $\{rz_0 : r > 0\}$.

$$
\begin{aligned}
&F'(\widehat{\mathbf{p}})\mathbf{p} \\
&= \operatorname{Re}\left\{\nu_E \int\limits_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot \overline{\mathbf{E}}_{\mathbf{p}} + \nu_H \int\limits_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d)\cdot \boldsymbol{\mu}^{-1}\operatorname{curl}\overline{\mathbf{E}}_{\mathbf{p}}\right\} \qquad (47)\\
&+ \nu\widehat{\mathbf{p}}\cdot\mathbf{p}.
\end{aligned}
$$

Looking at the above expression, we see that the *free* control $\mathbf{p}$ (i.e., the direction) appears implicitly via the mappings $\mathbf{p} \mapsto \overline{\mathbf{E}}_{\mathbf{p}}$ and $\mathbf{p} \mapsto \boldsymbol{\mu}^{-1}\operatorname{curl}\overline{\mathbf{E}}_{\mathbf{p}}$, a situation which is usually to be avoided mainly because of how inefficient would be a numerical scheme that requires a PDE solver to act at every iteration. The introduction of an adjoint state is a standard method in optimal control theory to make such dependencies explicit; here the procedure is less straightforward, since we have to somehow take into account the *split structure* of the state variable (40).

To this end, we define *two* adjoint states: a vector one and a scalar one, which respectively correspond to $\mathbf{Q}$ and η in (40).

Definition 1 *(Adjoint state(s)).* Let $\widehat{\mathbf{p}} \in \mathbb{R}^3$ be a given control with associated state $\mathbf{E} = \mathbf{E}_{\widehat{\mathbf{p}}}$. The problem to find $(\mathbf{T}, \Psi) \in \mathbf{V} \times W$ such that:

$$
\begin{aligned}
a^-[\mathbf{T},\mathbf{v}] &= \nu_E \int\limits_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot\overline{\mathbf{v}} + \nu_H \int\limits_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d)\cdot\boldsymbol{\mu}^{-1}\operatorname{curl}\overline{\mathbf{v}} \quad \forall \mathbf{v}\in\mathbf{V},\\
b[\Psi,\xi] &= \nu_E \int\limits_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot\nabla\overline{\xi} \quad \forall \xi\in W,
\end{aligned}
\tag{48}
$$

is called *adjoint equation* of the control problem to minimize (34) subject to (35) – (38). The functional spaces $\mathbf{V}, W$ have already been defined respectively in (19) and (12), $b[\cdot,\cdot]$ is the Hermitian form appearing in the weak formulation for η and $a^-[\cdot,\cdot]$ is the conjugate transpose of the sesquilinear form $a^+[\cdot,\cdot]$ appearing in the weak formulation for $\mathbf{Q}$: see (25), (22) and (13).

Corollary 3 *(Existence of adjoint states). For all given target fields $\mathbf{E}_d, \mathbf{H}_d \in \boldsymbol{L}^2(\Omega)$, for every fixed control $\widehat{\mathbf{p}} \in \mathcal{P}_{ad}$, the adjoint system (48) has a unique solution $(\mathbf{T}_{\widehat{\mathbf{p}}}, \Psi_{\widehat{\mathbf{p}}}) =: (\widehat{\mathbf{T}}, \widehat{\Psi}) \in \mathbf{V}\times W$; $\widehat{\mathbf{T}}, \widehat{\Psi}$ are respectively called first and second adjoint state associated with $\widehat{\mathbf{p}}$.*

This result again follows from the Lax and Milgram lemma because the sesquilinear forms on the LHS are coercive in the corresponding spaces.

We fix $\widehat{\mathbf{p}} \in \mathcal{P}_{ad}$; testing the weak formulations (48) with respectively $\mathbf{Q}_{\mathbf{p}-\widehat{\mathbf{p}}} \in \mathbf{V} \hookrightarrow \boldsymbol{H}(\operatorname{curl};\Omega)$ and $\eta_{\mathbf{p}-\widehat{\mathbf{p}}} \in W \hookrightarrow H^1(\Omega)$ and summing up the two terms, we get

$$
\begin{aligned}
a^-[\widehat{\mathbf{T}}, \mathbf{Q}_{\mathbf{p}-\widehat{\mathbf{p}}}] + b[\widehat{\Psi}, \eta_{\mathbf{p}-\widehat{\mathbf{p}}}]
&= \nu_E \int\limits_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot\overline{\mathbf{Q}}_{\mathbf{p}-\widehat{\mathbf{p}}} + \nu_E \int\limits_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot\nabla\overline{\eta}_{\mathbf{p}-\widehat{\mathbf{p}}}\\
&\quad + \nu_H \int\limits_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d)\cdot\boldsymbol{\mu}^{-1}\operatorname{curl}\overline{\mathbf{Q}}_{\mathbf{p}-\widehat{\mathbf{p}}}\\
&= \nu_E \int\limits_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot[\overline{\mathbf{Q}}_{\mathbf{p}-\widehat{\mathbf{p}}} + \nabla\overline{\eta}_{\mathbf{p}-\widehat{\mathbf{p}}}]
\end{aligned}
$$

$$
\begin{aligned}
&+\nu_H \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d)\cdot\boldsymbol{\mu}^{-1}(\operatorname{curl}\overline{\mathbf{E}}_{\mathbf{p}-\widehat{\mathbf{p}}} - \operatorname{curl}\overline{A}(\mathbf{p}-\widehat{\mathbf{p}}))\\
&= \nu_E \int_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot[\overline{\mathbf{E}}_{\mathbf{p}-\widehat{\mathbf{p}}} - \overline{A}(\mathbf{p}-\widehat{\mathbf{p}})]\\
&+\nu_H \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d)\cdot\boldsymbol{\mu}^{-1}(\operatorname{curl}\overline{\mathbf{E}}_{\mathbf{p}-\widehat{\mathbf{p}}} - \operatorname{curl}\overline{A}(\mathbf{p}-\widehat{\mathbf{p}})).
\end{aligned}
\tag{49}
$$

On the other hand, the sesquilinear forms $a^+[\cdot,\cdot], a^-[\cdot,\cdot]$ satisfy

$$
\overline{a^+[\mathbf{u},\mathbf{v}]} = a^-[\mathbf{v},\mathbf{u}] \qquad \forall \mathbf{u},\mathbf{v}\in\mathbf{V}
$$

while $b[\cdot,\cdot]$ is Hermitian and therefore rearranging the terms in (49) it follows that:

$$
\begin{aligned}
&\nu_E \int_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot\overline{\mathbf{E}}_{\mathbf{p}-\widehat{\mathbf{p}}} + \nu_H \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d)\cdot\boldsymbol{\mu}^{-1}\operatorname{curl}\overline{\mathbf{E}}_{\mathbf{p}-\widehat{\mathbf{p}}}\\
&= [\overline{\mathcal{G}(\widehat{\mathbf{T}}) + \widetilde{\mathcal{G}}(\widehat{\Psi})}]\cdot(\mathbf{p}-\widehat{\mathbf{p}}) + \nu_E \int_{B^c_{\mathbf{x}_0}} [(\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot\overline{A}(\mathbf{p}-\widehat{\mathbf{p}})]\\
&+\nu_H \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d)\cdot\boldsymbol{\mu}^{-1}\operatorname{curl}\overline{A}(\mathbf{p}-\widehat{\mathbf{p}}),
\end{aligned}
\tag{50}
$$

where the definitions of $\mathcal{G}, \widetilde{\mathcal{G}}$ are respectively given in (44), (42): they correspond to the linear mappings appearing on the right hand sides in the weak formulations for $\eta, \mathbf{Q}$, see (13), (22) and (44), (42).

The above expression is not yet completely satisfying since the *free* control $\mathbf{p}$ still somehow appears implicitly in the right hand side of (50). Nevertheless, we can still make use of the adjoint states to overcome this problem. Indeed we have:

$$
\begin{aligned}
\int_{B^c_{\mathbf{x}_0}} [(\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot\overline{A}(\mathbf{p}-\widehat{\mathbf{p}})] &= \int_{B^c_{\mathbf{x}_0}} \overline{A}^T(\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot(\mathbf{p}-\widehat{\mathbf{p}})\\
&= \left(\int_{B^c_{\mathbf{x}_0}} \overline{A}^T(\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\right)\cdot(\mathbf{p}-\widehat{\mathbf{p}})\\
&= \sum_{i=1}^{3}(\mathbf{p}-\widehat{\mathbf{p}})_i \int_{B^c_{\mathbf{x}_0}} \left(\sum_{j=1}^{n}(\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)_j(\overline{A}^T)_{ij}\right)\\
&= \sum_{i=1}^{3}(\mathbf{p}-\widehat{\mathbf{p}})_i \int_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{E}_d)\cdot\overline{A}^{(i)},
\end{aligned}
\tag{51}
$$

where $\overline{A}^{(i)}$ denotes the i-th column of the matrix $\overline{A}$. Similarly, for the last term in (50) we can write[7]:

$$\begin{aligned}
\int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \operatorname{curl} \overline{A}(\mathbf{p} - \widehat{\mathbf{p}}) = \\
&= \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \sum_{i=1}^{3} (\mathbf{p} - \widehat{\mathbf{p}})_i \operatorname{curl} \overline{A}^{(i)} \\
&= \sum_{i=1}^{3} (\mathbf{p} - \widehat{\mathbf{p}})_i \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\widehat{\mathbf{p}}} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \operatorname{curl} \overline{A}^{(i)}.
\end{aligned} \tag{52}$$

The above identities can be now exploited to eventually derive necessary (and sufficient) optimality conditions. Before doing that, let us define by $\mathcal{A}^{(i)}$ a suitable extension in $B_{\mathbf{x}_0}$ of the vector function $A^{(i)}$ whose components $A^{(i)}_j$ are given by:

$$A^{(i)}_j = -i\omega\mu_0[\Phi_{\mathbf{x}_0}\delta_{ij} + D_i D_j \Phi_{\mathbf{x}_0}].$$

Here, for *suitable extension* we mean that $\mathcal{A}^{(i)} \in \boldsymbol{H}(\operatorname{curl};\Omega)$. Moreover, for each $j = 1, \ldots 3$, let $u_j \in H^1(\Omega_I)$ be the solution of the following problem:

$$\begin{cases}
\operatorname{div}(\boldsymbol{\epsilon}_I \nabla u_j) = \operatorname{div}(\boldsymbol{\epsilon}_I A^{(j)}) & \text{in } \Omega_I \\
\boldsymbol{\epsilon}_I \nabla u_j \cdot \mathbf{n} = \boldsymbol{\epsilon}_I A^{(j)} \cdot \mathbf{n} & \text{on } \Gamma \\
u_j = 0 & \text{on } \Gamma_C,
\end{cases}$$

and set

$$\widetilde{u}_j := \begin{cases} u_j & \text{in } \Omega_I \\ 0 & \text{in } \Omega_C. \end{cases}$$

Then by construction

$$\mathcal{A}^{(j)} - \nabla \widetilde{u}_j \in \mathbf{V}$$

for each $j = 1, \ldots 3$, so that $\mathcal{A}^{(j)} - \nabla \widetilde{u}_j$ is now an admissible test function for $(48)_1$.

[7] In the first equality in (52), we use the fact that:

$$\operatorname{curl}(A\mathbf{q}) = \sum_{k=1}^{3} q_k \operatorname{curl} A^{(k)},$$

where $\mathbf{q}$ is a fixed vector of $\mathbb{R}^3$ and $A^{(k)}$ denotes the k-*th* column of the matrix $A = A(\mathbf{x})$. Using the Levi-Civita symbol, the LHS can be rewritten as:

$$\operatorname{curl}(A\mathbf{q}) = \partial_i(A_{jl}q_l)\epsilon_{ijk}\mathbf{e}_k = [q_l\partial_i A_{jl} + \partial_i q_l A_{jl}]\epsilon_{ijk}\mathbf{e}_k = q_l \partial_i A_{jl}\epsilon_{ijk}\mathbf{e}_k;$$

the RHS is equal to

$$\sum_{l=1}^{3} q_l \operatorname{curl} A^{(l)} = q_l \operatorname{curl} A^{(l)} = q_l \partial_i A^{(l)}_j \epsilon_{ijk}\mathbf{e}_k,$$

on the other hand, $A^{(l)}_j$ is the j-*th* component of the column vector $A^{(l)}$, namely A_{jl}.

Theorem 4 *(First order optimality conditions). Let $\mathbf{p}^* \in \mathcal{P}_{ad} \subset \mathbb{R}^3$ be an optimal control for problem (34) and let $\mathbf{E}_{\mathbf{p}^*}$ be the corresponding optimal electric field; then there exists a unique adjoint state $(\mathbf{T}^*, \Psi^*) \in (\mathbf{V} \times W)$ which solves (48), such that the following inequality holds:*

$$\operatorname{Re}\left\{\overline{\mathcal{G}(\mathbf{T}^*) + \widetilde{\mathcal{G}}(\Psi^*)} + \mathbf{a}^-[\mathbf{T}^*, \mathcal{A}] + \mathbf{b}[\Psi^*, \widetilde{u}] + \nu\mathbf{p}^*\right\} \cdot (\mathbf{p} - \mathbf{p}^*) \geq 0 \qquad \forall \mathbf{p} \in \mathcal{P}_{ad}, \tag{53}$$

where $\mathcal{G}, \widetilde{\mathcal{G}}$ are defined in (44), (42),

$$\mathbf{a}^-[\mathbf{T}^*, \mathcal{A}] := \begin{pmatrix} a^-[\mathbf{T}^*, \mathcal{A}^{(1)} - \nabla\widetilde{u}_1] \\ a^-[\mathbf{T}^*, \mathcal{A}^{(2)} - \nabla\widetilde{u}_2] \\ a^-[\mathbf{T}^*, \mathcal{A}^{(3)} - \nabla\widetilde{u}_3] \end{pmatrix}$$

and

$$\mathbf{b}[\Psi^*, \widetilde{u}] := \begin{pmatrix} b[\Psi^*, \widetilde{u}_1] \\ b[\Psi^*, \widetilde{u}_2] \\ b[\Psi^*, \widetilde{u}_3] \end{pmatrix}.$$

Conversely, if inequality (53) holds for some $\mathbf{p}^*$ and $\nu > 0$, then $\mathbf{p}^*$ is optimal for (34).

Proof. It is well known that for an optimal control $\mathbf{p}^*$, the inequality

$$F'(\mathbf{p}^*)(\mathbf{p} - \mathbf{p}^*) \geq 0 \qquad \forall \mathbf{p} \in \mathcal{P}_{ad} \tag{54}$$

holds. The fact that if $\nu > 0$ this variational inequality is both necessary and sufficient follows from the strict convexity of the objective functional. We shall show that (54) is actually equivalent to (53). The derivative of the cost functional (47) evaluated at $\widehat{\mathbf{p}} := \mathbf{p}^*$ in the direction $\mathbf{p} := \mathbf{p} - \mathbf{p}^*$ reads:

$$\begin{aligned} F'(\mathbf{p}^*)(\mathbf{p}-\mathbf{p}^*) &= \operatorname{Re}\left\{\nu_E \int_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\mathbf{p}^*} - \mathbf{E}_d) \cdot \overline{\mathbf{E}}_{\mathbf{p}-\mathbf{p}^*}\right\} \\ &+ \operatorname{Re}\left\{\nu_H \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\mathbf{p}^*} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \operatorname{curl} \overline{\mathbf{E}}_{\mathbf{p}-\mathbf{p}^*}\right\} \\ &+ \nu\mathbf{p}^* \cdot (\mathbf{p} - \mathbf{p}^*). \end{aligned} \tag{55}$$

Owing to (50), (51) and (52), we see that the first two addenda in (55) are equal to (disregarding the real part operator in front of the whole expression):

$$\begin{aligned} &[\overline{\mathcal{G}(\mathbf{T}^*) + \widetilde{\mathcal{G}}(\Psi^*)}] \cdot (\mathbf{p} - \mathbf{p}^*) \\ &+ \sum_{i=1}^{3} (\mathbf{p} - \mathbf{p}^*)_i \left\{\nu_E \int_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\mathbf{p}^*} - \mathbf{E}_d) \cdot \overline{A}^{(i)}\right\} \\ &+ \sum_{i=1}^{3} (\mathbf{p} - \mathbf{p}^*)_i \left\{\nu_H \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\mathbf{p}^*} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \operatorname{curl} \overline{A}^{(i)}\right\}. \end{aligned} \tag{56}$$

On the other hand, for each $i \in \{1,2,3\}$ we have by $(48)_1$:

$$
\begin{aligned}
&a^-[\mathbf{T}^*, \mathcal{A}^{(i)} - \nabla\widetilde{u}_i] \\
&\quad = \nu_E \int_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\mathbf{p}^*} - \mathbf{E}_d) \cdot \overline{(\mathcal{A}^{(i)} - \nabla\widetilde{u}_i)} + \nu_H \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\mathbf{p}^*} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \operatorname{curl} \overline{\mathcal{A}}^{(i)} \\
&\quad = \nu_E \int_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\mathbf{p}^*} - \mathbf{E}_d) \cdot \overline{A}^{(i)} + \nu_H \int_{B^c_{\mathbf{x}_0}} (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}_{\mathbf{p}^*} - \mathbf{H}_d) \cdot \boldsymbol{\mu}^{-1} \operatorname{curl} \overline{A}^{(i)} \\
&\qquad \underbrace{- \nu_E \int_{B^c_{\mathbf{x}_0}} (\mathbf{E}_{\mathbf{p}^*} - \mathbf{E}_d) \cdot \nabla\overline{\widetilde{u}}_i}_{=b[\Psi^*, \widetilde{u}_i]}
\end{aligned}
$$

since $A^{(i)} = \mathcal{A}^{(i)}|_{B^c_{\mathbf{x}_0}}$ by construction. The latter computation together with (56) gives the result. $\square$

Remark 4. If $\mathbf{p}^*$ lies in the interior of $\mathcal{P}_{ad}$, then by standard argument it can be shown that the explicit formula

$$
\mathbf{p}^* = -\frac{1}{\nu} \operatorname{Re}\left\{\overline{\mathcal{G}(\mathbf{T}^*) + \widetilde{\mathcal{G}}(\Psi^*)} + \mathbf{a}^-[\mathbf{T}^*, \mathcal{A}] + \mathbf{b}[\Psi^*, \widetilde{u}]\right\}
$$

holds.

Acknowledgments

I am grateful to the PhD school in Mathematics of the University of Trento for its support and funding, and I wish to thank my advisor Alberto Valli for suggesting me this problem as well as for the countless comments and corrections.

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