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Strong solutions for the stochastic Navier-Stokes equations on the 2D rotating sphere with stable Lévy noise

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ABSTRACT

The Navier-Stokes equation with rough data arises in many problems of fluid dynamics but mathematical analysis of such problems is notoriously difficult. In this paper we consider a two-dimensional fluid moving on the surface of a rotating sphere under the influence of an impulsive force that is very irregular in time. More precisely, we assume that the impulsive force is associated to a Brownian Motion subordinated by a stable subordinator. Then we prove the existence and uniqueness of a strong solution (in PDE sense) to the stochastic Navier-Stokes equations on the rotating 2-dimensional unit sphere perturbed by a stable Lévy noise. This strong solution turns out to exist globally in time.

1. Introduction

The deterministic Navier-Stokes system (NSEs) on the rotating sphere serves as a basic model in large scale ocean dynamics. Many authors have studied the NSEs on the unit spheres. Notably, Il'in and Filatov [18,16] tackled the well-posedness of these equations and identified the Hausdorff dimension of their global attractors [17]. Temam and Wang investigated the inertial forms of NSEs on the sphere while Teman and Ziane show that the NSEs on a 2D sphere is a limit of NSE defined on a spherical cell [29]. Our paper is concerned with the following stochastic Navier-Stokes equations (SNSEs) on a 2D rotating sphere:

$$\partial_t u + \nabla_u u - \nu \mathbf{L} u + \omega \times u + \nabla p = f + \eta(x,t), \quad \mathrm{div}\, u = 0, \quad u(0) = u_0, \tag{1.1}$$

where $\mathbf{L}$ is the stress tensor, ω is the Coriolis acceleration, f is the external force and η is the noise process that can be informally described as the derivative of an H-valued Lévy process. Rigorous definitions of all relevant quantities in this equation will be given in sections 2 and 3. To the best of our knowledge, there are only three papers which discuss stochastic Navier-Stokes equations on spheres [6,7,31]. All these were concerned with the Gaussian case. In particular, the authors in [6] proved the existence and uniqueness of

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weak solutions to (1.1) with additive Gaussian noise. Moreover, they proved that the associated random dynamical system is asymptotically compact, which induces the existence of both a compact random attractor and an invariant measure in their accompanying paper [7]. The author in [31] studied the Navier-Stokes system on spheres with a Gaussian kick force and a deterministic force. The main contribution was the existence and uniqueness of a time-invariant measure.

Much effort has been made in recent years to study the Navier-Stokes equations (and other important equations of mathematical physics and fluid dynamics) perturbed by impulsive noise. The most challenging is the case of cylindrical impulsive noise that is a model for very rough noise and leads, formally at least, to equations with very interesting ergodic properties. At present, it is not known how to obtain a rigorous theory for a general cylindrical impulsive noise. For this reason, a special case of cylindrical Lévy noise defined as a subordination of a standard cylindrical Wiener process by a stable subordinator has attracted a lot of attention. Let us note: the linear equation with general cylindrical impulsive noise has been recently studied by Riedle (see for instance [19] and reference therein) but this theory is not sufficiently developed to be used in the analysis of the stochastic Navier-Stokes equations.

Our paper is the first paper to discuss SNSEs on the sphere with a stable Lévy noise. There are three new features which distinguish our paper from other work in the literature on SNSEs on spheres and SNSEs with Lévy noise. First, the domain of consideration is a sphere. Second, the noise is of a stable type which is ruled out by many existing studies on stochastic PDEs with Lévy noise. Third, we present a new well-posedness result that holds for strong solutions which are sufficiently smooth.

The aim of our paper is to prove the existence and uniqueness of a global strong solution to (1.1). In particular, we prove that given a $\mathbb{L}^4$-valued noise, H-valued forcing f and small V-valued initial data, there exists a unique global strong solution in a PDE sense for the abstract stochastic Navier-Stokes equations on the 2D unit sphere perturbed by stable Lévy noise, which depends continuously on the initial data. The time interval of existence depends on the regularity of the forcing and the assumptions imposed on noise.

The paper is organised as follows: In section 2, we review the fundamental mathematical theory of the deterministic Navier-Stokes equations (NSEs) on the sphere. We state some known results without proofs. In section 3, we define the SNSEs on spheres. We start with some analytic facts; we introduce the driving noise process, which is a stable Lévy noise via subordination. The SNSEs are then decomposed into an Ornstein-Uhlenbeck (OU) process (associated with the linear part of the SNSEs) and nonlinear PDEs. In section 4, we prove a strong classical solution (see the proof of Theorem 3.11) for smooth initial data with sufficient regular noise following the classical lines in the proof of Theorem 3.1 [5].

2. Navier-Stokes equations on a rotating 2D unit sphere

The sphere is the simplest example of a compact Riemannian manifold without boundaries, hence one may employ the well-developed tools from Riemannian geometry to study objects on such a manifold. Nevertheless, all objects of interest in this thesis are defined explicitly under the spherical coordinates. The presentation here follows closely from Goldys et al. [6] and references therein.

2.1. Preliminaries

Let $\mathbb{S}^2$ be a 2D unit sphere in $\mathbb{R}^3$; that is $\mathbb{S}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| = 1\}$. An arbitrary point x on $\mathbb{S}^2$ can be parametrized in the spherical coordinates as

$$x = \hat{x}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi.$$

The corresponding angles θ and ϕ will be denoted by $\theta(x)$ and $\phi(x)$ respectively, or simply by θ and ϕ.

Let $e_\theta = e_\theta(\theta, \phi)$ and $e_\phi = e_\phi(\theta, \phi)$ be the standard unit tangent vectors of $\mathbb{S}^2$ at point $\hat{x}(\theta, \phi) \in \mathbb{S}^2$ in the spherical coordinates, that is,

$$e_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \quad e_\phi = (-\sin\phi, \cos\phi, 0).$$

We remark that

$$e_\theta = \frac{\partial \hat{x}(\theta,\phi)}{\partial \theta}, \quad e_\phi = \frac{1}{\sin\theta}\frac{\partial \hat{x}(\theta,\phi)}{\partial \phi},$$

where the second identity holds whenever $\sin\theta \neq 0$.

Our first objective is to give a meaning to all of the terms in the deterministic Navier-Stokes equations for the velocity field $u(\hat{x},t) = (u_\theta(\hat{x},t), u_\phi(\hat{x},t))$ of a geophysical fluid flow on the 2D rotating unit sphere $\mathbb{S}^2$ under the external force $f = (f_\theta, f_\phi) = f_\theta e_\theta + f_\phi e_\phi$. The motion of the fluid is governed by the equation

$$\partial_t u + \nabla_u u - \nu \mathbf{L} u + \omega \times u + \frac{1}{\rho}\nabla p = f, \quad \text{div } u = 0, \ u(x,0) = u_0. \tag{2.1}$$

Here ν and ρ are two positive constants denoting the viscosity and the density of the fluid. The normal vector field

$$\omega = 2\Omega\cos(\theta(x))x,$$

where $x = \hat{x}(\theta(x), \phi(x))$; Ω is the angular velocity of the Earth; and θ is the parameter representing the colatitude. Note that $\theta(x) = \cos^{-1}(x_3)$. In what follows we will identify ω with the corresponding scalar function ω defined by $\omega(x) = 2\Omega\cos(\theta(x))$. We will introduce now the other terms that appear in the equation (1.1). The surface gradient for a scalar function f on $\mathbb{S}^2$ is given by

$$\nabla f = \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{\sin\theta}\frac{\partial f}{\partial \phi} e_\phi, \quad 0 \le \theta \le \pi, \ 0 \le \phi \le 2\pi.$$

Unless specified otherwise, by a vector field on $\mathbb{S}^2$ we mean a tangential vector field, that is, a section of the tangent vector bundle of $\mathbb{S}^2$.

On the other hand, for a vector field $u = (u_\theta, u_\phi)$ on $\mathbb{S}^2$, that is $u = u_\theta e_\theta + u_\phi e_\phi$, one puts

$$\text{div}\, u = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(u_\theta \sin\theta) + \frac{\partial}{\partial\phi} u_\phi\right). \tag{2.2}$$

Given two vector fields u and v on $\mathbb{S}^2$, there exist vector fields $\tilde{u}$ and $\tilde{v}$ defined in some neighbourhood of the surface $\mathbb{S}^2$ and such that their restrictions to $\mathbb{S}^2$ are equal to u and v. More precisely, see Definition 3.31 in [11],

$$\tilde{u}|_{\mathbb{S}^2} = u : \mathbb{S}^2 \to T\mathbb{S}^2, \quad \text{and} \quad \tilde{v}|_{\mathbb{S}^2} = v : \mathbb{S}^2 \to T\mathbb{S}^2.$$

For $x \in \mathbb{R}^3$, we define the orthogonal projection $\pi_x : \mathbb{R}^3 \to T_x\mathbb{S}^2$ of x onto $T_x\mathbb{S}^2$, that is

$$\pi_x : \mathbb{R}^3 \ni y \mapsto y - (x \cdot y)x = -x \times (x \times y) \in T_x\mathbb{S}^2. \tag{2.3}$$

Lemma 2.1 *([7]). Suppose $\tilde{u}$ and $\tilde{v}$ are $\mathbb{R}^3$-valued vector fields on $\mathbb{S}^2$, and u, v are tangent vector fields on $\mathbb{S}^2$, defined by $u(x) = \pi_x(\tilde{u}(x))$ and $v(x) = \pi_x(\tilde{v}(x))$, $x \in \mathbb{S}^2$. Then the following identity holds:*

$$\pi_x(\tilde{u}(x) \times \tilde{v}(x)) = u(x) \times ((x \cdot v(x))x) + ((x \cdot u(x))x \times v(x), \quad x \in \mathbb{S}^2. \tag{2.4}$$

Proof. Let us fix $x \in \mathbb{S}^2$. Then one can decompose vectors $\tilde{u}$ and $\tilde{v}$ into tangential and normal components as follows:

$$\begin{aligned}\tilde{u} &= u + u^{\perp} \quad \text{with} \quad u \in T_x\mathbb{S}^2, \quad u^{\perp} = (u \cdot x)x,\\ \tilde{v} &= v + v^{\perp} \quad \text{with} \quad v \in T_x\mathbb{S}^2, \quad v^{\perp} = (v \cdot x)x.\end{aligned}$$

Since $u \times v$ is normal to $T_x\mathbb{S}^2$, $\pi_x(u \times v) = 0$. Likewise, $u^{\perp} \times v^{\perp} = 0$ since the cross-product of two parallel vectors yields the 0 vector. Hence, it follows that

$$\pi_x(\tilde{u} \times \tilde{v}) = \pi_x(u \times v + u \times v^{\perp} + u^{\perp} \times v) = u \times v^{\perp} + u^{\perp} \times v. \quad \square \tag{2.5}$$

We will denote by $\tilde{\nabla}$ the usual gradient in $\mathbb{R}^3$ and then we have

$$(\nabla f)(x) = \pi_x(\tilde{\nabla}\tilde{f}(x)). \tag{2.6}$$

The operator curl is defined by the formula

$$(\operatorname{curl} u)(x) = (I - \pi_x)((\tilde{\nabla} \times \tilde{u})(x)) = (x \cdot (\tilde{\nabla} \times \tilde{u})(x))x. \tag{2.7}$$

Let u be a tangent vector field on $\mathbb{S}^2$. Applying formula (2.5) to the vector fields $\tilde{u}$ and $\tilde{v} = \tilde{\nabla} \times \tilde{u}$, one gets

$$\begin{aligned}\pi_x(\tilde{u} \times (\tilde{\nabla} \times \tilde{u})) &= \tilde{u} \times (\tilde{\nabla} \times (u^{\perp} + u)\\ &= u \times ((\nabla \times u)^{\perp}) + u^{\perp} \times (\nabla \times u)\\ &= u \times ((x \cdot (\tilde{\nabla} \times \tilde{u}))x)\\ &= (x \cdot (\tilde{\nabla} \times \tilde{u}))(u \times x), \quad x \in \mathbb{S}^2.\end{aligned} \tag{2.8}$$

So, we can now define the curl of the vector field u on $\mathbb{S}^2$, by,

$$\operatorname{curl}\, u := \hat{x} \cdot (\tilde{\nabla} \times \tilde{u})|_{\mathbb{S}^2}. \tag{2.9}$$

Equations (2.9) and (2.4) together yield

$$\pi_x[\tilde{u} \times (\tilde{\nabla} \times \tilde{u})](x) = [u(x) \times x]\operatorname{curl} u(x), \quad x \in \mathbb{S}^2.$$

Therefore, we have the following:

Definition 2.2. Let u be a tangent vector field on $\mathbb{S}^2$, and let the vector field ψ be normal to $\mathbb{S}^2$. We set

$$\operatorname{curl} u = (\hat{x} \cdot (\tilde{\nabla} \times \tilde{u}))|_{\mathbb{S}^2}, \quad \operatorname{Curl} \psi = (\tilde{\nabla} \times \psi)|_{\mathbb{S}^2}. \tag{2.10}$$

The first equation above indicates a projection of $\nabla \times \tilde{u}$ onto the normal direction, while the second equation means a restriction of $\nabla \times \psi$ to the tangent field on $\mathbb{S}^2$. The definitions presented above do not depend on the extensions $\tilde{u}$ and $\tilde{\psi}$. A vector field ψ normal to $\mathbb{S}^2$ will often be identified with a scalar function on $\mathbb{S}^2$ when it is convenient to do so. The following expressions describe the relationships among Curl of a scalar function ψ, Curl of a normal vector field $\mathbf{w} = w\hat{x}$, and curl of a vector field v on $\mathbb{S}^2$.

$$\operatorname{Curl} \psi = -\hat{x} \times \nabla\psi, \quad \operatorname{Curl} \mathbf{w} = -\hat{x} \times \nabla w, \quad \operatorname{curl}\, v = -\operatorname{div}(\hat{x} \times v). \tag{2.11}$$

Let

$$(\nabla_v u)(x) = \pi_x \left(\sum_{i=1}^{3} \tilde{v}_i(x) \partial_i \tilde{u}(x) \right) = \pi_x \left((\tilde{v}(x) \cdot \tilde{\nabla}) \tilde{u}(x) \right), \quad x \in \mathbb{S}^2. \tag{2.12}$$

Invoking (2.4) and the formula

$$(\tilde{u} \cdot \tilde{\nabla})\tilde{u} = \tilde{\nabla} \frac{|\tilde{u}^2|}{2} - \tilde{u} \times (\tilde{\nabla} \times \tilde{u}),$$

we find that the covariant derivative $\nabla_u u$ takes the form

$$\nabla_u u = \nabla \frac{|u^2|}{2} - \pi_x(\tilde{u} \times (\tilde{\nabla} \times \tilde{u})).$$

In particular, using (2.4) we obtain

$$\nabla_u u = \nabla \frac{|u|^2}{2} - \pi_x(\tilde{u} \times (\tilde{\nabla} \times \tilde{u})).$$

The surface diffusion operator acting on vector fields on $\mathbb{S}^2$ is denoted by $\mathbf{\Delta}$ (known as the Laplace de Rham operator) and is defined as

$$\mathbf{\Delta} v = \nabla \text{div } v - \text{Curl curl } v. \tag{2.13}$$

Using (2.11) one can derive the following relations connecting the above operators:

$$\text{div Curl } v = 0, \quad \text{curl Curl v} = -\hat{x}\Delta v, \quad \mathbf{\Delta}\text{Curl } v = \text{Curl}\Delta v. \tag{2.14}$$

Next, we recall the definition of the Ricci tensor Ric of the 2D sphere $\mathbb{S}^2$. Since

$$\text{Ric} = \begin{pmatrix} E & F \\ F & C, \end{pmatrix}$$

where the coefficients E, F, G of the first fundamental form are given by

$$\begin{aligned} E &= x_\theta \cdot x_\theta = 1; \\ F &= x_\theta \cdot x_\phi = x_\phi \cdot x_\theta = 0; \\ C &= x_\phi \cdot x_\phi = \sin^2 \theta, \end{aligned}$$

we find that

$$\text{Ric} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}. \tag{2.15}$$

Finally we define the stress tensor $\mathbf{L}$ by

$$\mathbf{L} = \mathbf{\Delta} + 2\text{Ric},$$

where $\mathbf{\Delta}$ is the Laplace-de Rham operator.

2.2. Function spaces on the sphere

In what follows we denote by dS the surface measure on $\mathbb{S}^2$. In the spherical coordinates one has locally $dS = \sin\theta\, d\theta d\phi$. For $p \in [1, \infty)$, we denote by $L^p := L^p(\mathbb{S}^2, \mathbb{R})$ of p-integrable scalar function on $\mathbb{S}^2$, endowed with the norm

$$|v|_{L^p} = \left(\int_{\mathbb{S}^2} |v(x)|^p dS(x) \right)^{1/p}, \quad v \in L^p$$

For $p = 2$, the corresponding inner product is denoted by

$$(v_1, v_2) = (v_1, v_2)_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} v_1 v_2 \, dS.$$

On the other hand, we denote by $\mathbb{L}^p = \mathbb{L}^p(\mathbb{S}^2)$ the space $L^p(\mathbb{S}^2, T\mathbb{S}^2)$ of vector fields $v : \mathbb{S}^2 \to T\mathbb{S}^2$ endowed with the norm

$$|v|_{\mathbb{L}^p} = \left(\int_{\mathbb{S}^2} |v(x)|^p dS(x) \right)^{1/p}, \quad v \in \mathbb{L}^p$$

where, for $x \in \mathbb{S}^2$, $|v(x)|$ denotes the length of $v(x)$ in the tangent space $T_x\mathbb{S}^2$. For $p = 2$, the corresponding inner product is denoted by

$$(v_1, v_2) = (v_1, v_2)_{\mathbb{L}^2} = \int_{\mathbb{S}^2} v_1 \cdot v_2 \, dS.$$

In this paper, the induced norm on $\mathbb{L}^2(\mathbb{S}^2)$ is denoted by $|\cdot|$. For other inner product spaces, say V with the inner product $(\cdot,\cdot)_V$, the associated norm is denoted by $|\cdot|_V$.

The following identities hold for appropriate real valued scalar functions and vector fields on $\mathbb{S}^2$; see (2.4)-(2.6) in [16]:

$$(\nabla\psi, v) = -(\psi, \operatorname{div} v), \tag{2.16}$$

$$(\text{Curl } \psi, v) = (\psi, \text{curl } v), \tag{2.17}$$

$$(\text{Curlcurl } w, z) = (\text{curl } w, \text{curl } z). \tag{2.18}$$

In (2.17), the $\mathbb{L}^2(\mathbb{S}^2)$ inner product is used on the left hand side, while the $L^2(\mathbb{S}^2)$ is used on the right hand side. Throughout this paper, we identify a normal vector field $\mathbf{w}$ with a scalar field w and by $\mathbf{w} = \hat{x}w$. We hence put

$$(\psi, \mathbf{w}) := (\psi, w)_{L^2(\mathbb{S}^2)}, \quad \text{if } \mathbf{w} = \hat{x}w, \quad \psi, w \in L^2(\mathbb{S}^2). \tag{2.19}$$

Let us now introduce the Sobolev spaces $H^1(\mathbb{S}^2)$ and $\mathbb{H}^1(\mathbb{S}^2)$ of scalar functions and vector fields on $\mathbb{S}^2$. Let ψ be a scalar function and let u be a vector field on $\mathbb{S}^2$, respectively. For $s \geq 0$ we define

$$|\psi|^2_{H^1(\mathbb{S}^2)} = |\psi|^2_{L^2(\mathbb{S}^2)} + |\nabla\psi|^2_{L^2(\mathbb{S}^2)}, \tag{2.20}$$

and

$$|u|^2_{\mathbb{H}^1(\mathbb{S}^2)} = |u|^2 + |\nabla \cdot u|^2 + |\mathrm{Curl}\, u|^2 \,. \tag{2.21}$$

One has the following Poincaré inequality

$$\lambda_1 |u|^2 \leq |\mathrm{div}\, u|^2 + |\mathrm{Curl}\, u|^2, \quad u \in \mathbb{H}^1(\mathbb{S}^2), \tag{2.22}$$

where $\lambda_1 > 0$ is the first positive eigenvalue of the Laplace-Hodge operator; see below. By the Hodge decomposition theorem in Riemannian geometry [10], the space of C^∞ smooth vector fields on $\mathbb{S}^2$ can be decomposed into three components:

$$C^\infty(T\mathbb{S}^2) = \mathcal{G} \oplus \mathcal{V} \oplus \mathcal{H},$$

where

$$\mathcal{G} = \{\nabla\psi \in C^\infty(\mathbb{S}^2)\}, \quad \mathcal{V} = \{\mathrm{Curl}\psi \in C^\infty(\mathbb{S}^2)\},$$

and $\mathcal{H}$ is the finite-dimensional space of harmonic vector fields. Since the sphere is simply connected, that is, the map $\mathbb{S}^2 \to \mathbb{S}^2$ is a diffeomorphism, we have $\mathcal{H} = \{0\}$. The condition of orthogonality to $\mathcal{H}$ is dropped out. We introduce the following spaces:

$$\begin{aligned} H &:= \{u \in \mathbb{L}^2(\mathbb{S}^2) : \nabla \cdot u = 0\}, \\ V &:= H \cap \mathbb{H}^1(\mathbb{S}^2). \end{aligned} \tag{2.23}$$

In other words, H is the closure of the

$$\{u \in C^\infty(T\mathbb{S}^2) : \nabla \cdot u = 0\}$$

in the $\mathbb{L}^2$ norm $|u| = (u,u)^{1/2}$, where $u = (u_\theta, u_\phi)$ and

$$(u,v) = \int_{\mathbb{S}^2} (u_\theta v_\theta + u_\phi v_\phi)\, dS(x). \tag{2.24}$$

The space V is the closure of

$$\{u \in C^\infty(T\mathbb{S}^2) : \nabla \cdot u = 0\}$$

in the norm of $\mathbb{H}^1\left(\mathbb{S}^2\right)$. Since V is densely and continuously embedded into H, and H can be identified with its dual H', one has the following Gelfand triple:

$$V \subset H \cong H' \subset V'. \tag{2.25}$$

2.3. Stokes operator

We will recall first that the Laplace-Beltrami operator on $\mathbb{S}^2$

$$\Delta f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{2.26}$$

can be defined in terms of spherical harmonics $Y_{l,m}$ as follows (see also [32]). For $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$, we define

$$Y_{l,m}(\theta,\varphi)=\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\varphi},\quad m=-l,\cdots,l,\quad l=0,1,\cdots \tag{2.27}$$

with P_l^m being the associated Legendre polynomials. The family $\{Y_{l,m}: l=0,1,\dots,\ m=-l,\dots,l\}$ form an orthonormal basis in $L^2(\mathbb{S}^2)$ and we then can define the well known Laplace-Beltrami operator on $\mathbb{S}^2$ (2.26) by putting

$$\Delta Y_{l,m}=-l(l+1)Y_{l,m}.$$

Then one can extend by linearity to all functions $f\in L^2(\mathbb{S}^2)$ such that

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}l^2(l+1)^2\left(f,Y_{l,m}\right)^2_{L^2(\mathbb{S}^2)}<\infty\,.$$

We consider the following linear Stokes problem [6]. That is, given $f\in V'$, find $v\in V$ such that

$$\nu\text{Curlcurl}u-2\nu\text{Ric}(u)+\nabla p=f,\quad \operatorname{div}u=0. \tag{2.28}$$

By taking the inner product of the first equation above with a test field $v\in V$, and then using (2.18), the pressure term drops and we obtain

$$\nu(\operatorname{curl}u,\operatorname{curl}v)-2\nu(\operatorname{Ric}u,v)=(f,v)\quad\forall\, v\in V.$$

Without loss of generality, letting $\nu=1$, we define a bilinear form $a: V\times V\to\mathbb{R}$ by

$$a(u,v)=(-\mathbb{L}u,v). \tag{2.29}$$

By performing some elementary calculations, one can write (2.29) as follows:

$$a(u,v):=(\operatorname{curl}u,\operatorname{curl}v)-2(\operatorname{Ric}u,v),\quad u,\,v\in V. \tag{2.30}$$

In view of (2.21) and formula (2.15) for the Ricci tensor on $\mathbb{S}^2$, the bilinear form a satisfies

$$a(u,v)\le|u|_{\mathbb{H}^1}|v|_{\mathbb{H}^1} \tag{2.31}$$

and so it is continuous on V. So, by the Riesz representation theorem, there exists a unique operator $\mathcal{A}: V\to V'$ where V' is the dual of V, such that $a(u,v)=(\mathcal{A}u,v)$, for $\{u,v\}\in V$. Let us recall that by the results in [28], p. 1446, we also have

$$a(u,u)=|\operatorname{Def}u|_2^2,\quad u\in V$$

where Def is the deformation tensor (see [28] for more details). Then by the Poincaré inequality (2.22) we find that $a(u,u)\ge\alpha|u|_V^2$, for a certain $\alpha>0$, which implies that a is coercive in V. Hence, by the Lax-Milgram theorem, the operator $\mathcal{A}: V\to V'$ is an isomorphism. Let A be a restriction of $\mathcal{A}$ to H:

$$\begin{cases}D(A) & :=\{u\in V:\ \mathcal{A}u\in H\},\\ Au & :=\mathcal{A}u,\quad u\in D(A).\end{cases} \tag{2.32}$$

It is well known (see for instance [27], Theorem 2.2.3) that A is positive definite, self-adjoint in H, and $D(A^{1/2})=V$ with equivalent norms. Furthermore, for some positive constants c_1,c_2 we have

$$c_1|u|_{D(A)} \leq |Au| \leq c_2|u|_{D(A)}\,,$$
$$\langle Au, u\rangle = ((u,u)) = |u|_V = |\nabla u|^2 = |Du|^2, \quad u \in D(A). \tag{2.33}$$

The spectrum of A consists of an infinite sequence of eigenvalues λ_l. Using the stream function ψ_l for which $w_l = \text{Curl}\psi_{l,m}$ and identities (2.14), one can show that each λ_l is in fact the vector of eigenvalues of the Laplace-Beltrami operator Δ, that is $\lambda_l = l(l+1)$. Additionally, there exists an orthonormal basis $(\mathbf{Z}_{l,m})_{l\geq 1}$ of H consisting of the eigenvectors of A, where

$$\mathbf{Z}_{l,m} = \lambda_l^{-1/2}\text{Curl}Y_{l,m}, \quad l = 1,\ldots,m = -l,\ldots,l. \tag{2.34}$$

Therefore, for any $v \in H$, one has

$$v = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \hat{v}_{l,m}\mathbf{Z}_{l,m}, \quad \widehat{v}_{l,m} = \int_{\mathbb{S}^2} v\cdot \mathbf{Z}_{l,m}dS = (v, \mathbf{Z}_{l,m}). \tag{2.35}$$

An equivalent definition of the operator A can be given using the so-called Leray-Helmholtz projection P that is defined as an orthogonal projection from $\mathbb{L}^2(\mathbb{S}^2)$ onto H. Let $\mathbb{H}^2(\mathbb{S}^2)$ denote the domain of the Laplace-Hodge operator in H endowed with the graph norm. It can be shown from [13] that $D(A) = \mathbb{H}^2(\mathbb{S}^2) \cap V$ and $A = -P(\mathbf{\Delta} + 2\text{Ric})$. Therefore, we obtain an equivalent definition of the so-called Stokes operator on the sphere.

Definition 2.3. The Stokes operator A on the sphere is defined as

$$A : D(A) \subset H \to H, \quad A = -P(\mathbf{\Delta} + 2\text{Ric}), \quad D(A) = \mathbb{H}^2(\mathbb{S}^2) \cap V, \tag{2.36}$$

where $\mathbf{\Delta}$ is the Laplace-De Rham operator.

It can be shown that $V = D(A^{1/2})$ when endowed with the norm $|x|_V = |A^{1/2}x|$ and the inner product $((x,y)) = \langle Ax, y\rangle$. After identification of H with its dual space we have $V \subset H \subset V'$ with continuous dense injection. The dual pairing between V and V' is denoted by $(\cdot,\cdot)_{V\times V'}$. Moreover, there exist positive constants c_1, c_2 such that

$$c_1|u|_V^2 \leq (Au, u) \leq c_2|u|_V^2, \quad u \in D(A).$$

Let us now introduce the Sobolev spaces $H^s(\mathbb{S}^2)$ and $\mathbb{H}^2(\mathbb{S}^2)$ of scalar functions and vector fields on $\mathbb{S}^2$. Let ψ be a scalar function and let u be a vector field on $\mathbb{S}^2$, respectively. For $s \geq 0$ we define

$$|\psi|^2_{H^s(\mathbb{S}^2)} = |\psi|^2_{L^2(\mathbb{S}^2)} + |(-\Delta)^{s/2}\psi|^2_{L^2(\mathbb{S}^2)}, \tag{2.37}$$

and

$$|u|^2_{\mathbb{H}^s(\mathbb{S}^2)} = |u|^2 + |(-\mathbf{\Delta})^{s/2}u|^2, \tag{2.38}$$

where Δ is the Laplace-Beltrami operator and $\mathbf{\Delta}$ is the Laplace-de Rham operator on the sphere. Note that, for $k = 0,1,2,\cdots$ and $\theta \in (0,1)$ the space $H^{k+\theta}(\mathbb{S}^2)$ can be defined as the interpolation space between $H^k(\mathbb{S}^2)$ and $H^{k+1}(\mathbb{S}^2)$. One can apply the procedure given in [7] for $H^{k+\theta}(\mathbb{S}^2)$. The fractional power $A^{s/2}$ of the Stokes operator A in H for any $s \geq 0$ is given by

$$D(A^{s/2}) = \left\{ v \in H : v = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \hat{v}_{l,m} \mathbf{Z}_{l,m}, \ \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \lambda_l^s |\hat{v}_{l,m}|^2 < \infty \right\},$$

$$A^{s/2} v := \sum_{m=1}^{\infty} \sum_{m=-l}^{l} \lambda_l^{s/2} \hat{v}_{l,m} \mathbf{Z}_{l,m} \in H.$$

The Coriolis operator $\mathbf{C}_1 : \mathbb{L}^2(\mathbb{S}^2) \to \mathbb{L}^2(\mathbb{S}^2)$ is defined by the formula[1]

$$(\mathbf{C}_1 v)(x) = 2\Omega(x \times v(x))\cos\theta, \quad x \in \mathbb{S}^2. \tag{2.39}$$

It is clear from the above definition that $\mathbf{C}_1$ is a bounded linear operator defined on $\mathbb{L}^2(\mathbb{S}^2)$. In what follows we will need the operator $\mathbf{C} = P\mathbf{C}_1$ which is well defined and bounded in H. Furthermore, for $u \in H$,

$$(\mathbf{C}u, u) = (\mathbf{C}_1 u, Pu) = \int_{\mathbb{S}^2} 2\Omega\cos\theta((x \times u) \cdot u(x))dS(x) = 0. \tag{2.40}$$

In addition,

Lemma 2.4. *For any smooth function u and $s \geq 0$*

$$(\mathbf{C}u, A^s u) = 0. \tag{2.41}$$

Proof. The case $s = 0$ is obvious as in the line above, due to the fact that $(\omega \times u) \cdot u = 0$. For $s > 0$ we refer readers to Lemma 5 in [26]. $\square$

Let $X = H \cap \mathbb{L}^4\left(\mathbb{S}^2\right)$ be endowed with the norm

$$|v|_X = |v|_H + |v|_{\mathbb{L}^4(\mathbb{S}^2)},$$

then X is a Banach space. It is known that the Stokes operator A generates an analytic C_0-semigroup $\{e^{-tA}\}_{t\geq 0}$ in X (see Theorem A.1 in [6]). Since the Coriolis operator $\mathbf{C}$ is bounded on X, we can define in X an operator

$$\hat{A} = \nu A + \mathbf{C}, \quad D(\hat{A}) = D(A),$$

with $\nu > 0$.

Lemma 2.5. *Suppose that $V \subset H \cong H' \subset V'$ is a Gelfand triple of Hilbert spaces. If a function u being $L^2(0, T; V)$ and $\partial_t u$ belongs to $L^2(0, T; V')$ in weak sense, then u is almost everywhere equal to a continuous function from $[0, T]$ to H; the real function $|u|^2$ is absolutely continuous; and, in the weak sense one has*

$$\partial_t |u(t)|^2 = 2\langle \partial_t u(t), u(t)\rangle. \tag{2.42}$$

Proposition 2.6. *The operator $\hat{A}$ with the domain $D(\hat{A}) = D(A)$ generates a strongly continuous and analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ on X. In particular, there exist $M \geq 1$ and $\mu > 0$ such that*

$$|e^{-t\hat{A}}|_{\mathcal{L}(X,X)} \leq Me^{-\mu t}, \quad t \geq 0; \tag{2.43}$$

[1] The angular velocity vector of Earth is denoted by Ω consistent with geophysical fluid dynamics literature. It should not be confused with the notation for probability space Ω used in this paper.

and for any $\delta > 0$ there exists $M_\delta \geq 1$ such that

$$|\hat{A}^\delta e^{-t\hat{A}}|_{\mathcal{L}(X,X)} \leq M_\delta t^{-\delta} e^{-\mu t}, \quad t > 0. \tag{2.44}$$

Proof. See the proof of Proposition 5.3 in [6]. □

Now consider the trilinear form b on $V \times V \times V$, defined as

$$b(v,w,z) = (\nabla_v w, z) = \int_{\mathbb{S}^2} \nabla_v w \cdot z dS = \pi_x \sum_{i,j=1}^{3} \int_{\Omega} v_j D_i w_j z_j dx, \quad v,w,z \in V. \tag{2.45}$$

Using the following identity (see [6]),

$$2\nabla_w v = -\text{curl}(w \times v) + \nabla(w \cdot v) - v \,\text{div}\, w + w \,\text{div}\, v - v \times \text{curl}\, w - w \times \text{curl}\, v,$$

and equation (2.13), one can write the divergence free fields v, w, z in the trilinear form as follows:

$$b(v,w,z) = \frac{1}{2} \int_{\mathbb{S}^2} [-v \times w \cdot \text{curl}\, z + \text{curl}\, v \times w \cdot z - v \times \text{curl}\, w \cdot z] dS. \tag{2.46}$$

Now, we know that the bilinear form $B : V \times V \to V'$ is defined by

$$(B(u,v), w) = b(u,v,w) = \sum_{i,j=1}^{3} \int_{\Omega} u_i \frac{\partial (v_k)_j}{\partial x_i} u_j dx, \quad w \in V. \tag{2.47}$$

Moreover,

$$b(v,w,w) = 0, \quad b(v,z,w) = -b(v,w,z), \quad \text{for} \quad v \in V, w, z \in \mathbb{H}^1(\mathbb{S}^2), \tag{2.48}$$

and

$$|B(u,v), w| = |b(u,v,w)| \leq c|u||w|(|\text{curl}\, v|_{\mathbb{L}^\infty(\mathbb{S}^2)} + |v|_{\mathbb{L}^\infty(\mathbb{S}^2)}), \quad u \in H, v \in V, w \in H, \tag{2.49}$$

$$|B(u,v), w| = |b(u,v,w)| \leq c|u|^{1/2}|u|_V^{1/2}|v|^{1/2}|v|_V^{1/2}|w|_V, \quad u,v,w \in V, \tag{2.50}$$

$$|B(u,v), w| = |b(u,v,w)| \leq c|u|^{1/2}|u|_V^{1/2}|v|_V^{1/2}|Au|^{1/2}|w|, \ \forall\, u \in V, v \in D(A), w \in H, \ n = 2, \tag{2.51}$$

$$|b(u,v,w)| \leq c|u|_{\mathbb{L}^4(\mathbb{S}^2)}|v|_V|w|_{\mathbb{L}^4(\mathbb{S}^2)}, \quad v \in V, u, w \in \mathbb{H}^1(\mathbb{S}^2). \tag{2.52}$$

In view of (2.50), one has

$$\sup_{z \in V, |z|_V \neq 0} \frac{|(B(u,v), z)|}{|z|_V} = |B(u,v)|_{V'} \leq c|u|^{1/2}|u|_V^{1/2}|v|^{1/2}|v|_V^{1/2}$$

which implies

$$\begin{aligned} |B(u,u)|_{V'} &\leq c|u||u|_V, \\ |B(u,u)|_H &\leq c|u||u|_V, \end{aligned} \tag{2.53}$$

and

$$\sup_{z\in H,|z|_H\neq 0}\frac{|(B(u,v),z)|}{|z|_H}=|B(u,v)|_H\le c|u|^{1/2}|u|_V^{1/2}|v|^{1/2}|v|_V^{1/2},$$

which implies

$$|B(u,u)|_H\le c|u||u|_V. \tag{2.54}$$

In view of (2.51),

$$\sup_{z\in H,|z|_H\neq 0}\frac{|(B(u,v),z)|}{|z|_H}=|B(u,v)|_H\le c|u|^{1/2}|u|_V^{1/2}|u|^{1/2}|Au|^{1/2}$$

one has

$$|B(u,u)|_H\le c|u|^{1/2}|u|_V|Au|^{1/2}\le c|u|_V^{1/2}|u|_V|Au|^{1/2}\quad \forall\ u\in D(A). \tag{2.55}$$

In view of (2.52), b is a bounded trilinear map from $\mathbb{L}^4(\mathbb{S}^2)\times V\times \mathbb{L}^4(\mathbb{S}^2)$ to $\mathbb{R}$.

Lemma 2.7. *The trilinear map b can be uniquely extended from $V\times V\times V$ to a trilinear map*

$$b:(\mathbb{L}^4(\mathbb{S}^2)\cap H)\times \mathbb{L}^4(\mathbb{S}^2)\times V\to\mathbb{R}\,.$$

Finally, we recall the interpolation inequality (see [18], p. 12),

$$|u|_{\mathbb{L}^4(\mathbb{S}^2)}\le C|u|^{1/2}_{\mathbb{L}^2(\mathbb{S}^2)}|u|_V^{1/2}\,. \tag{2.56}$$

Inequality (2.50) is deduced from the following Sobolev embedding:

$$H^{1/2}=W^{1/2,2}(\mathbb{S}^2)\hookrightarrow \mathbb{L}^4(\mathbb{S}^2).$$

Then using (2.13), (2.16), (2.32) and (2.46), we arrive at the *weak solution* of the Navier-Stokes equations (2.2), which is a vector field $u\in L^2([0,T];V)$ with $u(0)=u_0$ that satisfies the weak form of (2.2):

$$(\partial_t u,v)+b(u,u,v)+\nu(\mathrm{curl}u,\mathrm{curl}v)-2\nu(\mathrm{Ric}\ u,v)+(\mathbf{C}u,v)=(f,v),\quad v\in V, \tag{2.57}$$

where the bilinear form is defined earlier. With a slight abuse of notation, we denote $B(u)=B(u,u)$ and $B(u)=\pi(u,\nabla u)$.

3. Stochastic Navier-Stokes equations on the 2D unit sphere

By adding a Lévy white noise to (2.1), we obtain the main equation in this paper:

$$\begin{aligned}\partial_t u+\nabla_u u-\nu\mathbf{L}u+\omega\times u+\nabla p&=f+\eta(x,t),\\ \mathrm{div}\ u=0,\ u(x,0)&=u_0,\ x\in\mathbb{S}^2.\end{aligned} \tag{3.1}$$

We assume that $u_0\in H$, $f\in V'$ and $\eta(x,t)$ is Lévy white noise. This noise process can informally be described as the derivative of an H-valued Lévy process that is rigorously defined in Lemma 3.7. Applying the Leray-Helmholtz projection, we can interpret equation (3.1) as an abstract stochastic differential equation in H

$$du(t) + Au(t) + B(u(t), u(t)) + \mathbf{C}u = f dt + GdL(t), \quad u(0) = u_0, \tag{3.2}$$

where L is an H-valued stable Lévy process, and $G : H \to H$ is a bounded operator. In order to study this equation we need to consider first some properties of stochastic convolution.

3.1. Stochastic convolution of β-stable noise

In this section we will recall a linear version of equation (3.2)

$$dz(t) + Az(t) + \mathbf{C}z(t) = GdL(t), \quad z(0) = 0\,. \tag{3.3}$$

Under appropriate assumptions formulated below, its solution takes the form

$$z(t) = \int_0^t e^{-\widehat{A}(t-s)} GdL(s), \tag{3.4}$$

where $\widehat{A} = A + \mathbf{C}$. Let W be a cylindrical Wiener process on a Hilbert space K continuously embedded into H, and let X be a $\beta/2$-stable subordinator.[2] Denote the stable distribution $S_\alpha(\sigma, \beta, \mu)$ consistent with page 9 in [25], where $\alpha \in (0, 2]$, $\sigma \geq 0$, $\beta \in [-1, 1]$, $\mu \in \mathbb{R}$. Then the process $L = W(X)$ is a symmetric cylindrical β-stable process in H.

We need the Ornstein-Uhlenbeck process (3.4) to take value in X. To this end, we need the following definition.

Definition 3.1. Let K be a separable Hilbert space and let X be not necessarily Hilbert. Let γ_K be the canonical cylindrical (finitely additive) Gaussian measure on K. A bounded linear operator $U : K \to X$ is said to be γ-radonifying iff $U(K)$ is a Borel Gaussian measure on X.

Assume that $G : H \to H$ is γ-radonifying. Then the process GL is a well defined Lévy process taking values in H. Under these assumptions the process z defined by (3.4) is a well defined H-valued process and moreover, it can be considered as a solution to the following integral equation:

$$z(t) = -\int_0^t e^{-(t-s)A}\mathbf{C}z(s)\, ds + \int_0^t e^{-(t-s)A} G\, dL(s). \tag{3.5}$$

With some abuse of notation, we will denote now by λ_l, eigenvalues of the Stokes operator A, $\lambda_1 \leq \lambda_2 \leq \cdots$; and by e_l, the corresponding eigenvectors that form an orthonormal basis in H. We will impose a stronger condition on the operator G. We will suppose that there exists a bounded sequence σ_l in $\mathbb{R}$ so that

$$Ge_l = \sigma_l e_l, \quad l = 1, 2, \ldots\,.$$

We will consider the process

$$z^0(t) = \int_0^t e^{-(t-s)A} GdL(s) = \sum_{l=1}^{\infty} z_l^0(t) e_l, \tag{3.6}$$

[2] See definition on p. 50, Eg 1.3.19 in [1].

where

$$z_l^0(t) = \int_0^t e^{-\lambda_l(t-s)} \sigma_l dL^l(s). \tag{3.7}$$

Lemma 3.2. *Suppose that there exists some $\delta > 0$ such that $\sum_{l\geq 1} |\sigma_l|^\beta \lambda_l^{\beta\delta} < \infty$. Then for all $p \in (0, \beta)$,*

$$\mathbb{E}|A^\delta L(t)|^p \leq C(\beta, p) \left(\sum_{l\geq 1} |\sigma_l|^\beta \lambda_l^{\beta\delta} \right)^{\frac{p}{\beta}} t^{\frac{p}{\beta}} < \infty. \tag{3.8}$$

Proof. Let $L(t) = \sum_{l\geq 1} L^l(t) e_l$, $t \geq 0$ be the cylindrical β-stable process on H, where e_l is the complete orthonormal system of eigenfunctions on H; and $L^1, L^2, \cdots, L^l$ are i.i.d. $\mathbb{R}$-valued, symmetric β-stable processes on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Now take a bounded sequence of real numbers $\sigma = (\sigma_l)_{l\in\mathbb{N}}$. Define

$$G_\sigma : H \to H; \quad G_\sigma u := \sum_{l=1}^\infty \sigma_l \langle u, e_l \rangle e_l,$$

where σ_l are chosen such that

$$G_\sigma L(t) = \sum_{l=1}^\infty \sigma_l \langle L^l(t), e_l \rangle e_l = \sum_{l=1}^\infty \sigma_l L^l(t) e_l.$$

To show (3.8), we follow the argument in the proof of Lemma 3.1 in [34] and Theorem 4.4 in [24]. Take a Rademacher sequence $\{r_l\}_{l\geq 1}$ in a new probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, that is, $\{r_l\}_{l\geq 1}$ are i.i.d. with $\mathbb{P}\{r_l = 1\} = \mathbb{P}\{r_l = -1\} = \frac{1}{2}$. By the following Khintchine inequality: for any $p > 0$, there exists some $C(p) > 0$ such that for an arbitrary real sequence $\{h_l\}_{l\geq 1}$,

$$\left(\sum_{l\geq 1} h_l^2 \right)^{1/2} \leq C(p) \left(\mathbb{E}' | \sum_{l\geq 1} r_l h_l|^p \right)^{1/p}.$$

Using this inequality, we get

$$\begin{aligned} \mathbb{E}|A^\delta L(t)|^q &= \mathbb{E} \left(\sum_{l\geq 1} \lambda_l^{2\delta} |\sigma_l|^2 |L^l(t)|^2 \right)^{p/2} \\ &\leq C\mathbb{E}\mathbb{E}' \left| \sum_{l\geq 1} r_l \lambda_l^\delta |\sigma_l| |L^l(t)| \right|^p \\ &= C\mathbb{E}'\mathbb{E} \left| \sum_{l\geq 1} r_l \lambda_l^\delta |\sigma_l| |L^l(t)| \right|^p, \end{aligned}$$

where $C = C(p)$. For any $\lambda \in \mathbb{R}$, and $|r_k| = 1$, the formula (4.7) of [24] yields

$$\mathbb{E} \exp \left\{ i\eta \sum_{l\geq 1} r_l \eta_l^\delta |\sigma_l| L^l(t) \right\} = \exp \left\{ -|\eta|^\delta \sum_{l\geq 1} |\sigma_l|^\beta \lambda_l^{\beta\delta} t \right\}.$$

Now we know that any symmetric β-stable random variable $X \sim \tilde{S}_\alpha(\sigma, 0, 0)$ satisfies

$$\mathbb{E}e^{i\eta X} = e^{-\sigma^\beta \eta^\beta}$$

for some $\beta \in (0, 2)$, $\eta \in \mathbb{R}$. Then, for any $p \in (0, \beta)$,

$$\mathbb{E}|X|^p = C(\beta, p)\sigma^p.$$

Since $\sum_{l\geq 1} |\sigma_l|^\beta \lambda_l^{\beta\delta} < \infty$, (3.8) holds. $\square$

Lemma 3.3. *Suppose that there exists $\delta > 0$ such that*

$$\sum_{l=1}^{\infty} |\sigma_l|^\beta \lambda_l^{\beta\delta} < \infty\,,$$

then for all $p \in (0, \beta)$ and $T > 0$

$$\mathbb{E} \sup_{0\leq t\leq T} |\hat{A}^\delta z(t)|^p \leq C\left(1 + T^{p(1-\delta)}\right) T^{p/\beta}. \tag{3.9}$$

Proof. It is proved in [34] that for $p > 1$

$$\mathbb{E} \sup_{0\leq t\leq T} |A^\delta z(t)|^p \leq CT^{p/\beta}\,. \tag{3.10}$$

In order to prove the lemma for the process z, we use formula (3.5). Let $Z = z - z^0$. Then (3.5) yields

$$\frac{dZ}{dt} = -AZ - C\left(Z + z^0\right) = -\widehat{A}Z - Cz^0, \quad Z(0) = 0\,.$$

Therefore

$$Z(t) = -\int_0^T e^{-(t-s)\widehat{A}} C z^0(s)\, ds, \quad t \geq 0\,.$$

Then, by the properties of analytic semigroups we find that

$$\begin{aligned}\left|\widehat{A}^\delta Z(t)\right| &\leq \int_0^t \left|\widehat{A}^\delta e^{-(t-s)\widehat{A}}\right| \left|Cz^0(s)\right| ds \\ &\leq \sup_{s\leq t}\left|Cz^0(s)\right| \int_0^t \frac{c}{(t-s)^\delta}\, ds \\ &\leq c_1 t^{1-\delta} \sup_{s\leq t}\left|Cz^0(s)\right| \\ &\leq c_1|C|t^{1-\delta} \sup_{s\leq t}\left|z^0(s)\right|.\end{aligned}$$

Since $\mathbf{C}$ is bounded, we have $D\left(\widehat{A}\right) = D(A)$ by Theorem 2.11 in [23]. Since $A \geq 0$ is self-adjoint, the domains of fractional powers can be identified as the complex interpolation spaces, see Section 1.15.3 of

[30]. Therefore, $D\left(A^\delta\right) = D\left(\widehat{A}^\delta\right)$ for every $\gamma \in (0,1)$, which yields the existence of constants, r_1, r_2 depending on δ only, such that

$$r_1\left|\widehat{A}^\delta x\right| \le \left|A^\delta x\right| \le r_2\left|\widehat{A}^\gamma x\right|, \quad x \in D\left(A^\gamma\right).$$

Using (3.10), we find that

$$\mathbb{E}\sup_{t\le T}\left|A^\delta Z(t)\right|^p \le c_1^p r_2^p |C|^p T^{p(1-\delta)}\mathbb{E}\sup_{s\le T}\left|z^0(s)\right|^p < \infty.$$

Now the lemma follows since $z(t) = Z(t) + z^0(t)$.

Finally, for completeness we prove the case $p \in (0,1)$ for the process z^0. As (3.9) is proved for $q \in (1,\beta)$, we fix $q \in (1,\beta)$ and then

$$\mathbb{E}\left(\sup_{0\le t\le T}|A^\delta z^0(t)|^q\right) \le CT^{q/\beta}.$$

Using the Hölder inequality (see for instance [14], p. 191) one has

$$\mathbb{E}(|X|^p\cdot 1) \le (\mathbb{E}X^{pq})^{1/q}.$$

We then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T}|A^\delta z^0(t)|^p\right)\\
&= \mathbb{E}\left(\left\{\sup_{0\le t\le T}|A^\delta z^0(t)|\right\}^p\right)\\
&\le \mathbb{E}\left(\left\{\sup_{0\le t\le T}|A^\delta z^0(t)|\right\}^{pq}\right)^{1/q}\\
&\le \mathbb{E}\left(\left\{\sup_{0\le t\le T}|A^\delta z^0(t)|\right\}^{q}\right)^{p/q}\\
&\le (C_1T^{q/\beta})^{p/q}\\
&= C_1^{p/q}T^{p/\beta}\\
&\le CT^{p/\beta}. \quad \square
\end{aligned}$$

Proposition 3.4 *(p.110 of [24]). Suppose $\sum_{l\ge1}\frac{\sigma_l^\beta}{\lambda+\alpha} < \infty$, then for any $0 < p < \beta$, $t \ge 0$,*

$$E|z^0(t)|^p \le \tilde{c}_p\left(\sum_{l=1}^\infty |\sigma_l|^\beta \frac{1-e^{-\beta(\lambda_l+\alpha)t}}{\beta(\lambda_l+\alpha)}\right)^{p/\beta},$$

where c_p depends on p and β. Moreover, as $\alpha \to \infty$,

$$\mathbb{E}|z^0(t)|^p \to 0.$$

Proof. In the spirit of the proof of Lemma 3.2, we follow the argument in the proof of Theorem 4.4 in [24]. Let $z^0(t)$ be the solution of

$$dz^0(t) + (A + \alpha I)z^0(t) = GdL(t), \quad z^0(0) = 0$$

which has the expression

$$\begin{aligned} z^0(t) &= \int_0^t S(t-s)GdL(s) \\ &= \sum_{l=1}^{\infty} \left(\int_0^t e^{-(\lambda_l+\alpha)(t-s)} \sigma_l dL_s^l \right) e_l, \end{aligned}$$

where we used the notation $S(t) = e^{-t(A+\alpha I)}$. Take a Rademacher sequence $\{r_l\}_{l\geq 1}$ in a new probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, that is $\{r_l\}_{l\geq 1}$ are i.i.d. with $\mathbb{P}(r_l = 1) = \mathbb{P}(r_l = -1) = \frac{1}{2}$. By the following Khintchine inequality: for any $p > 0$, there exists some $c_p > 0$ such that for any arbitrary real sequence $\{c_l\}_{l\in\mathbb{N}}$,

$$\left(\sum_{l\geq 1} c_l^2 \right)^{1/2} \leq c_p \left(\mathbb{E}' | \sum_{l\geq 1} r_l c_l |^p \right)^{1/p},$$

where c_p depends only on p.

Now fixing $\omega \in \Omega$, $t \geq 0$, we have

$$\left(\sum_{l\geq 1} |z_l^0(t,\omega)|^2 \right)^{1/2} \leq c_p (\mathbb{E}' | \sum_{l\geq 1} r_l z_l^0(t,\omega)|^p)^{1/p}.$$

Then

$$\begin{aligned} \mathbb{E}|z^0(t)|^p &= \left(\sum_{l=1}^{\infty} | \int_0^t e^{-(\lambda_l+\alpha)(t-s)} \sigma_l dL_s^l |^2 \right)^{\frac{p}{2}} \\ &\leq c_p^p \mathbb{E} \left(\mathbb{E}' | \sum_{l=1}^{\infty} r_l z_l^0(t)|^p \right) = c_p^p \mathbb{E}' \left(\mathbb{E} | \sum_{l=1} r_l z_t^l |^p \right) = c_p^p \mathbb{E}' \left(\mathbb{E} | \sum_{l=1}^{\infty} r_l \int_0^t e^{-(\lambda_l+\alpha)(t-s)} \sigma_l dL_s^l |^p \right). \end{aligned}$$

For any $t \geq 0$, $\kappa \in \mathbb{R}$ using the fact $|r_l| = 1$, $l \geq 1$ and formula (4.7) in [24],

$$\mathbb{E} e^{i\kappa \sum_{l\geq 1} r_l z_l^0(t)} = e^{-|\kappa|^\beta} \sum_{l\geq 1} |\sigma_l|^\beta \int_0^t e^{-\beta(\lambda_l+\alpha)(t-s)} ds.$$

Now we use (3.2) in [24]: If X is a symmetric β-stable r.v. with distribution $S(\beta, \gamma, 0)$ satisfying

$$\mathbb{E} e^{i\kappa X} = e^{-\gamma^\beta |\kappa|^\beta}$$

for some $\beta \in (0, 2)$ and any $\kappa \in \mathbb{R}$, then for any $p \in (0, \beta)$, one has

$$\mathbb{E} X^p = C(\beta, p)\gamma^p.$$

Since $\sum_{l\geq 1} \frac{\sigma_l^\beta}{\lambda_l+\alpha} < \infty$, the assertion follows. Furthermore, $\mathbb{E}|z_t|_p^0 \to 0$ as $\alpha \to \infty$. $\square$

Let us now recall the definition of Skorohod space $D = D([a, b]; E)$, which consists of a function $x : [0, T] \to E$ which admits a limit $x(t-)$ from the left at each point $t \in (0, T]$ and the limit $x(t+)$ from the right at each point $t \in (0, T]$. The Skorohod space can be endowed with a metric topology such that it becomes a complete separable metric space (see for instance, Billingsley [2]).

Here we present a Lemma that allows us to claim that the solution of SNSEs has càdlàg trajectories. The proof follows closely with Lemma 3.3 in [34].

Lemma 3.5. *Assume that for a certain $\delta \in [0, 1)$*

$$\sum_{l=1}^{\infty} |\sigma_l|^\beta \lambda_l^{\beta\delta} < \infty .$$

Then the process z defined by (3.7) has a version in $D\left([0, \infty]; D\left(A^\delta\right)\right)$.

Proof. By Lemma 3.3 we have

$$\mathbb{E} \sup_{0 \le t \le T} |A^\delta z(t)|^p < \infty$$

for any $p \in (0, \beta)$. Now, by Theorem 2.2 in [21] z^0 has a càdlàg modificationin V. By representation (3.5) the process z is càdlàg as well, and the proof of the Lemma is completed. □

Let $B : H \to H$ be a self-adjoint operator with the complete orthonormal system of eigenfunctions $(e_l) \subset L^p(\mathbb{S}^2)$ and the corresponding set of eigenvalues (λ_l). It follows from Theorem 2.3 of [8] that if, further, B has a compact inverse B^{-1}, then the operator $U^{-s} : H \to L^p(\mathbb{S}^2)$ is well-defined and γ-radonifying iff

$$\int_{\mathbb{S}^2} \left(\sum_l \lambda_l^{-2s} |e_l(x)|^2 \right)^{p/2} dS(x) < \infty. \tag{3.11}$$

We now present some results about the γ-radonifying property.

Lemma 3.6. *Let $\boldsymbol{\Delta}$ denote the Laplace-de Rham operator on $\mathbb{S}^2$ and $q \in (1, \infty)$. Then the operator*

$$(-\boldsymbol{\Delta} + 1)^{-s} : H \to L^q(\mathbb{S}^2) \text{ is } \gamma - \text{radonifying iff } s > 1/2.$$

Proof. See proof of Lemma 3.1 in [7]. □

Let $X = \mathbb{L}^4(\mathbb{S}^2) \cap H$ be the Banach space endowed with the norm

$$|x|_X = |x|_H + |x|_{\mathbb{L}^4(\mathbb{S}^2)}.$$

It follows from Lemma 3.6 that the operator

$$A^{-s} : H \to X \text{ is } \gamma - \text{radonifying iff } s > 1/2. \tag{3.12}$$

One has to choose X wisely, so that $U : K \to X$ is γ-radonifying in checking validity of subordinator condition as on page 156, [9]. The following is our standard assumption.

Assumption 1. A continuously embedded Hilbert space $K \subset H \cap \mathbb{L}^4$ is such that for any $\delta \in (0, 1/2)$,

$$A^{-\delta} : K \to H \cap \mathbb{L}^4 \quad \text{is } \gamma\text{-radonifying}. \tag{3.13}$$

It follows from (3.12) that if $K = D(A^s)$ for some $s > 0$, then Assumption 1 is satisfied.

Remark. Under the above assumption, we have the fact that $K \subset H$ and Banach space X is taken as $H \cap L^4$. In fact, space $K := Q^{1/2}(W)$ is the RKHS of noise $W(t)$ on $H \cap \mathbb{L}^4$ with the inner product $\langle \cdot, \cdot \rangle_K = \langle Q^{-1/2}x, Q^{-1/2}y \rangle_W$, $x, y \in K$. The notation Q denotes the covariance of the noise W.

Note: The parameters used in Lemma 3.6 and Assumption 1 are independent. In Lemma 3.6, we start with the whole space; a smaller exponent is required to map onto $H \cap \mathbb{L}^4(\mathbb{S}^2)$, so the assumption $s > 1/2$ is justified. On the other hand, in Assumption 1, we start with a smaller space, so a bigger exponent is required to map onto $H \cap \mathbb{L}^4(\mathbb{S}^2)$, so $\delta \in (0, 1/2)$.

Corollary. *In the framework of Proposition 2.6, let us additionally assume that there exists a separable Hilbert space $K \subset X$ such that the operator $A^{-\delta} : K \to X$ is γ-radonifying for some $\delta \in (0, \frac{1}{2})$. Then*

$$\int_0^\infty |e^{-tA}|^2_{R(K,X)} dt < \infty.$$

Proof. Since $e^{-tA} = A^\delta e^{-tA} A^{-\delta}$, it follows by Neidhardt [22] that

$$|e^{-tA}|_{R(K,X)} \le |A^\delta e^{-sA}|_{\mathcal{L}(X,X)} |A^{-\delta}|_{R(K,X)},$$

and then Proposition 2.6 yields finiteness of the integral. □

Let us recall what one means by M-type p Banach space (see for instance [4]). Suppose $p \in [1, 2]$ is fixed, then the Banach space E is called type p, iff there exists a constant $K_p(E) > 0$, such that

$$\mathbb{E} \left| \sum_{i=1}^n \xi_i x_i \right|^p \le K_p(E) \sum_{i=1}^n |x_i|^p,$$

for any finite sequence of symmetric i.i.d. random variables $\xi_1, \cdots, \xi_n : \Omega \to [-1, 1], n \in \mathbb{N}$, and any finite sequence $x_1, \cdots, x_n$ from E.

Moreover, a Banach space E is of martingale type p iff there exists $L_p(E) > 0$ such that for any E-valued martingale $\{M_n\}_{n=0}^N$ the following holds:

$$\sup_{n \le N} \mathbb{E}|M_n|^p \le L_p(E) \sum_{n=0}^N \mathbb{E}|M_n - M_{n-1}|^p.$$

The following is an abstract result from [15] which will be needed for the rest of this paper.

Lemma 3.7 *([15], Corollary 8.1). Assume that: $p \in (1, 2]$; X is a subordinator Lévy process from the class Sub(p); E is a separable type p Banach space; U is a separable Hilbert space; $E \subset U$; and $W = (W(t), t \ge 0)$ is an U-valued Wiener process.*

Define a U-valued Lévy process as

$$L(t) = W(X(t)), \quad t \ge 0.$$

Then the E-valued process

$$z(t) = \int_0^t e^{-(t-s)(\mathbf{A}+\alpha I)} dL(s)$$

is well defined. Moreover, with probability 1, for all $T > 0$,

$$\int_0^T |z(t)|_E^p dt < \infty,$$

$$\int_0^T |z(t)|_{L^4}^4 dt < \infty.$$

The following existence and regularity result is a version of the result in [9].

Theorem 3.8. *Let the process L be defined in the same way as in Lemma 3.7. Assume that one of the following conditions is satisfied:*

(i) $p \in (0,1]$ *or*
(ii) *the Banach space E is separable and of martingale type p for a certain $p \in (1,2]$.*

Then the process

$$z_\alpha(t) = \int_{-\infty}^t e^{-(t-s)(\hat{A}+\alpha I)} dL(s) \tag{3.14}$$

is well defined in E for all $t > 0$. Moreover, if $p \in (1,2]$, then the process z of (3.14) is càdlàg.

Proof. As $S = (S(t), t \geq 0)$ is a C_0 semigroup in the separable martingale type p-Banach space E, there exists a Hilbert space H as the reproducing Kernel Hilbert space of $W(1)$ such that the embedding $i : H \hookrightarrow E$ is γ-radonifying. The proof of this theorem is a straightforward application of Theorem 4.1 and 4.4 in [9]. □

In order to obtain well-posedness of (3.1), one needs some regularity on the noise term. Fortunately, this becomes attainable using Lemma 3.7. In view of this, we construct the driving Lévy noise $L = L(t)$ by subordinating a cylindrical Wiener process W on a Hilbert space H as defined in (2.23). Let $\{W^l(t), t \geq 0\}$ be a sequence of independent standard one-dimensional Wiener processes on some given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The cylindrical Wiener process on H is defined by

$$W(t) := \sum_l W^l(t) e_l,$$

where e_l is the complete orthonormal system of eigenfunctions on H.

For $\beta \in (0,2)$, let $X(t)$ be an independent symmetric $\beta/2$-stable subordinator. That is, an increasing, one dimensional Lévy process with the Laplace Transform

$$\mathbb{E}e^{-rX(t)} = e^{-t|r|^{\beta/2}}, \quad r > 0.$$

The subordinated cylindrical Wiener process $\{L(t), t \geq 0\}$ on H is defined by

$$L(t) := W(X(t)), \quad t \geq 0.$$

Note in general that $L(t)$ does not belong to H. More precisely, $L(t)$ lives on some larger Hilbert space U with the γ-radonifying embedding $H \hookrightarrow U$. Now, let us consider the abstract Itô equation in (3.2) (which we restate here) in $H = L^2(\mathbb{S}^2)$:

$$du(t) + \nu Au(t)dt + B(u(t), u(t))dt + \mathbf{C}u = fdt + GdL(t), \quad u(0) = u_0. \tag{3.15}$$

Writing (3.2) into the usual mild form, one has

$$u(t) = S(t)u_0 - \int_0^t S(t-s)B(u(s))ds + \int_0^t S(t-s)fds + \int_0^t S(t-s)GdL(s), \tag{3.16}$$

where $S(t)$ is an analytic C_0 semigroup $(e^{-t\hat{A}})$ generated by $\hat{A} = \nu A + \mathbf{C}$, and A is the Stokes operator in H. Note that $\hat{A}$ is a strictly positive self-adjoint operator in H, that is $A : D(A) \subset H \to H$, $\hat{A} = \hat{A}^* > 0$, $\langle Av, v\rangle \geq \gamma|v|^2$ for any $v \in D(A)$ for some $\gamma > 0$ and $v \neq 0$. The operator $G : H \to H$ is a bounded linear operator. For a fixed $\alpha > 0$ we introduce the process

$$z_\alpha(t) := \int_0^t e^{-(t-s)(\alpha+\widehat{A})}GdL(t) \tag{3.17}$$

that solves the OU equation

$$dz_\alpha(t) + (\nu A + \mathbf{C} + \alpha)z_\alpha(t)dt = GdL(t), \quad t \geq 0\,. \tag{3.18}$$

Now let $v(t) = u(t) - z_\alpha(t)$. Then

$$\begin{cases} dv(t) + \nu A(u(t) - z_\alpha(t))dt + B(u(t))dt + \mathbf{C}(u - z_\alpha(t))dt - \alpha z_\alpha(t)dt = fdt, \\ v(0) = v_0. \end{cases}$$

The problem becomes

$$\begin{cases} dv(t) + \nu Av(t)dt + B(v(t) + z_\alpha(t))dt + \mathbf{C}v(t)dt - \alpha z_\alpha(t)dt = fdt, \\ v(0) = v_0. \end{cases}$$

Converting into the standard form,

$$\begin{cases} \frac{d^+}{dt}v(t) + (\nu A + \mathbf{C})v(t) = f - B(v(t) + z_\alpha(t)) + \alpha z_\alpha(t), \\ v(0) = v_0, \end{cases} \tag{3.19}$$

where $\frac{d^+v}{dt}$ is the right-hand derivative of $v(t)$ at t. The solution to equation (3.19) will be understood in the mild sense, that is as a solution to the integral equation

$$v(t) = S(t)v(0) + \int_0^t S(t-s)(f - B(v(s) + z_\alpha(s)) + \alpha z_\alpha(s))ds, \tag{3.20}$$

with $v_0 = u_0 - z_\alpha(0)$.

For brevity, we write z_α as z. Let us now explain what is meant by a solution of (3.2). Finally all these enter into the definition of (3.17).

Definition 3.9. Suppose that $z \in L^4_{\mathrm{loc}}([0,T); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in H$, $f \in V'$.

Let $T > 0$. A weak solution to (3.19) is a function $v \in C([0,T); H) \cap L^2_{\mathrm{loc}}([0,T); V)$ such that for any $\phi \in V$

$$\partial_t(v, \phi) = (v_0, \phi) - \nu(v, A\phi) - b(v+z, v+z, \phi) - (\mathbf{C}v, \phi) + (\alpha z + f, \phi). \tag{3.21}$$

Equivalently, (3.19) holds as an equality in V' for a.e. $t \in [0,T]$.

It is easy to check that if the assumptions of this definition hold, and v is a mild solution then it is also a weak solution.

Now if $f \in H$, and the following regularity is satisfied,

$$v \in L^\infty(0,T;V) \cap L^2(0,T;D(A)), \tag{3.22}$$

then the solution becomes strong. More precisely,

Definition 3.10 *(Strong solution).* Suppose that $z \in L^4_{\mathrm{loc}}([0,T); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in V$, $f \in H$. We say that v is a *strong solution* of the stochastic Navier-Stokes equations (3.19) on the time interval $[0,T]$ if u is a weak solution of (3.19) and in addition

$$v \in L^\infty(0,T;V) \cap L^2(0,T;D(A)). \tag{3.23}$$

Given this definition to (3.2) it is enough to prove the existence and uniqueness of equation (3.19).

Before stating the main theorem of this paper, we recall, for the reader's convenience, the standing assumptions of this work that have been made earlier.

Assumptions

- $G : H \to H$ is γ-radonifying and so GL defines a Lévy process taking values in H. Moreover, the condition (3.6) satisfies, namely,

$$Ge_l = \sigma_l e_l, \quad l = 1, 2, \cdots.$$

- Assumption 1 holds.
- $L(t)$ is a subordinated Wiener process as defined.

All three assumptions above go into (3.15) and (3.16). From that, one obtains (3.19) and (3.20) that goes into Definition 3.9.

The main theorems proved in this paper are the following.

Theorem 3.11. *Assume that $\alpha \geq 0$, $z \in L^4_{loc}([0,\infty); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $f \in H$ and $v_0 \in H$. Then, there exists a unique solution of (3.20) in the space $C(0,T;H) \cap L^2(0,T;V)$ which belongs to $C(h,T;V) \cap L^2_{loc}(h,T;D(A))$ for all $h > 0$ and $T > 0$. Moreover, if $v_0 \in V$, then $v \in C(0,T;V) \cap L^2_{loc}(0,T;D(A))$ for all $T > 0$. In particular, $v(T, z_n)u_n^0 \to v(T, z_n)u_0$ in H. Moreover, if*

$$\sum_{l=1}^{\infty} |\sigma_l|^\beta \lambda_l^{\beta/2} < \infty\,,$$

then the theorem holds.

Theorem 3.12. *Assume that $\alpha \geq 0$, $z \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$, $f \in H$ and $v_0 \in H$. Then, there exists a $\mathbb{P}$-a.s. unique solution of (3.2) in the space $D(0,T;H) \cap L^2(0,T;V)$, which belongs to $D(\epsilon,T;V) \cap L^2_{loc}(\epsilon,T;D(A))$ for all $\epsilon > 0$, and $T > 0$. Moreover, if $v_0 \in V$, then $u \in D(0,T;V) \cap L^2_{loc}(0,T;D(A))$ for all $T > 0$, $\omega \in \Omega$. Moreover, if*

$$\sum_{l=1}^{\infty} |\sigma_l|^\beta \lambda_l^{\beta/2} < \infty\,,$$

then the theorem holds.

4. Proof of Theorem 3.11: strong solutions

Suppose now $f \in H$. In what proceeds we will show that if $u_0 \in V$ then we obtain a more regular solution, and deduce that if $v_0 \in H$ then $v(t) \in V$ for every $t > 0$. In this paper, we will construct a unique global strong solution (in the sense of Definition 3.10).

The proof of Theorem 3.11 follows closely to Theorem 3.1 in [5]. However in the proof in [5] there is no Coriolis force and additive noise, whereas here there are. In particular, our constants in the proof now depend on $|F(t)|$, $|z(t)|$ and $|z(t)|_V$, but not on the Coriolis term due to the antisymmetric condition $(\mathbf{C}v, Av) = 0$.

Remark. One can alternatively prove Theorem 3.11 via the usual Galerkin approximation.

4.1. Existence and uniqueness of a strong solution with $v_0 \in V$

The following function spaces are introduced for convenience.

Definition 4.1. The spaces

$$X_T := C(0,T;H) \cap L^2(0,T;V), \tag{4.1}$$

$$Y_T = C(0,T;V) \cap L^2(0,T;D(A)) \tag{4.2}$$

are endowed with the norms

$$|\cdot|_{X_T} := |\cdot|_{C(0,T;H)} + |\cdot|_{L^2(0,T;V)},$$
$$|\cdot|_{Y_T} := |\cdot|_{C(0,T;V)} + |\cdot|_{L^2(0,T;D(A))}.$$

Or explicitly,

$$|f|^2_{X_T} = \sup_{0\leq t\leq T} |f(t)|^2 + \int_0^T |f(s)|^2_V ds,$$

$$|f|^2_{Y_T} = \sup_{0\leq t\leq T} |f(t)|^2_V + \int_0^T |Af(s)|^2 ds.$$

Let $\mathcal{K}$ be the map in Y_T defined by

$$\mathcal{K}(u)(t) = \int_0^t S(t-s)B(u(s))ds, \quad t \in [0,T],\ u \in Y_T. \tag{4.3}$$

The following is a crucial lemma for the proof of existence and uniqueness.

Lemma 4.2. *There exists $c > 0$ such that for every $u, v \in Y_T$,*

$$|\mathcal{K}(u)|^2_{Y_T} \le c|u|^2_{Y_T}\sqrt{T}, \tag{4.4}$$

$$|\mathcal{K}(u) - \mathcal{K}(v)|^2_{Y_T} \le c|u - v|^2_{Y_T}(|u|^2_{Y_T} + |v|^2_{Y_T})\sqrt{T}. \tag{4.5}$$

Proof. To prove this Lemma, we apply the arguments from [3]. Let $\mathcal{K}$ be defined as in (4.3). By the maximum regularity we have,

$$\int_0^T \left|\frac{d\mathcal{K}}{dt}\right|^2_H dt + \int_0^T |A\mathcal{K}(t)|^2_H\, dt \le C \int_0^T |B(u(t))|^2_H\, dt$$

By the trace theorem, see for instance [33], one has

$$\sup_{t\le T} |\mathcal{K}(t)|^2_V \le \int_0^T \left|\frac{d\mathcal{K}}{dt}\right|^2_H dt + \int_0^T |A\mathcal{K}(t)|^2_H\, dt$$

Then we have

$$\begin{aligned}
\sup_{0\le t\le T} |\mathcal{K}(u)(t)|_V &\le C \int_0^T |B(u(t))|^2_H\, dt \\
&\le C \int_0^T |u|^2_V |u|_V |Au| dt \quad \text{by (2.55)} \\
&\le C \sup_{0\le t\le T} |u(t)|^2_V \int_0^T |u(t)|_V |Au(t)| dt \\
&\le C \sup_{0\le t\le T} |u(t)|^3 \sqrt{T} \left(\int_0^T |Au(t)|^2\, dt\right)^{1/2}
\end{aligned}$$

Now if $x, y \ge 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, p, $q > 0$, then

$$xy \le \frac{x^p}{p} + \frac{y^q}{q} \quad \text{Young's inequality}$$

Now take

$$x = \sup_{0\le t\le T} |u(t)|^3, \quad y = \left(\int_0^T |Au(t)|^2\, dt\right)^{1/2}$$

$$p = 4/3, \quad q = 4$$

We have

$$
\begin{aligned}
\sup_{0\le t\le T} |\mathcal{K}(u)(t)|_V &\le C \int_0^T |B(u(t))|_H^2 \, dt \\
&\le C \sup_{0\le t\le T} |u(t)|^3 \sqrt{T} \left(\int_0^T |Au(t)|^2 \, dt \right)^{1/2} \\
&\le C\sqrt{T} \left\{ \frac{3}{4} \sup_{0\le t\le T} |u(t)|^4 + \frac{1}{4} \left(\int_0^T |Au(t)|^2 \right)^2 \right\} \\
&\le C\sqrt{T} \left\{ \frac{3}{4} |u|_{Y_T}^4 + \frac{1}{4} |u|_{Y_T}^4 \right\} \\
&\le C\sqrt{T} |u|_{Y_T}^4
\end{aligned}
$$

Similarly, to prove (4.5), we have

$$
H(t) := \mathcal{K}(u)(t) - \mathcal{K}(v)(t) = \int_0^t S(t-s)(B(u(s)) - B(v(s))) \, ds
$$

and therefore by the maximum regularity we have

$$
\int_0^T \left| \frac{dH}{dt} \right|_H^2 dt + \int_0^T |AH(t)|_H^2 \, dt \le C \int_0^T |B(u(t)) - B(v(t))|_H^2 \, dt
$$

By the same argument

$$
\int_0^T |A(\mathcal{K}(u)(t) - \mathcal{K}(v)(t))|_H^2 \, dt \le C \int_0^T |B(u(t)) - B(v(t))|_H^2 \, dt
$$

Finally we get

$$
|\mathcal{K}(u) - \mathcal{K}(v)|_{Y_T}^2 \le 2C \int_0^T |B(u(t)) - B(v(t))|_H^2 \, dt
$$

Now note that

$$
\begin{aligned}
|\mathcal{K}(u) - \mathcal{K}(v)|_{Y_T}^2 &\le C_1 \int_0^T |B(u) - B(v)|_H^2 \, dt \\
&\le C_1 \int_0^T |B(u(t) - v(t), u(t)) + B(v(t), u(t) - v(t))|_H^2 \, dt \\
&\le C_1 \int_0^T C_2 |u(t) - v(t)|_V^2 |u(t)|_V |Au(t)| + C_3 |u(t) - v(t)|_V^2 |v(t)|_V |Av(t)| \, dt
\end{aligned}
$$

We focus on the two terms under the integral. Using similar arguments from the proof of (4.4), we have

$$\begin{aligned}
&\int_0^T |u(t)-v(t)|_V^2 |u(t)|_V |Au(t)|\,dt \\
&\le \sup_{0\le t\le T} |u(t)-v(t)|_V^2 \int_0^T |u(t)|_V |Au(t)|\,dt \\
&\le \sup_{0\le t\le T} |u(t)-v(t)|_V^2 \sqrt{T} \left(\int_0^T |u(t)|_V^2 |Au(t)|^2\,dt \right)^{1/2} \\
&\le \sup_{0\le t\le T} |u(t)-v(t)|_V^2 \sqrt{T} \sup_{0\le t\le T} |u(t)|_V \left(\int_0^T |Au(t)|^2\,dt \right)^{1/2} \\
&\le \frac{1}{2} \sup_{0\le t\le T} |u-v|_V^2 \sqrt{T} \left\{ \sup_{0\le t\le T} |u(t)|_V^2 + \int_0^T |Au(t)|^2\,dt \right\} \\
&\le \frac{1}{2} |u-v|_{Y_T}^2 \sqrt{T} |u|_{Y_T}^2 \\
&\le \frac{1}{2} \sqrt{T} |u-v|_{Y_T}^2 |u|_{Y_T}^2
\end{aligned}$$

Similarly, we can show

$$\begin{aligned}
&\int_0^T |u(t)-v(t)|_V^2 |v(t)|_V |Av(t)|\,dt \\
&\le \sup_{0\le t\le T} |u(t)-v(t)|_V^2 \int_0^T |v(t)|_V |Av(t)|\,dt \\
&\le \sup_{0\le t\le T} |u(t)-v(t)|_V^2 \sqrt{T} \left(\int_0^T |v(t)|_V^2 |Av(t)|^2\,dt \right)^{1/2} \quad dt \\
&\dots \\
&\le \sqrt{T} |u(t)-v(t)|_{Y_T}^2 |v|_{Y_T}^2
\end{aligned}$$

Combine the above, the claim in (4.5) follows. □

Lemma 4.3. *Assume that $\alpha \ge 0$, $z \in L^4_{loc}([0,\infty); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $f \in H$ and $v_0 \in V$. Then, there exists a unique solution of (3.16) in the space $C(0,T;V) \cap L^2(0,T;D(A))$ for all $T > 0$.*

Proof. First let us prove local existence and uniqueness. Let $Y_\tau = C(0,\tau;V) \cap L^2(0,\tau;D(A))$ be equipped with the norm

$$|f|_{Y_\tau}^2 = \sup_{t\le\tau} |f(t)|^2 + \int_0^\tau |Af(s)|^2\,ds,$$

and let Γ be a nonlinear mapping in Y_τ as

$$(\Gamma v)(t) = S(t)v_0 + \int_0^t S(t-s)(f - B(v(s)+z(s)) + \alpha z(s))ds.$$

Now recall the following classical result due to Lions, see [5] or [20]:

$$\begin{cases} A1 & S(\cdot)v_0 \in Y_\tau,\ \forall\ v_0 \in H, \tau > 0; \\ A2 & \text{The map } t \mapsto x(t) = \int_0^t S(t-s)f(s)ds \text{ belongs to } Y_\tau \text{ for all } L^2(0,\tau;H); \\ A3 & \text{The mapping } f \mapsto x \text{ is continuous from } L^2(0,\tau;H) \text{ to } Y_\tau. \end{cases}$$

Note, our assumption $z(t) \in L^4([0,\infty); L^4(\mathbb{S}^2) \cap H)$ implies that $z(t) \in Y_\tau$ as $z(t)$ is square integrable and V can be continuously embedded into $L^4(\mathbb{S}^2)$.

The first step is to show Γ is well defined. Using assumptions A1 and A2 and the assumption for $z(t)$, together with Young's inequality, one can show that

$$|\Gamma|^2_{Y_\tau} \le c\,|S(t)v_0|^2_{Y_\tau} + c\left|\int_0^t S(t-s)B(v(s)+z(s))ds\right|^2_{Y_\tau} + c\left|\int_0^t S(t-s)fds\right|^2_{Y_\tau} + c\alpha\left|\int_0^t S(t-s)z(s)\right|^2_{Y_\tau},$$

for some different constant c. Now due to A_1 and A_2, the first and third terms are finite. Due to A_2 and the trilinear inequality (2.52) the second term is finite. The last term is also finite due to the assumption on $z(t)$ that

$$|\Gamma|^2_{Y_\tau} \le c_1 + c_2|v|^4_{Y_\tau} + c_3 + c_4. \tag{4.6}$$

Whence the map Γv is well defined in Y_τ, and Γ maps Y_τ into itself.

Now we have

$$\begin{aligned} &|\Gamma(v_1) - \Gamma(v_2)|^2_{Y_\tau} \\ &\le |\int_0^\tau S(t-s)(B(v_1(s)+z(s)) - B(v_2(s)+z))ds|^2_{Y_\tau} \\ &\le c_6|v_1 - v_2|^2_{Y_\tau}(|v_1+z|^2_{Y_\tau} + |v_2+z|^2_{Y_\tau})\sqrt{\tau}, \end{aligned}$$

for all v_1, v_2 and z in Y_τ. Therefore, for sufficiently small $\tau > 0$, Γ is a contraction in a closed ball of Y_τ, yielding existence and uniqueness of a local solution of (3.20) in Y_τ. That is, the solutions are bounded in V on some short time interval $[0,\tau)$.

Remark. If the following map

$$(\Gamma u)(t) = S(t)u_0 - \int_0^t S(t-s)B(u(s))ds + \int_0^t S(t-s)fds + \int_0^t S(t-s)GdL(s)$$

is used to prove contraction, then one would have to assume

$$\int_0^T |Az(t)|^2 dt < \infty.$$

The local existence and uniqueness results indicate that the solution can be extended up to the maximal lifetime $T_{f,z}$ and then is well defined on the right-open interval $[0, T_{f,z})$. Next, we will prove the local solution may be continued to the global solution which is valid for all $t > 0$, in the class of weak solutions satisfying a certain energy inequality. This is consistent with the results for the 2D NSEs that, a strong solution exists globally in time and is unique. See for instance Theorem 7.4 of Foias and Temam [12].

It suffices to find a uniform a priori estimate for the solution v in the space Y_{T_0} such that for any $T_0 \in [0, T_{f,z})$:

$$|v|^2_{Y_{T_0}} \leq C \quad \text{for all} \quad T_0 \in [0, T_{f,z}), \tag{4.7}$$

where C is independent of T_0. This uniform a priori estimate, along with the local existence-uniqueness proved earlier, yields the unique global solution u in $Y_{T,z}$. Moreover, this solution exists globally in time. Hence one can deduce that the solution is well defined up to the time $t = T_{f,z}$. At this point in time the iterated process could be repeated and the solution can be found on $[T_{f,z}, 2T_{f,z}]$ and so forth. Hence the solution could be found in $C(0, \infty; V) \cap L^2_{\text{loc}}(0, \infty; D(A))$. To prove (4.7), we first need to show

$$|v|_{X_{T_0}} \leq c_0.$$

Toward that end, we work with a modified version of (3.19)

$$\begin{cases} \partial_t v + \nu A v = -B(v) - B(v, z) - B(z, v) - \mathbf{C}v + F, \\ v(0) = v_0, \end{cases} \tag{4.8}$$

where $F = -B(z) + \alpha z + f$ is an element of H, since the H norm of all of its three terms is bounded. Now multiplying both sides by v, and integrating over $\mathbb{S}^2$, one obtains

$$\begin{aligned} \partial_t |v|^2 + \nu |v|^2_V &= -b(v, v, v) - b(v, z, v) - b(z, v, v) - (\mathbf{C}v, v) + \langle F, v \rangle \\ &= b(v, v, z) + (F, v). \end{aligned}$$

Now by (2.50), one has

$$|b(v, v, z)| \leq c|v||v|_V|z|.$$

Then applying Young's inequality with $a = \sqrt{\frac{\epsilon}{2}}|v|_V$ and $b = |v|\sqrt{\frac{2}{\epsilon}}|z|_V$, it follows that

$$\leq \frac{\epsilon |v|^2_V}{4} + \frac{1}{\epsilon}|v|^2|z|^2_V.$$

On the other hand,

$$(F(t), v(t)) = |F(t)||v(t)| \leq \frac{1}{\epsilon}|F(t)|^2 + \frac{\epsilon}{4}|v|^2.$$

So that

$$\partial_t |v(t)|^2 + (2\nu - \frac{\epsilon}{2})|v(t)|^2_V \leq \frac{2}{\epsilon}|v|^2|z|^2_V + \frac{2}{\epsilon}|F(t)|^2 + \frac{\epsilon}{2}|v|^2 \tag{4.9}$$

for all $\epsilon > 0$.

By integrating in t from 0 to T, after simplifying, one obtains

$$\int_0^T |v(t)|_V^2\, dt \leq \frac{1}{2\nu - \frac{\epsilon}{2}} \left(|v(0)|^2 + \frac{2}{\epsilon}\int_0^T |v(t)|^2 |z(t)|_V^2 dt + \frac{2}{\epsilon}\int_0^T |F(t)|^2 dt + \frac{\epsilon}{2}\int_0^T |v(t)|^2 dt \right) \leq K_1. \tag{4.10}$$

Since $v(0) = u_0$,

$$K_1 = K_1(u_0, F, \nu, T, z).$$

On the other hand, by integrating (4.9) in t from 0 to s, $0 < s < T$, we obtain

$$|v(s)|^2 \leq |u_0|^2 + \frac{2}{\epsilon}\int_0^s |v(t)|^2 |z(t)|_V^2 dt + \frac{2}{\epsilon}\int_0^s |F(t)|^2 dt + \frac{\epsilon}{2}\int_0^s |v(t)|^2 dt,$$

$$\sup_{s\in[0,T_{f,z}]} |v(s)|^2 \leq K_2,$$

$$K_2 = K_2(u_0, F, \nu, T, z) = (2\nu - \frac{\epsilon}{2})K_1.$$

Hence, for any ϵ such that $\frac{\epsilon}{2} < 2\nu$, applying the Gronwall lemma to

$$\partial_t |v|^2 \leq \left(\frac{2}{\epsilon}|z|_V^2 + \frac{\epsilon}{2}\right)|v|^2 + \frac{2}{\epsilon}|F(t)|^2,$$

one obtains

$$|v(t)|^2 \leq |v(0)|^2 \exp\left(\int_0^t \frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2} d\tau\right)|v|^2 + \int_0^t \frac{2}{\epsilon}|F(s)|^2 \exp(\int_s^t \left(\frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}\right) d\tau) ds.$$

And so

$$\sup_{t\in[0,T_{f,z}]} |v(t)|^2 \leq |v(0)|^2 \exp\left(\int_0^{T_{f,z}} \frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2} d\tau\right) + \int_0^{T_{f,z}} \frac{2}{\epsilon}|F(s)|^2 \exp(\int_s^{T_{f,z}} \left(\frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}\right) d\tau) ds.$$

To avoid clumsiness, we write momentarily $T_{f,z} = T$. Let

$$\psi_T(z) = \exp\left(\int_0^T \frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2} d\tau\right) < \infty, \quad c_F = \int_0^T \frac{2}{\epsilon}|F(s)|^2 \exp\left(\int_s^T \left(\frac{2}{\epsilon}|z(\tau)|_V^2 + \frac{\epsilon}{2}\right) d\tau\right) ds. \tag{4.11}$$

So

$$\sup_{0\leq t\leq T} |v(t)|^2 \leq |v(0)|^2 \psi_T(z) + c_F, \tag{4.12}$$

which implies

$$v \in L^\infty([0,T]; H). \tag{4.13}$$

Now integrating

$$\partial_t |v|^2 + \nu |v|_V^2 \leq \left(\frac{2}{\epsilon} |z|_V^2 + \frac{\epsilon}{2} \right) |v|^2 + \frac{2}{\epsilon} |F(t)|^2, \tag{4.14}$$

from 0 to T, one gets

$$|v(T)|^2 + \nu \int_0^T |v(t)|_V^2 dt \leq \left(\psi_T(z)|v(0)|^2 + c_F \right) \int_0^T \left(\frac{2}{\epsilon} |z(t)|^2 + \frac{\epsilon}{2} \right) dt + \frac{2}{\epsilon} \int_0^T |F(t)|^2 dt + |v(0)|^2, \tag{4.15}$$

which implies

$$v \in L^2([0,T]; V), \tag{4.16}$$

and v is indeed a weak solution. To show that $v \in C([0,T]; H)$, note that $A : V \to V'$ is bounded and $Av \in L^2([0,T]; V')$. Then $F \in L^2([0,T]; V')$ since $z \in L^4([0,T]; L^4(\mathbb{S}^2) \cap H)$ which can be continuously embedded into V', and the terms $B(z)$, $B(v,z)$, $B(z,v)$ are all in $L^2([0,T]; V')$. Combining these facts along with (4.16) and invoking Lemma 4.1 of [6], we conclude that $v \in C([0,T]; H)$.

The uniform a priori estimate (4.15) implies that the solution is well defined up to time $t = T_{f,z}$. The iterative process may be repeated starting from $t = T_{f,z}$ with the initial condition $z(t)$. The solution is uniquely extended to $[0, 2T_{f,z}]$ and so on to an arbitrarily large time.

Now, multiplying both sides of (4.8) with Av, and noting again the classical fact that $\frac{1}{2}\partial_t |v(t)|^2 = (\partial_t v(t), v(t))$ and $(Cv, Av) = 0$, integrating over $\mathbb{S}^2$, one obtains:

$$\begin{aligned} &(\partial_t v, Av) + \nu (Av, Av) = -b(v,v,Av) - b(v,z,Av) - b(z,v,Av) + \langle F(t), Av(t) \rangle \\ \Longrightarrow\ &\frac{1}{2} \frac{d^+}{dt} |v|^2 + \nu |Av|^2 = -b(v(t), v(t), Av(t)) - b(v(t), z(t), Av(t)) - b(z(t), v(t), Av(t)) + \langle F(t), Av(t) \rangle. \end{aligned} \tag{4.17}$$

Now,

$$\begin{aligned} |b(v,v,Av)| &\leq C|v|^{\frac{1}{2}} |v|_V |Av|^{\frac{3}{2}} \quad \forall\, v \in V, v \in D(A), \\ |b(v,z,Av)| &\leq C|v|^{\frac{1}{2}} |v|_V^{\frac{1}{2}} |z|_V^{\frac{1}{2}} |Av|^{\frac{3}{2}} \quad \forall\, v \in V, v \in D(A), \\ |b(z,v,Av)| &\leq C|z|^{\frac{1}{2}} |z|^{\frac{1}{2}} |v|_V^{\frac{1}{2}} |Av|^{\frac{3}{2}} \quad \forall\, z \in V, v \in D(A). \end{aligned}$$

Also,

$$(F(t), Av(t)) \leq \frac{\epsilon}{4} |Av(t)|^2 + \frac{1}{\epsilon} |F(t)|^2.$$

Furthermore, using Young's inequality with the choice $p = \frac{4}{3}$ and $ab = (\epsilon p)^{\frac{1}{p}} |Av|^{3/2} \epsilon p)^{-\frac{1}{p}} |v|^{1/2} |v|_V$, the above estimates of the three bilinear terms become:

$$\begin{aligned} |b(v,v,Av)| &\leq C|v|^{\frac{1}{2}} |v|_V |Av|^{\frac{3}{2}} \\ &\leq \epsilon |Av|^2 + C(\epsilon) |v|^2 |v|_V^4, \\ |b(v,z,Av)| &\leq C|v|^{\frac{1}{2}} |v|_V^{\frac{1}{2}} |z|_V^{\frac{1}{2}} |Av|^{\frac{3}{2}} \\ &\leq \epsilon |Av|^2 + C(\epsilon) |v|^2 |v|_V^2 |z|_V^2, \end{aligned}$$

$$
\begin{aligned}
|b(z,v,Av)| &\leq C|z|^{\frac{1}{2}}|z|_V^{\frac{1}{2}}|v|_V^{\frac{1}{2}}|Av|^{\frac{3}{2}} \\
&\leq \epsilon|Av|^2 + C(\epsilon)|z|^2|z|_V^2|v|_V^2.
\end{aligned}
$$

Therefore,

$$
\frac{d^+}{dt}|v|_V^2 + (2\nu - 3\epsilon - \frac{\epsilon}{4})|Av|^2 \leq C(\varepsilon)(|v|^2|v|_V^4 + |v|^2|v|_V^2|z|_V^2 + |z|^2|z|_V^2|v|_V^2) + \frac{1}{\epsilon}|F(t)|^2. \tag{4.18}
$$

Momentarily dropping the term $|Av(t)|^2$, we have the differential inequality

$$
\begin{aligned}
&y' \leq a + \theta y, \\
&y(t) = |v|_V^2, \quad a(t) = \frac{1}{\nu}|F(t)|^2, \theta(t) = C(\epsilon)(|v|^2|v|_V^2 + |v|^2|z|_V^2 + |z|^2|z|_V^2).
\end{aligned}
$$

Then for any ϵ such that $\epsilon < \frac{8}{13}\nu$, using the Gronwall lemma, one has

$$
\frac{d^+}{dt}\left(y(t)\exp\left\{-\int_0^t \theta(\tau)d\tau\right\}\right) \leq a(t)\exp\left\{-\int_0^t \theta(\tau)d\tau\right\} ds
$$

$$
\begin{aligned}
|v(t)|_V^2 \leq |v(0)|_V^2 \exp&\left(\int_0^t C(\epsilon)(|v(\tau)|^2|v(\tau)|_V^2 + |v(\tau)|^2|z(\tau)|_V^2 + |z(\tau)|^2|z(\tau)|_V^2)d\tau\right) \\
&+ \frac{1}{\nu}\int_0^t |F(s)|^2 \exp\left(\int_s^t C(\epsilon)(|v(\tau)|^2|v(\tau)|_V^2 + |v(\tau)|^2|z(\tau)|_V^2 + |z(\tau)|^2|z(\tau)|_V^2)d\tau\right) ds
\end{aligned}
$$

$$
\sup_{t\in[0,T]} |v(t)|_V^2 \leq K_3, \tag{4.19}
$$

$$
K_3 = K_3(u_0, F, \nu, T, z) = \left(|v(0)|_V^2 + \frac{1}{\nu}\int_0^T |F(s)|^2 ds\right)\exp(C(\epsilon)K_2K_1),
$$

which implies

$$
v \in L^\infty(0,T;V). \tag{4.20}
$$

Let us now come back to (4.18), which we integrate from 0 to T. After simplifying, we have

$$
\int_0^T |Av(t)|^2 dt \leq K_4,
$$

and

$$
\begin{aligned}
K_4 &= K_4(u_0, F, \nu, z, T) \\
&= \frac{1}{2\nu - 3\epsilon - \frac{\epsilon}{4}}(|u_0|^2 + C(\epsilon)\sup_{t\in[0,T]}|v(t)|^2|v(t)|_V^4 + C(\epsilon)\sup_{t\in[0,T]}|v(t)|^2|v(t)|_V^2|z(t)|_V^2 \\
&\quad + C(\epsilon)\sup_{t\in[0,T]}|z(t)|^2|z(t)|_V^2|v(t)|_V^2 + \frac{1}{\epsilon}\int_0^T |F(t)|^2)dt.
\end{aligned}
$$

As

$$\sup_{t\in[0,T]} |v(t)|^2 \le K_2,$$
$$\sup_{t\in[0,T]} |v(t)|_V^4 \le K_3^2,$$
$$|z(t)|_V^2 \le C_1,$$
$$\sup_{t\in[0,T]} |z(t)|^2 \le C_2.$$

So,

$$K_4 = \frac{1}{2\nu - 3\epsilon - \frac{\epsilon}{4}}(|u_0|^2 + C(\epsilon)K_2K_3^2 + C(\epsilon)K_2K_3C_1 + C(\epsilon)C_2C_1K_3 + \frac{1}{\epsilon}\int_0^T |F(t)|^2)dt.$$

This implies

$$v \in L^2(0, T_{f,z}; D(A)). \tag{4.21}$$

It remains to show that $v \in C([0,T];V)$. Note, the fact that the solution with $v_0 \in V$ is in $L^2([0,T];V)$ implies that a.e. on $[0,T]$, $v(t) \in V$. Moreover, since $v(t) \in C([0,T];H)$ as previously deduced, and is unique as proved in step 1, it follows that $u \in C([0,T];V)$.

Together with the uniform a priori estimate, the local existence-uniqueness shown in step 1, allows us to conclude that there exists a unique $u \in C(0,\infty;H) \cap L^2(0,\infty;V) \subset C(0,\infty;V) \cap L^2(0,\infty;D(A))$, for any given $u_0 \in V$, $f \in H$, $z(t) \in L^4_{\text{loc}}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$. Moreover, our promising a priori bound (4.19) yields $T = \infty$. □

4.2. Existence and uniqueness of a strong solution with $v_0 \in H$

Corollary. *If $f \in H$, $v_0 \in H$, $z(t) \in L^4_{loc}([0,\infty);\mathbb{L}^4(\mathbb{S}^2) \cap H)$, then $v(t) \in V$ for all $t > 0$.*

We follow the proof in [5]. The idea stems from the standard approximation method commonly used in PDE theory. In view of the a priori estimate (4.18) one takes an approximated solution to (3.16) in Y_T. Then one shows the approximates converge. Finally one shows that the limit function indeed satisfies (3.16).

Let $(v_{0,n}) \subset V$ be a sequence converging to v_0 in H. For all $n \in \mathbb{N}$, let v_n be a solution of equation (3.16) in Y_T corresponding to the initial data $v_{0,n}$. Similar to the case when $v_0 \in V$, one can find a constant such that $|v_n|_{X_T} \le c, \quad \forall n \in \mathbb{N}$. Following the same lines as in the proofs of (4.13) and (4.16), v_n can be proved to be a weak solution.

Moreover, for $n, m \in \mathbb{N}$, take $v_{n,m} = v_n - v_m$ with $v^0_{n,m} = v^0_n - v^0_m$. Then $v_{n,m}$ is the solution of

$$\begin{cases} \partial_t v_{n,m} + \nu A v_{n,m} = -B(v_{n,m}, z) - B(z, v_{n,m}) - B(v_{n,m}, v_n) - B(v_m, v_{n,m}) - \mathbf{C} v_{n,m}, \\ v_{n,m}(0) = v^0_n - v^0_m. \end{cases} \tag{4.22}$$

Multiplying both sides of (4.22) by $v_{n,m}$ and integrating against $v_{n,m}$, using Lemma 2.5 and (2.48) and noting (2.40), one obtains

$$\partial_t |v_{n,m}|^2 + 2\nu |v_{n,m}|_V^2 = -2b(v_{n,m}, z, v_{n,m}) - 2b(v_{n,m}, v_n, v_{n,m}).$$

Since $|b(w,w,z)| \leq C|w||w|_V|z|_V$ and $|b(w,w,v)| \leq C|w||w|_V|v|_V$

$$\leq C|v_{n,m}||v_{n,m}|_V(|z|_V + |v_n|_V)$$

Then using the Young's inequality with $a = \epsilon|v_{n,m}|_V$ and $b = \frac{C}{\sqrt{\epsilon}}|v_{n,m}|(|z|_V + |v_n|_V)$,

$$\leq \frac{\epsilon|v_{n,m}|_V}{2} + \frac{C}{2\epsilon}|v_{n,m}|^2(|z|_V^2 + |v_n|_V^2). \tag{4.23}$$

Therefore, for any $\epsilon > 0$ such that $\frac{\epsilon}{2} < 2\nu$, one applies the Gronwall lemma to obtain

$$\partial_t|v_{n,m}|^2 \leq \frac{C}{2\epsilon}(|z|_V^2 + |v_n|_V^2)|v_{n,m}|^2.$$

Combining this with $v_{n,m}^0 = v_n^0 - v_m^0$, it is easy to show that

$$|v_{n,m}(t)|^2 \leq |v_{n,m}(0)|^2 \exp\left(\frac{C}{2\epsilon}(\int_0^T |z(t)|_V^2 + |v_n(t)|_V^2)|v_{n,m}(t)|^2 dt\right) < \infty,$$

as $\int_0^T |z(t)|_V^2 + |v_n(t)|_V^2 < \infty$. Hence $v_{n,m}$ converges in T, and is therefore Cauchy in T. That is, for any $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $|v_n - v_m| < \epsilon$ whenever $n, m \geq N$.

Let the limit of v_n be v. It remains to show v indeed satisfies (3.16).

Let v_n be the solution to

$$v_n(t) = S(t)v_{0,n} - \int_0^t S(t-s)(B(u_n(s)))ds + \alpha\int_0^t z_n(s)ds, \tag{4.24}$$

where $z_n(t) = \int_0^t S(t-s)GdL_n(t)$. We would like to show that

$$\lim_{n\to\infty} u_n(t) = S(t)u_0 - \int_0^t S(t-s)(B(u(s)))ds + \int_0^t S(t-s)fds + \alpha\int_0^t z(s)ds. \tag{4.25}$$

Assume $f_n \to f$ in $L^2(0,T;H)$, $z_n = \int_0^t S(t-s)GdL_n(t) \to z$ in $L^4([0,T];L^4(\mathbb{S}^2)\cap H)$, we would like to check if

$$\lim_{n\to\infty} B(u_n) = B(u) \quad \text{in} \quad H. \tag{4.26}$$

For this, note first that

$$\begin{aligned} \left||u_n|_V^2 - |u|_V^2\right| &= |(u_n,u_n) - (u,u)| \\ &= |(u_n,u_n)_V - (u,u_n)_V + (u,u_n)_V - (u,u)_V| \\ &= |(u_n,u_n)_V - (u,u_n)_V| + |(u,u_n)_V - (u,u)_V| \\ &\leq |u_n - u|_V|u_n|_V + |u|_V|u_n - u|_V. \end{aligned}$$

Now $|u_n|_V$ is Cauchy and is therefore bounded. So u_n converges to u in V as $n \to \infty$. Then using (2.50) one deduces that

$$
\begin{aligned}
&|B(u_n) - B(u)| \\
&= |B(u_n, u_n) - B(u_n, u) + B(u_n, u) - B(u, u)| \le C(|u_n|_V^2 + |u_n|_V^2 |u| + |u|_V^2) \to C|u|_V^2.
\end{aligned}
$$

Now analogous to the earlier proof of contraction we have,

$$
\begin{aligned}
&|B(u_n(s)) - B(u(s))|_{Y_T}^2 \\
&\le \left| \int_0^t S(t-s)(B(u_n(s)) - B(u(s)))ds \right|_{Y_T}^2 \\
&\le c \int_0^T |B(u_n(s)) - B(u(s))|^2 ds \\
&\le c|u|_T^2 \sqrt{T}.
\end{aligned}
$$

Therefore, $B(u_n) - B(u)$ is in $L^2(0, T; H)$. Now by the continuity argument again, one has

$$
\lim_{n\to\infty} \int_0^T S(t-s)B(u_n(s))ds = \int_0^T S(t-s)B(u(s))ds,
$$

and

$$
\lim_{n\to\infty} \int_0^T S(t-s)f_n(s)ds = \int_0^T S(t-s)f(s)ds.
$$

Combining the above with the assumptions that

$$
\lim_{n\to\infty} S(t)u_{0,n} = S(t)u_0,
$$
$$
\lim_{n\to\infty} z_n(t) = z(t),
$$

one deduces that

$$
\lim_{n\to\infty} v_n(t) = v(t),
$$

and there exists a solution to (3.16). However, the solution constructed as the limits of u_n leaves open the possibility that there is more than one limit. So we will now prove u is unique. The idea is analogous to proving (4.23). Nevertheless we detail as follows. Suppose v_1, v_2 are two solutions of (3.19) with the same initial condition. Let $w = v_1 - v_2$, then w satisfies

$$
\begin{cases}
\partial_t w + \nu A w = -B(w, z) - B(z, w) - B(w, v_1) - B(v_2, w), \\
w(0) = 0.
\end{cases} \tag{4.27}
$$

Multiplying (4.27) on both sides with w and integrating against w, using the identities $\partial_t |v(t)|^2 = 2\langle \partial_t v(t), v(t)\rangle$ again in Temam and (2.48), one gets

$$
\partial_t |w|^2 + 2\nu |w|_V^2 = -2b(w, z, w) - 2b(w, v_1, w).
$$

Since $|b(w,w,z)| \le C|w||w|_V|z|_V$ and $|b(w,w,v)| \le C|w||w|_V|v|_V$

$$\le C|w||w|_V(|z|_V + |v_1|_V).$$

Then via usual Young's inequality with $a = \sqrt{\epsilon}|w|_V$ and $b = \frac{C}{\sqrt{\epsilon}}|w|(|z|_V + |v_n|_V)$

$$\le \frac{\epsilon|w|_V}{2} + \frac{C}{2\epsilon}|w|^2(|z|_V^2 + |v_1|_V^2). \tag{4.28}$$

Therefore, for any $\epsilon > 0$ such that $\frac{\epsilon}{2} < 2\nu$, one applies the Gronwall lemma to

$$\partial_t |w|^2 \le \frac{C}{2\epsilon}(|z|_V^2 + |v_1|_V^2)|w|^2,$$

and combining with $w_0 = v_{1,0} - v_{2,0} = 0$, it follows from the Gronwall lemma that

$$|w(t)|^2 \le |w(0)|^2 \exp\left(\frac{C}{2\epsilon}(\int_0^T |z(t)|_V^2 + |v_1(t)|_V^2)|w(t)|^2 dt\right) < \infty$$

as $\int_0^T |z(t)|_V^2 + |v_1(t)|^2 dt < \infty$. Now, since $w(0) = 0$, necessarily $w(t)$ must be 0.

It remains to show $v \in C((0,T;V)$, as observed from the above energy inequality (4.23). The solution starts with an initial condition $v_0 \in H$ belonging to $L^2(0,T;V)$. This implies that almost everywhere in $(0,T]$, there must exist a time point ϵ ($\epsilon < T$) such that $u(\epsilon) \in V$. Then one may repeat step two onto another interval $[\epsilon, 2\epsilon], [2\epsilon, 3\epsilon]$, and soon over the whole $[\epsilon, \infty]$. Finally we obtain that $u \in C([\epsilon,T];V) \cap L^2([\epsilon,T];D(A))$ for all $\epsilon > 0$. Note that $T = \infty$ as implied from the a priori estimate.

In summary, in this section, we have proved:

Lemma 4.4. *Assuming that $\alpha \ge 0$, $z \in L^4_{loc}([0,\infty); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $f \in H$ and $v_0 \in H$. Then, there exists a unique solution of (3.20) in the space $C(0,T;H) \cap L^2(0,T;V)$, which belongs to $C(\epsilon,T;V) \cap L^2_{loc}(\epsilon,T;D(A))$ for all $\epsilon > 0$ and $T > 0$.*

Combining Lemma 4.4 with 4.3, we have proved Theorem 3.11.

Remark. Continuous dependence on v_0, z and f is implied from the point where local existence and uniqueness is attained and hence holds also for global solutions.

Remark. The proof of Theorem 3.11 shows that the solution v, starting from $v_0 \in H$, belongs to V for a.e. $t \ge t_0$. If we take any $\bar{t} \ge t_0$ such that $v(\bar{t}) \in V$, the solution is extended over the interval $[t_0, t_0 + \epsilon]$ and is found to be in $D(A)$ as well. One may repeat this step over another interval $[t_0+\epsilon, t_0+2\epsilon]$, $[t_0+2\epsilon, t_0+3\epsilon] \cdots$. Thus, we obtain that $v \in C([t_0+\epsilon,\infty);V) \cap L^2_{\rm loc}(t_0+\epsilon, D(A))$.

Furthermore, provided the noise does not degenerate, based on the condition given in the following, we obtained the existence and uniqueness results for the solution to the original equation (3.2).

If

$$\sum_l \lambda^{\frac{\beta}{2}} |\sigma_l|^\beta < \infty, \tag{4.29}$$

then by Lemma 3.5 the process z has a version which has left limits and is right continuous in V. Recall that $u_t := v_t + z_t$ and for each $T > 0$, define

$$Z_T(\omega) := \sup_{0 \le t \le T} |z_t(\omega)|_V, \quad \omega \in \Omega. \tag{4.30}$$

If (4.29) holds then by Lemma 3.3 we have

$$\mathbb{E} Z_T < \infty.$$

Hence there exists a measurable set $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and

$$Z_T(\omega) < \infty, \quad \omega \in \Omega_0 .$$

Finally, let us study (3.2) for $\omega \in \Omega_0$. Since $z(\cdot, \omega) \in D([0, \infty); V)$, it is of course $z(\cdot, \omega) \in D([0, \infty); H)$. Therefore, by Theorem 3.11, $u(\cdot, \omega) = v(\cdot, \omega) + z \cdot (\omega)$ is the unique cádlág solution to (3.2). So, we extend the existence theorem of a strong solution for u. Moreover, for $\omega \in \Omega_0$ we find that $u(\cdot, \omega) = v(\cdot, \omega) + z(\cdot, \omega)$ is the unique solution to (3.2) in $D([0, \infty); H) \cap D([0, \infty); V)$ which belongs to $D([h, \infty); V) \cap L^2_{\text{loc}}(h, \infty); D(A))$ for all $h > 0$. If $u_0 \in V$, then $u \in D([h, \infty); V) \cap L^2_{\text{loc}}([h, \infty); D(A))$ for all $h > 0$, $T > 0$.

This completes the proof of Theorem 3.12.

Since the solution is constructed using the Banach Fixed Point Theorem, the continuous dependence on initial data is implied from the existence-uniqueness proof of a strong solution in the above line. Moreover, our existence-uniqueness results work naturally when the initial time $t_0 \in \mathbb{R}$ other than 0.

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