



Point vortex approximation for 2D Navier–Stokes equations driven by space-time white noise



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ABSTRACT

We show that the system of point vortices, perturbed by a certain transport type noise, converges weakly to the vorticity form of 2D Navier–Stokes equations driven by the space-time white noise.

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1. Introduction

The purpose of this paper is to show that a particle system of stochastic point vortices converges, as the number of particles goes to infinity, to the vorticity form of the 2D Navier–Stokes equations driven by the space-time white noise:

$$d\omega + u \cdot \nabla \omega dt = \Delta \omega dt + \sqrt{2} \nabla^\perp \cdot dW, \quad \omega_0 \stackrel{d}{\sim} \text{white noise on } \mathbb{T}^2. \quad (1.1)$$

Here $u = (u_1, u_2)$ is a divergence free vector field on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\omega = \nabla^\perp \cdot u = \partial_2 u_1 - \partial_1 u_2$ is the vorticity. The equation (1.1) in velocity-pressure variables reads as

$$\begin{aligned} du + (u \cdot \nabla u + \nabla p) dt &= \Delta u dt + \sqrt{2} dW, \\ \operatorname{div} u &= 0, \end{aligned}$$

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which has been studied intensively in the last three decades, see for instance [2,7,9,3,20,4,19,21] among others. This equation has an invariant measure given by some Gaussian measure μ which is supported by any Sobolev or Besov spaces of negative order. It was shown in [7, Theorem 5.2] that, for μ -a.s. starting points in some Besov space, the above equation has a unique solution with continuous paths; moreover, if the initial data is a random variable with distribution μ , then the solution is a stationary process.

To motivate our study we begin by considering the vorticity form of the 2D Euler equation:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \omega|_{t=0} = \omega_0.$$

This is a nonlinear transport equation in which u is expressed by ω via the Biot–Savart law:

$$u(x) = (K * \omega)(x) = \langle \omega, K(x - \cdot) \rangle,$$

where K is the Biot–Savart kernel on \mathbb{T}^2 . We refer the readers to [11, Introduction] for a list of well posedness results on this equation. In particular, we are interested in the case when ω_0 has the form $\omega_0^N(dx) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{X_0^i}(dx)$, where $\xi_i \in \mathbb{R}$ are the vortex intensities and $X_0^i \in \mathbb{T}^2$ are some distinct points. According to [17, Section 4.4], the above equation can be interpreted as the finite dimensional dynamics on $(\mathbb{T}^2)^N$:

$$\frac{dX_t^{i,N}}{dt} = \frac{1}{\sqrt{N}} \sum_{j=1, j \neq i}^N \xi_j K(X_t^{i,N} - X_t^{j,N}) \quad (1.2)$$

with initial condition $X_0^{i,N} = X_0^i, i = 1, \dots, N$. This system is not necessarily well posed: an explicit example was given in [17, Section 4.2] which shows that three different vortex points starting from certain positions collapse to one point in finite time. Nevertheless, the above system of equations admits a unique solution for $(\text{Leb}_{\mathbb{T}^2}^{\otimes N})$ -a.e. starting point in $(\mathbb{T}^2)^N$.

Based on the above result, the first author of the current paper considered the system (1.2) with random initial data ω_0^N which converges weakly to the white noise on \mathbb{T}^2 (see [11, Section 3.2] or the statements below (1.3) of the current paper for more details). Denoting by $\omega_t^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{X_t^{i,N}}$; he proved in [11, Theorem 24] that the family of processes $\{\omega_t^N\}$ has a subsequence which converges weakly to some ω , with continuous paths in $H^{-1-}(\mathbb{T}^2) = \cap_{s>0} H^{-1-s}(\mathbb{T}^2)$, such that ω_t is a white noise on \mathbb{T}^2 for all $t > 0$; furthermore, the process ω solves the weak vorticity formulation of 2D Euler equations. We refer to [11, Theorem 25] for more general results and to [12] for extensions to stochastic settings. On the other hand, we considered in the recent paper [14] the following stochastic 2D Euler equation

$$d\omega + u \cdot \nabla \omega dt = 2\varepsilon_N \sum_{0 < |k| \leq N} e_k \frac{k^\perp}{|k|^2} \cdot \nabla \omega \circ dW^k,$$

where k runs over $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$ and $k^\perp = (k_2, -k_1)$, $\varepsilon_N = \left(\sum_{0 < |k| \leq N} \frac{1}{|k|^2}\right)^{-1/2} \sim (\log N)^{-1/2}$, $\{e_k\}_k$ is the orthonormal basis of sine and cosine functions (see (2.2)) and $\{W^k\}_k$ are independent standard Brownian motions. It was shown that this model, hyperbolic in nature, converges to the parabolic equation (1.1) above.

Motivated by the above discussions, for any $N \geq 1$, we shall study in the current paper the stochastic point vortex system

$$dX_t^{i,N} = \frac{1}{\sqrt{N}} \sum_{j=1, j \neq i}^N \xi_j K(X_t^{i,N} - X_t^{j,N}) dt + 2\varepsilon_N \sum_{0 < |k| \leq N} \frac{k^\perp}{|k|^2} e_k(X_t^{i,N}) \circ dW_t^k \quad (1.3)$$

with random vortex intensities and initial positions. More precisely, assume that the vortex intensities $\{\xi_i\}_{i \in \mathbb{N}}$ is a family of i.i.d. $N(0, 1)$ -random variables, and $\{X_0^i\}_{i \in \mathbb{N}}$ is an i.i.d. sequence of \mathbb{T}^2 -uniformly distributed random variables; moreover, the two families $\{\xi_i\}_{i \in \mathbb{N}}$ and $\{X_0^i\}_{i \in \mathbb{N}}$ are independent. Then, as $N \rightarrow \infty$, the random measure $\omega_0^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{X_0^i}$ converges weakly to the white noise on \mathbb{T}^2 . Our main result in this paper can be stated as follows.

Theorem 1.1. *Let $T > 0$ and $\{\xi_i\}_{i \in \mathbb{N}}$ and $\{X_0^i\}_{i \in \mathbb{N}}$ be given as above. For any $N \geq 1$, the solution $X_t^N = (X_t^{1,N}, \dots, X_t^{N,N})$ to (1.3) with random initial data $X_0^{i,N} = X_0^i$ ($1 \leq i \leq N$) is a well defined process on $(\mathbb{T}^2)^N$. As $N \rightarrow \infty$, the processes $\omega_t^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{X_t^{i,N}}$ ($t \in [0, T]$) converge weakly to a process $\{\omega_t\}_{t \in [0, T]}$ which is the unique stationary solution of (1.1).*

This result will be proved in Section 2 by following the general idea of [14], but we need some L^2 -boundedness estimate on a sequence of functionals of ω_0^N , which is done in the appendix.

We conclude the introduction with a brief discussion. The above convergence result is proved by a compactness argument and thus we do not have a rate of convergence, which is in general a difficult problem for equations with non-Lipschitz and non-monotone nonlinearities. The difficulty comes also from the fact that the solutions to (1.1) considered here are distribution-valued processes. Nevertheless, there are a few works in the literature dealing with other types of numerical approximations for SPDEs with such kind of non-linearities, in more regular regimes where solutions are function-valued processes. Dörsek [10] considered the approximation of the marginal distribution of the solution to the stochastic Navier-Stokes equations on the two-dimensional torus by high order numerical methods. He obtained the convergence rate for a splitting scheme and the method of cubature on Wiener space applied to a spectral Galerkin discretization; see [5] for related studies on the time-splitting algorithm and the implicit and semi-implicit Euler algorithms for the 2D Navier-Stokes equations. In the earlier work [1], Alabert and Gyöngy considered the stochastic Burgers equation on the interval $[0, 1]$, with Dirichlet boundary condition and driven by space-time white noise. They studied a finite difference scheme for the stochastic Burgers equation and estimated the rate of convergence to the solution of the equation. We refer to the paper [15] and the references therein for some recent progress in this respect.

2. Convergence of the stochastic point vortex systems

First, we give a more detailed description of our setting for which we need some notations. Recall the two sequences $\{\xi_i\}_{i \in \mathbb{N}}$ and $\{X_0^i\}_{i \in \mathbb{N}}$ of random variables defined above Theorem 1.1; for every $N \in \mathbb{N}$, let

$$\lambda_N^0 = (N(0, 1) \otimes \text{Leb}_{\mathbb{T}^2})^{\otimes N}$$

be the law of the random vector $((\xi_1, X_0^1), \dots, (\xi_N, X_0^N))$. The measure-valued vorticity field

$$\omega_0^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{X_0^i}$$

can be regarded as a random variable taking values in the space $H^{-1-}(\mathbb{T}^2) = \bigcap_{s>0} H^{-1-s}(\mathbb{T}^2)$ with the law μ_N^0 , where $H^r(\mathbb{T}^2)$ ($r \in \mathbb{R}$) is the usual Sobolev space on \mathbb{T}^2 . Denote by $\mathcal{M}(\mathbb{T}^2)$ the space of signed measures on \mathbb{T}^2 with finite variation, and

$$\mathcal{M}_N(\mathbb{T}^2) = \{\mu \in \mathcal{M}(\mathbb{T}^2) \mid \exists X \subset \mathbb{T}^2 \text{ such that } \#(X) = N \text{ and } \text{supp}(\mu) = X\}.$$

We can define the map $\mathcal{T}_N : (\mathbb{R} \times \mathbb{T}^2)^N \rightarrow \mathcal{M}_N(\mathbb{T}^2) \subset H^{-1-}(\mathbb{T}^2)$ as

$$((\xi_1, X_0^1), \dots, (\xi_N, X_0^N)) \mapsto \omega_0^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{X_0^i}, \quad (2.1)$$

then it holds that

$$\mu_N^0 = (\mathcal{T}_N)_\# \lambda_N^0 = \lambda_N^0 \circ \mathcal{T}_N^{-1}.$$

It is proved in [11, Proposition 21] that, as $N \rightarrow \infty$, ω_0^N converges in law to the white noise ω_{WN} on \mathbb{T}^2 .

We denote by

$$e_k(x) = \sqrt{2} \begin{cases} \cos(2\pi k \cdot x), & k \in \mathbb{Z}_+^2, \\ \sin(2\pi k \cdot x), & k \in \mathbb{Z}_-^2, \end{cases} \quad x \in \mathbb{T}^2, \quad (2.2)$$

where $\mathbb{Z}_+^2 = \{k \in \mathbb{Z}_0^2 : (k_1 > 0) \text{ or } (k_1 = 0, k_2 > 0)\}$ and $\mathbb{Z}_-^2 = -\mathbb{Z}_+^2$. Next, define

$$\sigma_k(x) = \frac{1}{\sqrt{2}} \frac{k^\perp}{|k|^2} e_k(x), \quad k \in \mathbb{Z}_0^2, \quad (2.3)$$

where $k^\perp = (k_2, -k_1)$. Then $\{\sigma_k : k \in \mathbb{Z}_0^2\}$ constitutes a CONS of $L_0^2(\mathbb{T}^2, \mathbb{T}^2)$, the space of square integrable and divergence free vector fields on \mathbb{T}^2 with zero mean. Let $\{W_t^k\}_{k \in \mathbb{Z}_0^2}$ be a sequence of independent standard Brownian motions, which are independent of $\{\xi_i\}_{i \in \mathbb{N}}$ and $\{X_0^i\}_{i \in \mathbb{N}}$. For $N \geq 1$, define $\Lambda_N = \{k \in \mathbb{Z}_0^2 : |k| \leq N\}$. Now, the stochastic point vortex system (1.3) can be written as follows: for $i = 1, \dots, N$,

$$dX_t^{i,N} = \frac{1}{\sqrt{N}} \sum_{j=1, j \neq i}^N \xi_j K(X_t^{i,N} - X_t^{j,N}) dt + 2\sqrt{2} \varepsilon_N \sum_{k \in \Lambda_N} \sigma_k(X_t^{i,N}) \circ dW_t^k \quad (2.4)$$

with the initial condition $X_0^{i,N} = X_0^i$. Denote by

$$\Delta_N = \{(x_1, \dots, x_N) \in (\mathbb{T}^2)^N : \text{there are } i \neq j \text{ such that } x_i = x_j\}$$

the generalized diagonal of $(\mathbb{T}^2)^N$ and $\Delta_N^c = (\mathbb{T}^2)^N \setminus \Delta_N$. Moreover, for any $\phi \in C^\infty(\mathbb{T}^2)$, set

$$H_\phi(x, y) = \frac{1}{2} K(x - y) \cdot (\nabla \phi(x) - \nabla \phi(y)), \quad x, y \in \mathbb{T}^2,$$

with the convention that $H_\phi(x, x) = 0$. It is well known that, for all $x \in \mathbb{T}^2 \setminus \{0\}$, $K(-x) = -K(x)$ and $|K(x)| \leq C/|x|$ for some constant $C > 0$; thus H_ϕ is symmetric and

$$\|H_\phi\|_\infty \leq C \|\nabla^2 \phi\|_\infty. \quad (2.5)$$

We have the following result.

Proposition 2.1. *For a.s. value of $((\xi_1, X_0^1), \dots, (\xi_N, X_0^N))$, the process $(X_t^{1,N}, \dots, X_t^{N,N})$ is well defined in Δ_N^c for all $t \geq 0$, and the associated random measure-valued vorticity*

$$\omega_t^N = \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i \delta_{X_t^{i,N}}$$

satisfies the equation below: for all $\phi \in C^\infty(\mathbb{T}^2)$,

$$\begin{aligned}
\langle \omega_t^N, \phi \rangle &= \langle \omega_0^N, \phi \rangle + \int_0^t \langle \omega_s^N \otimes \omega_s^N, H_\phi \rangle ds + \int_0^t \langle \omega_s^N, \Delta \phi \rangle ds \\
&\quad + 2\sqrt{2}\varepsilon_N \sum_{k \in \Lambda_N} \int_0^t \langle \omega_s^N, \sigma_k \cdot \nabla \phi \rangle dW_s^k.
\end{aligned} \tag{2.6}$$

The stochastic process ω_t^N is stationary in time, with the law μ_N^0 at any time $t \geq 0$.

Proof. The assertions are the same as [12, Proposition 2.3]; we only present a brief derivation of (2.6). For every $i \in \{1, \dots, N\}$, the Itô formula yields

$$d\phi(X_t^{i,N}) = \frac{1}{\sqrt{N}} \sum_{j \neq i} \xi_j K(X_t^{i,N} - X_t^{j,N}) \cdot \nabla \phi(X_t^{i,N}) dt + 2\sqrt{2}\varepsilon_N \sum_{k \in \Lambda_N} (\sigma_k \cdot \nabla \phi)(X_t^{i,N}) \circ dW_t^k.$$

Therefore,

$$\begin{aligned}
d\langle \omega_t^N, \phi \rangle &= \frac{1}{N} \sum_{1 \leq i \neq j \leq N} \xi_i \xi_j K(X_t^{i,N} - X_t^{j,N}) \cdot \nabla \phi(X_t^{i,N}) dt \\
&\quad + 2\sqrt{2}\varepsilon_N \sum_{k \in \Lambda_N} \langle \omega_t^N, \sigma_k \cdot \nabla \phi \rangle \circ dW_t^k.
\end{aligned}$$

Using the definition of H_ϕ we can write

$$d\langle \omega_t^N, \phi \rangle = \langle \omega_t^N \otimes \omega_t^N, H_\phi \rangle dt + 2\sqrt{2}\varepsilon_N \sum_{k \in \Lambda_N} \langle \omega_t^N, \sigma_k \cdot \nabla \phi \rangle \circ dW_t^k.$$

Transforming it into the Itô equation and rewriting in the integral form give us

$$\begin{aligned}
\langle \omega_t^N, \phi \rangle &= \langle \omega_0^N, \phi \rangle + \int_0^t \langle \omega_s^N \otimes \omega_s^N, H_\phi \rangle ds + 2\sqrt{2}\varepsilon_N \sum_{k \in \Lambda_N} \int_0^t \langle \omega_s^N, \sigma_k \cdot \nabla \phi \rangle dW_s^k \\
&\quad + 4\varepsilon_N^2 \sum_{k \in \Lambda_N} \int_0^t \langle \omega_s^N, \sigma_k \cdot \nabla (\sigma_k \cdot \nabla \phi) \rangle ds.
\end{aligned}$$

It remains to simplify the last term. We have $\sigma_k \cdot \nabla (\sigma_k \cdot \nabla \phi) = \text{Tr}[(\sigma_k \otimes \sigma_k) \nabla^2 \phi]$ since $\sigma_k \cdot \nabla \sigma_k \equiv 0$ for any $k \in \mathbb{Z}_0^2$. The equation (2.6) is a consequence of the following equality:

$$\sum_{k \in \Lambda_N} \sigma_k \otimes \sigma_k = \frac{1}{4} \varepsilon_N^{-2} I_2, \tag{2.7}$$

where I_2 is the (2×2) -unit matrix. This identity was proved in [14, Lemma 2.6]; we present the proof here for the reader's convenience. We have

$$\begin{aligned}
S_N(x) &:= \sum_{k \in \Lambda_N} \sigma_k(x) \otimes \sigma_k(x) = \sum_{k \in \Lambda_N \cap \mathbb{Z}_+^2} \frac{k^\perp \otimes k^\perp}{|k|^4} [\cos^2(2\pi k \cdot x) + \sin^2(2\pi k \cdot x)] \\
&= \sum_{k \in \Lambda_N \cap \mathbb{Z}_+^2} \frac{1}{|k|^4} \begin{pmatrix} k_2^2 & -k_1 k_2 \\ -k_1 k_2 & k_1^2 \end{pmatrix} = \frac{1}{2} \sum_{k \in \Lambda_N} \frac{1}{|k|^4} \begin{pmatrix} k_2^2 & -k_1 k_2 \\ -k_1 k_2 & k_1^2 \end{pmatrix}.
\end{aligned}$$

So S_N is independent of x . First, we have

$$S_N^{1,2} = -\frac{1}{2} \sum_{k \in \Lambda_N} \frac{k_1 k_2}{|k|^4} = 0$$

since we can sum the four terms involving (k_1, k_2) , $(-k_1, k_2)$, $(k_1, -k_2)$, $(-k_1, -k_2)$ at one time. Next,

$$S_N^{1,1} = \frac{1}{2} \sum_{k \in \Lambda_N} \frac{k_2^2}{|k|^4} = \frac{1}{2} \sum_{k \in \Lambda_N} \frac{k_1^2}{|k|^4} = S_N^{2,2}$$

since the points (k_1, k_2) and (k_2, k_1) appear in pair. Therefore,

$$S_N^{1,1} = S_N^{2,2} = \frac{1}{4} \sum_{k \in \Lambda_N} \frac{k_1^2 + k_2^2}{|k|^4} = \frac{1}{4} \sum_{k \in \Lambda_N} \frac{1}{|k|^2} = \frac{1}{4} \varepsilon_N^{-2}.$$

Hence we obtain (2.7). \square

Let Q^N be the law of ω^N on $\mathcal{X} = C([0, T], H^{-1}(\mathbb{T}^2))$, $N \geq 1$. We want to show that the family $\{Q^N\}_{N \geq 1}$ is tight in \mathcal{X} , for which we need the following integrability properties of ω_t^N that are proved in [11, Lemma 23] (except the second estimate which can be proved similarly to the first one).

Lemma 2.2. Assume $f : \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{R}$ and $g : \mathbb{T}^2 \rightarrow \mathbb{R}$ are bounded and measurable, and f is symmetric. Then, for every $p \geq 1$ and $\delta > 0$, there are constants $C_p, C_{p,\delta} > 0$ such that for all $N \geq 1$ and $t \in [0, T]$,

$$\mathbb{E}[|\langle \omega_t^N \otimes \omega_t^N, f \rangle|^p] \leq C_p \|f\|_\infty^p, \quad \mathbb{E}[|\langle \omega_t^N, g \rangle|^p] \leq C_p \|g\|_\infty^p, \quad \mathbb{E}[\|\omega_t^N\|_{H^{-1-\delta}}^p] \leq C_{p,\delta}.$$

Moreover,

$$\mathbb{E}[\langle \omega_t^N \otimes \omega_t^N, f \rangle^2] = \frac{3}{N} \int f^2(x, x) dx + \frac{N-1}{N} \left[\int f(x, x) dx \right]^2 + \frac{2(N-1)}{N} \iint f^2(x, y) dx dy.$$

With these estimates in hand, we can follow the arguments at the beginning of [12, Section 3] to show the tightness of $\{Q^N\}_{N \geq 1}$ in \mathcal{X} . To this end, we need to prove that $\{Q^N\}_{N \geq 1}$ is bounded in probability in $W^{1/3,4}(0, T; H^{-\kappa}(\mathbb{T}^2))$ for some $\kappa > 5$, and in $L^{p_0}(0, T; H^{-1-\delta}(\mathbb{T}^2))$ for any $p_0 > 0$ and $\delta > 0$.

First, by Lemma 2.2, for all $N \in \mathbb{N}$,

$$\mathbb{E} \left[\int_0^T \|\omega_t^N\|_{H^{-1-\delta}}^{p_0} dt \right] = \int_0^T \mathbb{E} [\|\omega_t^N\|_{H^{-1-\delta}}^{p_0}] dt \leq C_{p_0, \delta} T. \quad (2.8)$$

This implies the boundedness in probability of $\{Q^N\}_{N \geq 1}$ in $L^{p_0}(0, T; H^{-1-\delta}(\mathbb{T}^2))$ for any $p_0 > 0$ and $\delta > 0$.

Next, to show that $\{Q^N\}_{N \geq 1}$ is bounded in probability in $W^{1/3,4}(0, T; H^{-\kappa}(\mathbb{T}^2))$ with $\kappa > 5$, it suffices to prove

$$\sup_{N \geq 1} \mathbb{E} \left[\int_0^T \|\omega_t^N\|_{H^{-\kappa}}^4 dt + \int_0^T \int_0^T \frac{\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4}{|t-s|^{7/3}} dt ds \right] < \infty.$$

The expectation of the first part is finite by the estimate (2.8), thus we focus on the second part. We need the following result whose proof looks very similar to [14, Lemma 2.5]. The difference between them is that here the processes ω_t^N are random point vortices, while the processes in [14, Lemma 2.5] have white noise as marginal distribution.

Lemma 2.3. *There exists $C > 0$ such that for any $N \geq 1$ and $\phi \in C^\infty(\mathbb{T}^2)$, we have*

$$\mathbb{E}[\langle \omega_t^N - \omega_s^N, \phi \rangle^4] \leq C(t-s)^2(\|\nabla \phi\|_\infty^4 + \|\nabla^2 \phi\|_\infty^4).$$

Proof. By (2.6), one has

$$\begin{aligned} \langle \omega_t^N - \omega_s^N, \phi \rangle &= \int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle dr + \int_s^t \langle \omega_r^N, \Delta \phi \rangle dr \\ &\quad + 2\sqrt{2}\varepsilon_N \sum_{k \in \Lambda_N} \int_s^t \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle dW_r^k. \end{aligned} \quad (2.9)$$

First, Hölder's inequality leads to

$$\begin{aligned} \mathbb{E} \left[\left(\int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle dr \right)^4 \right] &\leq (t-s)^3 \mathbb{E} \left[\int_s^t \langle \omega_r^N \otimes \omega_r^N, H_\phi \rangle^4 dr \right] \\ &\leq (t-s)^3 \int_s^t C \|\nabla^2 \phi\|_\infty^4 dr = C(t-s)^4 \|\nabla^2 \phi\|_\infty^4, \end{aligned} \quad (2.10)$$

where in the second step we used Lemma 2.2 and (2.5). In the same way,

$$\mathbb{E} \left[\left(\int_s^t \langle \omega_r^N, \Delta \phi \rangle dr \right)^4 \right] \leq (t-s)^3 \mathbb{E} \left[\int_s^t \langle \omega_r^N, \Delta \phi \rangle^4 dr \right] \leq C(t-s)^4 \|\Delta \phi\|_\infty^4. \quad (2.11)$$

Next, by Burkholder's inequality,

$$\begin{aligned} \mathbb{E} \left[\left(\varepsilon_N \sum_{k \in \Lambda_N} \int_s^t \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle dW_r^k \right)^4 \right] &\leq C\varepsilon_N^4 \mathbb{E} \left[\left(\int_s^t \sum_{k \in \Lambda_N} \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle^2 dr \right)^2 \right] \\ &\leq C\varepsilon_N^4 (t-s) \int_s^t \mathbb{E} \left[\left(\sum_{k \in \Lambda_N} \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle^2 \right)^2 \right] dr. \end{aligned}$$

Cauchy's inequality and Lemma 2.2 imply that

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{k \in \Lambda_N} \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle^2 \right)^2 \right] &= \sum_{k, l \in \Lambda_N} \mathbb{E} [\langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle^2 \langle \omega_r^N, \sigma_l \cdot \nabla \phi \rangle^2] \\ &\leq \sum_{k, l \in \Lambda_N} [\mathbb{E} \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle^4]^{1/2} [\mathbb{E} \langle \omega_r^N, \sigma_l \cdot \nabla \phi \rangle^4]^{1/2} \\ &\leq C \left(\sum_{k \in \Lambda_N} \|\sigma_k \cdot \nabla \phi\|_\infty^2 \right)^2 \leq \tilde{C} \|\nabla \phi\|_\infty^4 \left(\sum_{k \in \Lambda_N} \|\sigma_k\|_\infty^2 \right)^2. \end{aligned}$$

Note that

$$\sum_{k \in \Lambda_N} \|\sigma_k\|_\infty^2 = \sum_{k \in \Lambda_N} \frac{1}{|k|^2} = \varepsilon_N^{-2},$$

hence,

$$\mathbb{E} \left[\left(\sum_{k \in \Lambda_N} \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle^2 \right)^2 \right] \leq C \|\nabla \phi\|_\infty^4 \varepsilon_N^{-4}.$$

This implies

$$\mathbb{E} \left[\left(\varepsilon_N \sum_{k \in \Lambda_N} \int_s^t \langle \omega_r^N, \sigma_k \cdot \nabla \phi \rangle dW_r^k \right)^4 \right] \leq C(t-s)^2 \|\nabla \phi\|_\infty^4.$$

Combining this estimate with (2.9)–(2.11), we obtain the desired result. \square

Applying Lemma 2.3 with $\phi(x) = e_k(x)$ leads to

$$\mathbb{E} [|\langle \omega_t^N - \omega_s^N, e_k \rangle|^4] \leq C(t-s)^2 |k|^8, \quad k \in \mathbb{Z}_0^2.$$

As a result, by Cauchy's inequality,

$$\begin{aligned} \mathbb{E} (\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4) &= \mathbb{E} \left[\left(\sum_k (1 + |k|^2)^{-\kappa} |\langle \omega_t^N - \omega_s^N, e_k \rangle|^2 \right)^2 \right] \\ &\leq \left(\sum_k (1 + |k|^2)^{-\kappa} \right) \sum_k (1 + |k|^2)^{-\kappa} \mathbb{E} [|\langle \omega_t^N - \omega_s^N, e_k \rangle|^4] \\ &\leq \tilde{C}(t-s)^2 \sum_k (1 + |k|^2)^{-\kappa} |k|^8 \leq \hat{C}(t-s)^2, \end{aligned}$$

since $2\kappa - 8 > 2$ due to the choice of κ . Consequently,

$$\mathbb{E} \left[\int_0^T \int_0^T \frac{\|\omega_t^N - \omega_s^N\|_{H^{-\kappa}}^4}{|t-s|^{7/3}} dt ds \right] \leq \hat{C} \int_0^T \int_0^T \frac{|t-s|^2}{|t-s|^{7/3}} dt ds < \infty.$$

The proof of the boundedness in probability of $\{Q^N\}_{N \geq 1}$ in $W^{1/3,4}(0, T; H^{-\kappa}(\mathbb{T}^2))$ is complete.

Combining this result with (2.8) and the discussions below Lemma 2.2, we conclude that $\{Q^N\}_{N \geq 1}$ is tight in $\mathcal{X} = C([0, T], H^{-1}(\mathbb{T}^2))$.

Since we are dealing with the SDEs (2.6), we need to consider Q^N together with the distribution of Brownian motions. Although we use only finitely many Brownian motions in (2.6), here we consider for simplicity the whole family $\{(W_t^k)_{0 \leq t \leq T} : k \in \mathbb{Z}_0^2\}$. To this end, we assume $\mathbb{R}^{\mathbb{Z}_0^2}$ is endowed with the metric

$$d_{\mathbb{Z}_0^2}(a, b) = \sum_{k \in \mathbb{Z}_0^2} \frac{|a_k - b_k| \wedge 1}{2^{|k|}}, \quad a, b \in \mathbb{R}^{\mathbb{Z}_0^2}.$$

Then $(\mathbb{R}^{\mathbb{Z}_0^2}, d_{\mathbb{Z}_0^2})$ is separable and complete (see [6, Example 1.2, p. 9]). The distance in $\mathcal{Y} := C([0, T], \mathbb{R}^{\mathbb{Z}_0^2})$ is given by

$$d_{\mathcal{Y}}(w, \hat{w}) = \sup_{t \in [0, T]} d_{\mathbb{Z}_0^2}(w(t), \hat{w}(t)), \quad w, \hat{w} \in \mathcal{Y},$$

which makes \mathcal{Y} a Polish space. Denote by \mathcal{W} the law on \mathcal{Y} of the sequence of independent Brownian motions $\{(W_t^k)_{0 \leq t \leq T} : k \in \mathbb{Z}_0^2\}$.

To simplify the notations, we write $W. = (W_t)_{0 \leq t \leq T}$ for the whole sequence of processes $\{(W_t^k)_{0 \leq t \leq T} : k \in \mathbb{Z}_0^2\}$ in \mathcal{Y} . Denote by P^N the joint law of $(\omega^N, W.)$ on $\mathcal{X} \times \mathcal{Y}$, $N \geq 1$. Since the marginal laws $\{Q^N\}_{N \in \mathbb{N}}$ and $\{W.\}$ are respectively tight on \mathcal{X} and \mathcal{Y} , we conclude that $\{P^N\}_{N \in \mathbb{N}}$ is tight on $\mathcal{X} \times \mathcal{Y}$. By Skorokhod's representation theorem, there exist a subsequence $\{N_i\}_{i \in \mathbb{N}}$ of integers, a probability space $(\hat{\Theta}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and stochastic processes $(\hat{\omega}^{N_i}, \hat{W}^{N_i})$ on this space with the corresponding laws P^{N_i} , and converging $\hat{\mathbb{P}}$ -a.s. in $\mathcal{X} \times \mathcal{Y}$ to a limit $(\hat{\omega}, \hat{W}.)$. We are going to prove that $\hat{\omega}.$, or more precisely another closely related process, solves the vorticity form of the Navier–Stokes equation with a suitable cylindrical Brownian motion.

We want to identify the approximating processes on the new probability space as random point vortices. For this purpose, we follow the discussions above [11, Lemma 28] and enlarge the probability space $(\hat{\Theta}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, so that it contains certain independent random variables we need. The rough idea is to apply a random permutation to an $(\mathbb{R} \times \mathbb{T}^2)^N$ -valued random variable which corresponds, via the mapping (2.1), to a random variable with values in $\mathcal{M}_N(\mathbb{T}^2)$, see the end of **Step 1** in the proof of [11, Lemma 28] for more details. Denote by $(\tilde{\Theta}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ a probability space on which, for every $N \geq 1$, we define a uniformly distributed random permutation $\tilde{s}_N : \tilde{\Theta} \rightarrow \Sigma_N$, where Σ_N is the permutation group of order N . Define the product probability space

$$(\Theta, \mathcal{F}, \mathbb{P}) = (\hat{\Theta} \times \tilde{\Theta}, \hat{\mathcal{F}} \otimes \tilde{\mathcal{F}}, \hat{\mathbb{P}} \otimes \tilde{\mathbb{P}}) \quad (2.12)$$

and the new processes

$$(\omega^{N_i}, W^{N_i}) = (\hat{\omega}^{N_i}, \hat{W}^{N_i}) \circ \pi_1, \quad (\omega, W) = (\hat{\omega}, \hat{W}) \circ \pi_1, \quad s_N = \tilde{s}_N \circ \pi_2,$$

where π_1 and π_2 are the projections on $\hat{\Theta} \times \tilde{\Theta}$. Here, we slightly abuse the notations by denoting the final probability spaces and processes like the original ones. In the sequel we always consider the processes on the new probability space.

First, by Proposition 2.1, it is easy to show

Lemma 2.4. *The new process $\omega.$ is stationary and for every $t \in [0, T]$, the law μ_t of ω_t on $H^{-1-}(\mathbb{T}^2)$ is the white noise measure μ .*

Similarly to [12, Lemma 3.5], we can identify the structure of $\omega_t^{N_i}$ as a sum of Dirac masses.

Lemma 2.5. *The process $\omega_t^{N_i}$ on the new probability space can be represented in the form $\frac{1}{\sqrt{N_i}} \sum_{j=1}^{N_i} \xi_j \delta_{X_t^{j, N_i}}$, where*

$$((\xi_1, X_0^{1, N_i}), \dots, (\xi_{N_i}, X_0^{N_i, N_i})) \quad (2.13)$$

is a random vector with law $\lambda_{N_i}^0$ and $(X_t^{1, N_i}, \dots, X_t^{N_i, N_i})$ solves the stochastic system (2.4) with the initial condition $(X_0^{1, N_i}, \dots, X_0^{N_i, N_i})$ and new Brownian motions $\{(W_t^{N_i, k}) : k \in \Lambda_{N_i}\}$ defined above.

As a consequence (cf. Proposition 2.1), for any $i \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{T}^2)$, the new process ω^{N_i} satisfies \mathbb{P} -a.s., for all $t \in [0, T]$,

$$\begin{aligned} \langle \omega_t^{N_i}, \phi \rangle &= \langle \omega_0^{N_i}, \phi \rangle + \int_0^t \langle \omega_s^{N_i} \otimes \omega_s^{N_i}, H_\phi \rangle ds + \int_0^t \langle \omega_s^{N_i}, \Delta \phi \rangle ds \\ &\quad + 2\sqrt{2} \varepsilon_{N_i} \sum_{k \in \Lambda_{N_i}} \int_0^t \langle \omega_s^{N_i}, \sigma_k \cdot \nabla \phi \rangle dW_s^{N_i, k}. \end{aligned} \quad (2.14)$$

Remark 2.6. Using the a.s. convergence of ω^{N_i} to ω in $C([0, T], H^{-1-}(\mathbb{T}^2))$, we can show that the quantities in the first line of (2.14) converge respectively in $L^2(\Theta, \mathbb{P})$ to

$$\langle \omega_t, \phi \rangle, \quad \langle \omega_0, \phi \rangle, \quad \int_0^t \langle \omega_r \otimes \omega_r, H_\phi \rangle dr, \quad \int_0^t \langle \omega_r, \Delta \phi \rangle dr,$$

see [12, Proposition 3.6] for details. Here, for a white noise ω_r , the quantity $\langle \omega_r \otimes \omega_r, H_\phi \rangle$ can be defined as an $L^2(\Theta, \mathbb{P})$ -limit of some approximating sequence, see [11, Theorem 8]. However, the term involving stochastic integrals does not converge strongly to some limit. Therefore, we can only seek for a weaker form of convergence.

Before proceeding further, we introduce some notations. By $\Lambda \Subset \mathbb{Z}_0^2$ we mean that Λ is a finite set. Let $\Pi_\Lambda : H^{-1-}(\mathbb{T}^2) \rightarrow \text{span}\{e_k : k \in \Lambda\}$ be the projection operator: $\Pi_\Lambda \omega = \sum_{l \in \Lambda} \langle \omega, e_l \rangle e_l$. We shall use the family of cylindrical functions below:

$$\mathcal{FC}_b^2 = \{F(\omega) = f(\langle \omega, e_l \rangle; l \in \Lambda) \text{ for some } \Lambda \Subset \mathbb{Z}_0^2 \text{ and } f \in C_b^2(\mathbb{R}^\Lambda)\},$$

where \mathbb{R}^Λ is the $(\#\Lambda)$ -dimensional Euclidean space. To simplify the notations, sometimes we write the cylindrical functions as $F = f \circ \Pi_\Lambda$, and for $l, m \in \Lambda$, $f_l(\omega) = (\partial_l f)(\Pi_\Lambda \omega)$ and $f_{l,m}(\omega) = (\partial_l \partial_m f)(\Pi_\Lambda \omega)$. Denote by \mathcal{L}_∞ the generator of the equation (1.1): for any cylindrical function $F = f \circ \Pi_\Lambda$ with $\Lambda \Subset \mathbb{Z}_0^2$,

$$\mathcal{L}_\infty F = 4\pi^2 \sum_{l \in \Lambda} |l|^2 [f_{l,l}(\omega) - f_l(\omega) \langle \omega, e_l \rangle] - \langle u(\omega) \cdot \nabla \omega, DF \rangle, \quad (2.15)$$

where the drift part

$$\langle u(\omega) \cdot \nabla \omega, DF \rangle = - \sum_{l \in \Lambda} f_l(\omega) \langle \omega \otimes \omega, H_{e_l} \rangle.$$

Finally, we introduce the notation

$$C_{k,l} = \frac{k^\perp \cdot l}{|k|^2}, \quad k, l \in \mathbb{Z}_0^2. \quad (2.16)$$

We have the following useful identity (cf. [13, Lemma 3.4] for the proof):

$$\sum_{k \in \Lambda_N} C_{k,l}^2 = \frac{1}{2} \varepsilon_N^{-2} |l|^2. \quad (2.17)$$

Now we prove that the limit ω is a martingale solution of the operator \mathcal{L}_∞ .

Proposition 2.7. For any $F \in \mathcal{FC}_b^2$,

$$M_t^F := F(\omega_t) - F(\omega_0) - \int_0^t \mathcal{L}_\infty F(\omega_s) ds \quad (2.18)$$

is an $\mathcal{F}_t = \sigma(\omega_s : s \leq t)$ -martingale.

Proof. The proof below is analogous to that of [14, Proposition 2.9], but the processes $\tilde{\omega}_t^{N_i}$ involved there are processes of white noises on \mathbb{T}^2 , while here $\omega_t^{N_i}$ are random point vortices. Recall the CONS defined in

(2.2). Taking $\phi = e_l$ in (2.14) for some $l \in \mathbb{Z}_0^2$, we have

$$\begin{aligned} d\langle \omega_t^{N_i}, e_l \rangle &= \langle \omega_t^{N_i} \otimes \omega_t^{N_i}, H_{e_l} \rangle dt - 4\pi^2 |l|^2 \langle \omega_t^{N_i}, e_l \rangle dt \\ &\quad + 2\sqrt{2} \varepsilon_{N_i} \sum_{k \in \Lambda_{N_i}} \langle \omega_t^{N_i}, \sigma_k \cdot \nabla e_l \rangle dW_t^{N_i, k}. \end{aligned} \quad (2.19)$$

Therefore, for $l, m \in \mathbb{Z}_0^2$,

$$d\langle \omega_t^{N_i}, e_l \rangle \cdot d\langle \omega_t^{N_i}, e_m \rangle = 8\varepsilon_{N_i}^2 \sum_{k \in \Lambda_{N_i}} \langle \omega_t^{N_i}, \sigma_k \cdot \nabla e_l \rangle \langle \omega_t^{N_i}, \sigma_k \cdot \nabla e_m \rangle dt.$$

It is easy to show that $\sigma_k \cdot \nabla e_l = \sqrt{2}\pi C_{k,l} e_k e_{-l}$; hence

$$\begin{aligned} \langle \omega_t^{N_i}, \sigma_k \cdot \nabla e_l \rangle \langle \omega_t^{N_i}, \sigma_k \cdot \nabla e_m \rangle &= 2\pi^2 C_{k,l} C_{k,m} \langle \omega_t^{N_i}, e_k e_{-l} \rangle \langle \omega_t^{N_i}, e_k e_{-m} \rangle \\ &= 2\pi^2 C_{k,l} C_{k,m} \left[\langle \omega_t^{N_i}, e_k e_{-l} \rangle \langle \omega_t^{N_i}, e_k e_{-m} \rangle - \delta_{l,m} \right] \\ &\quad + 2\pi^2 \delta_{l,m} C_{k,l}^2. \end{aligned}$$

As a result,

$$\begin{aligned} d\langle \omega_t^{N_i}, e_l \rangle \cdot d\langle \omega_t^{N_i}, e_m \rangle &= 16\pi^2 \varepsilon_{N_i}^2 \sum_{k \in \Lambda_{N_i}} C_{k,l} C_{k,m} \left[\langle \omega_t^{N_i}, e_k e_{-l} \rangle \langle \omega_t^{N_i}, e_k e_{-m} \rangle - \delta_{l,m} \right] dt \\ &\quad + 8\pi^2 \delta_{l,m} |l|^2 dt, \end{aligned}$$

where in the last step we have used (2.17). To simplify the notations, we denote by

$$R_{l,m}(\omega_t^{N_i}) = 8\pi^2 \sum_{k \in \Lambda_{N_i}} C_{k,l} C_{k,m} \left[\langle \omega_t^{N_i}, e_k e_{-l} \rangle \langle \omega_t^{N_i}, e_k e_{-m} \rangle - \delta_{l,m} \right].$$

Recall that $\omega_t^{N_i}$ has the law $\mu_{N_i}^0$ for any $t \in [0, T]$, thus $R_{l,m}(\omega_t^{N_i})$ is bounded in $L^2([0, T] \times \Theta)$ by Proposition 3.1 in the appendix. Finally, we get

$$d\langle \omega_t^{N_i}, e_l \rangle \cdot d\langle \omega_t^{N_i}, e_m \rangle = 2\varepsilon_{N_i}^2 R_{l,m}(\omega_t^{N_i}) dt + 8\pi^2 \delta_{l,m} |l|^2 dt. \quad (2.20)$$

By the Itô formula and (2.19), (2.20),

$$\begin{aligned} dF(\omega_t^{N_i}) &= df(\langle \omega_t^{N_i}, e_l \rangle; l \in \Lambda) \\ &= \sum_{l \in \Lambda} f_l(\omega_t^{N_i}) \left[\langle \omega_t^{N_i} \otimes \omega_t^{N_i}, H_{e_l} \rangle - 4\pi^2 |l|^2 \langle \omega_t^{N_i}, e_l \rangle \right] dt \\ &\quad + 2\sqrt{2} \varepsilon_{N_i} \sum_{l \in \Lambda} f_l(\omega_t^{N_i}) \sum_{k \in \Lambda_{N_i}} \langle \omega_t^{N_i}, \sigma_k \cdot \nabla e_l \rangle dW_t^{N_i, k} \\ &\quad + \sum_{l, m \in \Lambda} f_{l,m}(\omega_t^{N_i}) [\varepsilon_{N_i}^2 R_{l,m}(\omega_t^{N_i}) + 4\pi^2 \delta_{l,m} |l|^2] dt. \end{aligned}$$

Recalling the operator \mathcal{L}_∞ defined in (2.15), the above formula can be rewritten as

$$dF(\omega_t^{N_i}) = \mathcal{L}_\infty F(\omega_t^{N_i}) dt + \varepsilon_{N_i}^2 \zeta_t^{N_i} dt + dM_t^{N_i}, \quad (2.21)$$

where, by Proposition 3.1,

$$\zeta_t^{N_i} = \sum_{l,m \in \Lambda} f_{l,m}(\omega_t^{N_i}) R_{l,m}(\omega_t^{N_i})$$

is bounded in $L^2([0, T] \times \Theta)$ since $\{f_{l,m}\}_{l,m \in \Lambda}$ are bounded, and the martingale part

$$dM_t^{N_i} = 2\sqrt{2}\varepsilon_{N_i} \sum_{l \in \Lambda} f_l(\omega_t^{N_i}) \sum_{k \in \Lambda_{N_i}} \langle \omega_t^{N_i}, \sigma_k \cdot \nabla e_l \rangle dW_t^{N_i,k}.$$

Note that $M_t^{N_i}$ is a martingale w.r.t. the filtration

$$\mathcal{F}_t^{N_i} = \sigma(\omega_s^{N_i}, W_s^{N_i} : s \leq t),$$

where we denote by $W_s^{N_i} = \{W_s^{N_i,k}\}_{k \in \mathbb{Z}_0^2}$.

Next, we show that the formula (2.21) converges as $i \rightarrow \infty$ in a suitable sense. To this end, we follow the argument of [8, p. 232]. Fix any $0 < s < t \leq T$. Take a real valued, bounded and continuous function $\varphi : C([0, s], H^{-1-} \times \mathbb{R}^{\mathbb{Z}_0^2}) \rightarrow \mathbb{R}$. By (2.21), we have

$$\mathbb{E} \left[\left(F(\omega_t^{N_i}) - F(\omega_s^{N_i}) - \int_s^t \mathcal{L}_\infty F(\omega_r^{N_i}) dr - \varepsilon_{N_i}^2 \int_s^t \zeta_r^{N_i} dr \right) \varphi(\omega^{N_i}, W^{N_i}) \right] = 0.$$

Since $F \in \mathcal{FC}_b^2$ and $\omega_t^{N_i}$ has the law $\mu_{N_i}^0$ for all $t \in [0, T]$, by Lemma 2.2, all the terms in the round bracket are square integrable. Recall that, \mathbb{P} -a.s., (ω^{N_i}, W^{N_i}) converges to (ω, W) in $C([0, T], H^{-1-} \times \mathbb{R}^{\mathbb{Z}_0^2})$. Repeating the treatment of the term $I_3^{N_i}$ in the proof of [12, Proposition 3.6], we can show the convergence of the term involving the nonlinear part in $\mathcal{L}_\infty F$; the other terms are simple. Thus, letting $i \rightarrow \infty$ in the above equality yields

$$\mathbb{E} \left[\left(F(\omega_t) - F(\omega_s) - \int_s^t \mathcal{L}_\infty F(\omega_r) dr \right) \varphi(\omega, W) \right] = 0.$$

The arbitrariness of $0 < s < t$ and $\varphi : C([0, s], H^{-1-} \times \mathbb{R}^{\mathbb{Z}_0^2}) \rightarrow \mathbb{R}$ implies that M^F is a martingale with respect to the filtration $\mathcal{G}_t = \sigma(\omega_s, W_s : s \leq t)$, $t \in [0, T]$. For any $0 \leq s < t \leq T$, we have $\mathcal{F}_s \subset \mathcal{G}_s$, thus

$$\mathbb{E}(M_t^F | \mathcal{F}_s) = \mathbb{E}[\mathbb{E}(M_t^F | \mathcal{G}_s) | \mathcal{F}_s] = \mathbb{E}[M_s^F | \mathcal{F}_s] = M_s^F,$$

since M_s^F is adapted to \mathcal{F}_s . \square

At this stage, taking the cylinder functions $F(\omega) = \langle \omega, e_l \rangle$ and $F(\omega) = \langle \omega, e_l \rangle \langle \omega, e_m \rangle$ ($l, m \in \mathbb{Z}_0^2$) and using Lévy's characterization of Brownian motions, it is easy to show (see [14, Proposition 2.10] for details)

Proposition 2.8. *There exists a family of independent standard Brownian motions $\{W_t^k : t \geq 0\}_{k \in \mathbb{Z}_0^2}$ such that ω and $W = \sum_{k \in \mathbb{Z}_0^2} W_t^{-k} e_k \frac{k^\perp}{|k|}$ solve (1.1) in the following sense: for any $k \in \mathbb{Z}_0^2$,*

$$d\langle \omega_t, e_k \rangle = 2\sqrt{2}\pi|k| dW_t^k - (4\pi^2|k|^2 \langle \omega_t, e_k \rangle - \langle \omega_t \otimes \omega_t, H_{e_k} \rangle) dt.$$

In the remaining part of this section, we follow the arguments at the end of [14, Section 2]. We can rewrite (1.1) in the velocity variable $u = u(\omega)$ as follows:

$$du + b(u) dt = Au dt + \sqrt{2} dW. \quad (2.22)$$

Here, $b(u) = \mathcal{P} \operatorname{div}(u \otimes u)$ and $Au = \mathcal{P} \Delta u$, in which \mathcal{P} is the orthogonal projection onto the space of divergence free vector fields on \mathbb{T}^2 . It is clear that u has trajectories in $C([0, T], H^-(\mathbb{T}^2))$, that is, in $C([0, T], H^{-\delta}(\mathbb{T}^2))$ for any $\delta > 0$. As mentioned at the beginning of this paper, the above equation has been studied intensively in the last three decades. We deduce from Lemma 2.4 and Proposition 2.8 that the process u is a stationary solution to (2.22) in the sense of [7, Definition 4.1]. Let us remark that this definition is based only on the Sobolev regularity of $u \in C([0, T], H^-(\mathbb{T}^2))$; the definition of the nonlinear part $b(u)$ is based on the Galerkin approximation and coincides with our definition, as explained by [14, Theorem A.12] in terms of the vorticity variable.

Similarly to the arguments in [14, Proposition 2.11] (see also [18, Section 3.5]), we can prove

Proposition 2.9. *The uniqueness in law holds for stationary solutions to (2.22).*

Proof. By [16, Theorem 3.14], it is sufficient to show that the pathwise uniqueness holds for stationary solutions of (2.22). Let u_i ($i = 1, 2$) be two stationary solutions to the equation (2.22) in the sense of [7, Definition 4.1], which are defined on the same probability space $(\Theta, \mathcal{F}, \mathbb{P})$, with the same initial data $u_1(0) = u_2(0) = u(0)$ (\mathbb{P} -a.s.) and the same cylindrical Brownian motion $W(t)$, $0 \leq t \leq T$. Then, for $i = 1, 2$, \mathbb{P} -a.s.,

$$u_i(t) = u(0) - \int_0^t b(u_i(s)) ds + \int_0^t Au_i(s) ds + \sqrt{2} W(t), \quad 0 \leq t \leq T.$$

These equations can be rewritten as

$$u_i(t) = e^{tA} u(0) - \int_0^t e^{(t-s)A} b(u_i(s)) ds + \sqrt{2} \int_0^t e^{(t-s)A} dW(s).$$

We extend $W(\cdot)$ to be a two-sided cylindrical Brownian motion on \mathbb{R} (possibly at the price of enlarging $(\Theta, \mathcal{F}, \mathbb{P})$) and define

$$Z(t) = \sqrt{2} \int_{-\infty}^t e^{(t-s)A} dW(s).$$

It is well known that Z is a stationary process with paths in $C([0, T], B_{p,\rho}^\sigma)$ for any $\sigma < 0$, $\rho \geq p \geq 2$ (cf. the last line on p. 196 of [7]). Here, for any $s \in \mathbb{R}$, $B_{p,\rho}^s$ is the Besov space on \mathbb{T}^2 . Note that

$$\sqrt{2} \int_0^t e^{(t-s)A} dW(s) = Z(t) - e^{tA} Z(0),$$

we arrive at

$$u_i(t) - Z(t) = e^{tA} (u(0) - Z(0)) - \int_0^t e^{(t-s)A} b(u_i(s)) ds, \quad i = 1, 2. \quad (2.23)$$

As in [7, Theorem 5.2, p. 196], let $\alpha, \beta, p, \rho, \sigma$ be such that

$$\frac{2}{p} > \alpha > -\sigma > 0, \rho = p \geq 2, \beta \geq 1, -\frac{1}{2} + \frac{1}{p} < \frac{\alpha}{2} - \frac{1}{\beta} < \frac{\sigma}{2}.$$

Using these parameters, we define the following space

$$\mathcal{E} = L^\beta(0, T; B_{p,\rho}^\alpha) \cap C([0, T], B_{p,\rho}^\sigma).$$

Since for any $t \in [0, T]$, $u_i(t)$ is distributed as $\mathcal{N}(0, (-A)^{-1}) = \otimes_{k \in \mathbb{Z}_0^2} \mathcal{N}(0, 1/(4\pi^2|k|^2))$, one has $u_i(t) \in B_{p,\rho}^\sigma$, \mathbb{P} -a.s. (see [3, Proposition 3.1]). We also have $Z(0) \in B_{p,\rho}^\sigma$ (\mathbb{P} -a.s.), thus by [7, Lemma 6.1], we obtain that, \mathbb{P} -a.s., $[0, T] \ni t \mapsto e^{tA}(u(0) - Z(0)) \in \mathcal{E}$. Next, for any $\gamma \geq 1$ and $\varepsilon > 0$, since

$$\mathbb{E} \left(\int_0^T \|b(u_i(t))\|_{H^{-1-\varepsilon}}^\gamma dt \right) = \int_0^T \mathbb{E} (\|b(u_i(t))\|_{H^{-1-\varepsilon}}^\gamma) dt,$$

using estimates on the operator $b(\cdot)$ and the regularity provided by the Gaussian marginal of $u_i(\cdot)$, we can prove $b(u_i(\cdot)) \in L^\gamma(0, T; H^{-1-\varepsilon})$ (\mathbb{P} -a.s.), see the arguments on the top of p. 197 in [7] for details. Therefore, [7, Lemma 6.2] gives us that $\int_0^t e^{(t-s)A} b(u_i(s)) ds \in \mathcal{E}$. Combining these discussions with the equations (2.23), we deduce that $u_i - Z \in \mathcal{E}$ (\mathbb{P} -a.s.) for $i = 1, 2$. By [7, Theorem 5.2, p. 196] (see in particular the arguments on p. 200 after the proof), we obtain, \mathbb{P} -a.s., $u_1(t) = u_2(t)$ for all $t \in [0, T]$. Thus the pathwise uniqueness holds for stationary solutions to (2.22). \square

Recall that $\{Q^N\}_{N \geq 1}$ are the distributions of $(\omega_t^N)_{0 \leq t \leq T}$. Now we can prove the main result of this paper.

Proof of Theorem 1.1. The first assertion follows from Proposition 2.1. Next, by Proposition 2.8, the limit of any weakly convergent subsequence of $\{Q^N\}_{N \geq 1}$ is a stationary solution to (1.1); moreover, it is shown in Proposition 2.9 that the stationary solutions to (1.1) are unique in law. Thus, we deduce the second assertion from the tightness of the family $\{Q^N\}_{N \geq 1}$. \square

3. Appendix

Recall the expressions of ω_0^N in (2.1) and of $C_{k,l}$ in (2.16). In this part we prove the following technical result.

Proposition 3.1. *For any $l, m \in \mathbb{Z}_0^2$ fixed, the sequence of random variables*

$$R_{l,m}(\omega_0^N) = \sum_{k \in \Lambda_N} C_{k,l} C_{k,m} (\langle \omega_0^N, e_k e_l \rangle \langle \omega_0^N, e_k e_m \rangle - \delta_{l,m})$$

is bounded in $L^2(\Theta, \mathcal{F}, \mathbb{P})$.

The proof of the above assertion follows the idea of [13, Appendix 6], with some combinatorial flavor here. Since l, m are fixed, we write R_N instead of $R_{l,m}(\omega_0^N)$ for simplicity. We deal with the two cases $l \neq m$ and $l = m$ in the two subsections separately.

3.1. Case 1: $l \neq m$

The definition of ω_0^N yields

$$R_N = \frac{1}{N} \sum_{k \in \Lambda_N} C_{k,l} C_{k,m} \sum_{r,s=1}^N \xi_r \xi_s (e_k e_l)(X_0^r)(e_k e_m)(X_0^s),$$

therefore,

$$\begin{aligned} R_N^2 &= \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r,s,r',s'=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \xi_r \xi_s \xi_{r'} \xi_{s'} \\ &\quad \times (e_k e_l)(X_0^r)(e_k e_m)(X_0^s)(e_{k'} e_l)(X_0^{r'})(e_{k'} e_m)(X_0^{s'}). \end{aligned}$$

Recall that the two families $\{\xi_r\}_{r \geq 1}$ and $\{X_0^r\}_{r \geq 1}$ are independent, and $\{\xi_r\}_{r \geq 1}$ is an i.i.d. sequence of $N(0,1)$ random variables, while $\{X_0^r\}_{r \geq 1}$ consists of i.i.d. \mathbb{T}^2 -valued uniform random variables. We have

$$\begin{aligned} \mathbb{E} R_N^2 &= \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r,s,r',s'=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}(\xi_r \xi_s \xi_{r'} \xi_{s'}) \\ &\quad \times \mathbb{E}[(e_k e_l)(X_0^r)(e_k e_m)(X_0^s)(e_{k'} e_l)(X_0^{r'})(e_{k'} e_m)(X_0^{s'})] \end{aligned}$$

and by the Isserlis–Wick theorem,

$$\begin{aligned} \mathbb{E}(\xi_r \xi_s \xi_{r'} \xi_{s'}) &= \mathbb{E}(\xi_r \xi_s) \mathbb{E}(\xi_{r'} \xi_{s'}) + \mathbb{E}(\xi_r \xi_{r'}) \mathbb{E}(\xi_s \xi_{s'}) + \mathbb{E}(\xi_r \xi_{s'}) \mathbb{E}(\xi_s \xi_{r'}) \\ &= \delta_{r,s} \delta_{r',s'} + \delta_{r,r'} \delta_{s,s'} + \delta_{r,s'} \delta_{s,r'}. \end{aligned}$$

As a result, we can write

$$\mathbb{E} R_N^2 = S_1 + S_2 + S_3. \quad (3.1)$$

3.1.1. The quantity S_1

We have

$$S_1 = \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r,r'=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k^2 e_l e_m)(X_0^r)(e_{k'}^2 e_l e_m)(X_0^{r'})].$$

Note that X_0^r and $X_0^{r'}$ are independent if $r \neq r'$, hence

$$\begin{aligned} S_1 &= \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{1 \leq r \neq r' \leq N} C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k^2 e_l e_m)(X_0^r)] \mathbb{E}[(e_{k'}^2 e_l e_m)(X_0^{r'})] \\ &\quad + \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k^2 e_{k'}^2 e_l^2 e_m^2)(X_0^r)]. \end{aligned} \quad (3.2)$$

We denote the two terms by $S_{1,1}$ and $S_{1,2}$, respectively.

First, since X_0^r ($r \in \mathbb{N}$) is a uniformly distributed random variable on the torus \mathbb{T}^2 , we obtain

$$\begin{aligned} S_{1,1} &= \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{1 \leq r \neq r' \leq N} C_{k,l} C_{k,m} C_{k',l} C_{k',m} \int e_k^2 e_l e_m \, dx \int e_{k'}^2 e_l e_m \, dx \\ &= \frac{N^2 - N}{N^2} \sum_{k,k' \in \Lambda_N} C_{k,l} C_{k,m} C_{k',l} C_{k',m} \int e_k^2 e_l e_m \, dx \int e_{k'}^2 e_l e_m \, dx \\ &= \left(1 - \frac{1}{N}\right) \left(\sum_{k \in \Lambda_N} C_{k,l} C_{k,m} \int e_k^2 e_l e_m \, dx \right)^2. \end{aligned}$$

Note that $C_{-k,l} = -C_{k,l}$ and $e_k^2 + e_{-k}^2 \equiv 2$ for any $k \in \mathbb{Z}_0^2$, we have

$$\sum_{k \in \Lambda_N} C_{k,l} C_{k,m} e_k^2 = \sum_{k \in \Lambda_N \cap \mathbb{Z}_+^2} (C_{k,l} C_{k,m} e_k^2 + C_{-k,l} C_{-k,m} e_{-k}^2) = 2 \sum_{k \in \Lambda_N \cap \mathbb{Z}_+^2} C_{k,l} C_{k,m} \quad (3.3)$$

is a constant. This implies

$$S_{1,1} = 0 \quad (3.4)$$

since $\int e_l e_m \, dx = 0$ for $l \neq m$.

Regarding the term $S_{1,2}$, we have

$$\begin{aligned} S_{1,2} &= \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \int e_k^2 e_{k'}^2 e_l^2 e_m^2 \, dx \\ &= \frac{1}{N} \sum_{k,k' \in \Lambda_N} C_{k,l} C_{k,m} C_{k',l} C_{k',m} \int e_k^2 e_{k'}^2 e_l^2 e_m^2 \, dx. \end{aligned}$$

As $|e_k(x)| \leq \sqrt{2}$ for all $x \in \mathbb{T}^2$ and $k \in \mathbb{Z}_0^2$, we deduce that

$$|S_{1,2}| \leq \frac{16}{N} \sum_{k,k' \in \Lambda_N} \frac{|l|^2 |m|^2}{|k|^2 |k'|^2} = \frac{16}{N} |l|^2 |m|^2 \left(\sum_{k \in \Lambda_N} \frac{1}{|k|^2} \right)^2 \leq C(l, m) \frac{(\log N)^2}{N}.$$

Combining the above estimate with (3.2) and (3.4), we arrive at

$$|S_1| \leq C_1 \frac{(\log N)^2}{N} \quad \text{for all } N \geq 2. \quad (3.5)$$

3.1.2. The quantity S_2

We have

$$S_2 = \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r,s=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k e_{k'} e_l^2)(X_0^r)(e_k e_{k'} e_m^2)(X_0^s)].$$

Similar to (3.2), the above quantity can be decomposed as

$$\begin{aligned} S_2 &= \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{1 \leq r \neq s \leq N} C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k e_{k'} e_l^2)(X_0^r)] \mathbb{E}[(e_k e_{k'} e_m^2)(X_0^s)] \\ &\quad + \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k^2 e_{k'}^2 e_l^2 e_m^2)(X_0^r)], \end{aligned}$$

which are denoted as $S_{2,1}$ and $S_{2,2}$. Note that

$$|S_{2,2}| = |S_{1,2}| \leq C_1 \frac{(\log N)^2}{N} \quad \text{for all } N \geq 2.$$

Next, using the fact that X_0^r is uniformly distributed on \mathbb{T}^2 and the Cauchy inequality,

$$\begin{aligned} |S_{2,1}| &= \left| \left(1 - \frac{1}{N}\right) \sum_{k,k' \in \Lambda_N} C_{k,l} C_{k,m} C_{k',l} C_{k',m} \int e_k e_{k'} e_l^2 dx \int e_k e_{k'} e_m^2 dx \right| \\ &\leq \left[\sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \left(\int e_k e_{k'} e_l^2 dx \right)^2 \right]^{1/2} \left[\sum_{k,k' \in \Lambda_N} C_{k,m}^2 C_{k',m}^2 \left(\int e_k e_{k'} e_m^2 dx \right)^2 \right]^{1/2}. \end{aligned}$$

It suffices to estimate one of the two terms. Intuitively, the quantity

$$I_N := \sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \left(\int e_k e_{k'} e_l^2 dx \right)^2 \quad (3.6)$$

is bounded as $N \rightarrow \infty$ due to the fact that the integral $\int e_k e_{k'} e_l^2 dx \neq 0$ imposes a constraint on k and k' , e.g. $k = k'$ or $2l = k + k'$. Such constraint reduces the degree of freedom of k and k' , and implies

$$I_N \leq C_l \sum_{k \in \Lambda_N} \frac{1}{|k|^4} \leq C_l \sum_{k \in \mathbb{Z}_0^2} \frac{1}{|k|^4} \quad \text{for all } N \geq 1.$$

We refer the readers to [13, Section 6.1.2] for details.

To summarize, we obtain

$$|S_2| \leq C_2 \left(1 + \frac{(\log N)^2}{N} \right). \quad (3.7)$$

3.1.3. The quantity S_3

Similar computations as above lead to

$$\begin{aligned} S_3 &= \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r,s=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k e_{k'} e_l e_m)(X_0^r)(e_k e_{k'} e_l e_m)(X_0^s)] \\ &= \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{1 \leq r \neq s \leq N} C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k e_{k'} e_l e_m)(X_0^r)] \mathbb{E}[(e_k e_{k'} e_l e_m)(X_0^s)] \\ &\quad + \frac{1}{N^2} \sum_{k,k' \in \Lambda_N} \sum_{r=1}^N C_{k,l} C_{k,m} C_{k',l} C_{k',m} \mathbb{E}[(e_k^2 e_{k'}^2 e_l^2 e_m^2)(X_0^r)]. \end{aligned}$$

Again, the last quantity is dominated by a constant multiple of $(\log N)^2/N$. The first one on the right hand side is equal to

$$\left(1 - \frac{1}{N}\right) \sum_{k,k' \in \Lambda_N} C_{k,l} C_{k,m} C_{k',l} C_{k',m} \left(\int e_k e_{k'} e_l e_m dx \right)^2,$$

which, due to the same reason as for the term (3.6), is bounded in N . Therefore, we still have

$$|S_3| \leq C_3 \left(1 + \frac{(\log N)^2}{N} \right).$$

Combining the above inequality with (3.1), (3.5) and (3.7), we conclude the assertion in the first case $l \neq m$.

3.2. Case 2: $l = m$

In this case,

$$R_N = \sum_{k \in \Lambda_N} C_{k,l}^2 (\langle \omega_0^N, e_k e_l \rangle^2 - 1).$$

Consequently,

$$\mathbb{E} R_N^2 = \sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \mathbb{E} (\langle \omega_0^N, e_k e_l \rangle^2 \langle \omega_0^N, e_{k'} e_l \rangle^2 - \langle \omega_0^N, e_k e_l \rangle^2 - \langle \omega_0^N, e_{k'} e_l \rangle^2 + 1). \quad (3.8)$$

By the definition of ω_0^N ,

$$\begin{aligned} \mathbb{E} (\langle \omega_0^N, e_k e_l \rangle^2) &= \frac{1}{N} \sum_{r,s=1}^N \mathbb{E} (\xi_r \xi_s) \mathbb{E} [(e_k e_l)(X_0^r)(e_k e_l)(X_0^s)] \\ &= \frac{1}{N} \sum_{r=1}^N \mathbb{E} [(e_k^2 e_l^2)(X_0^r)] = \int e_k^2 e_l^2 dx. \end{aligned}$$

As a result,

$$\sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \mathbb{E} (\langle \omega_0^N, e_k e_l \rangle^2) = \left(\sum_{k' \in \Lambda_N} C_{k',l}^2 \right) \sum_{k \in \Lambda_N} C_{k,l}^2 \int e_k^2 e_l^2 dx. \quad (3.9)$$

Similar to (3.3),

$$\sum_{k \in \Lambda_N} C_{k,l}^2 e_k^2 = 2 \sum_{k \in \Lambda_N \cap \mathbb{Z}_+^2} C_{k,l}^2 = \sum_{k \in \Lambda_N} C_{k,l}^2 = \frac{1}{2} \varepsilon_N^{-2} |l|^2, \quad (3.10)$$

where the last step is due to (2.17). Substituting this result into (3.9) yields

$$\sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \mathbb{E} (\langle \omega_0^N, e_k e_l \rangle^2) = \frac{1}{4} \varepsilon_N^{-4} |l|^4.$$

Analogously,

$$\sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \mathbb{E} (\langle \omega_0^N, e_{k'} e_l \rangle^2) = \frac{1}{4} \varepsilon_N^{-4} |l|^4.$$

Combining these facts with (3.8), we obtain

$$\mathbb{E} R_N^2 = \sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \mathbb{E} (\langle \omega_0^N, e_k e_l \rangle^2 \langle \omega_0^N, e_{k'} e_l \rangle^2) - \frac{1}{4} \varepsilon_N^{-4} |l|^4. \quad (3.11)$$

Now we compute the expectation on the right hand side of (3.11). We have

$$\langle \omega_0^N, e_k e_l \rangle^2 \langle \omega_0^N, e_{k'} e_l \rangle^2 = \frac{1}{N^2} \sum_{r,s,r',s'=1}^N \xi_r \xi_s \xi_{r'} \xi_{s'} (e_k e_l)(X_0^r)(e_{k'} e_l)(X_0^s)(e_{k'} e_l)(X_0^{r'})(e_{k'} e_l)(X_0^{s'}).$$

The Isserlis–Wick theorem implies

$$\begin{aligned} \mathbb{E}(\langle \omega_0^N, e_k e_l \rangle^2 \langle \omega_0^N, e_{k'} e_l \rangle^2) &= \frac{1}{N^2} \sum_{r,r'=1}^N \mathbb{E}[(e_k^2 e_l^2)(X_0^r)(e_{k'}^2 e_l^2)(X_0^{r'})] \\ &\quad + \frac{2}{N^2} \sum_{r,s=1}^N \mathbb{E}[(e_k e_{k'} e_l^2)(X_0^r)(e_k e_{k'} e_l^2)(X_0^s)] \\ &=: J_1 + J_2. \end{aligned} \quad (3.12)$$

First,

$$J_1 = \frac{1}{N^2} \sum_{1 \leq r \neq r' \leq N} \mathbb{E}[(e_k^2 e_l^2)(X_0^r)] \mathbb{E}[(e_{k'}^2 e_l^2)(X_0^{r'})] + \frac{1}{N^2} \sum_{r=1}^N \mathbb{E}[(e_k^2 e_k^2 e_l^4)(X_0^r)]$$

which are denoted by $J_{1,1}$ and $J_{1,2}$, respectively. Note that

$$J_{1,1} = \left(1 - \frac{1}{N}\right) \int e_k^2 e_l^2 dx \int e_{k'}^2 e_l^2 dx$$

and

$$J_{1,2} = \frac{1}{N} \int e_k^2 e_{k'}^2 e_l^4 dx \leq \frac{16}{N}.$$

Moreover,

$$\sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \cdot J_{1,1} = \left(1 - \frac{1}{N}\right) \left(\sum_{k \in \Lambda_N} C_{k,l}^2 \int e_k^2 e_l^2 dx \right)^2 = \frac{1}{4} \left(1 - \frac{1}{N}\right) \varepsilon_N^{-4} |l|^4,$$

where the last step is due to (3.10). Therefore,

$$\sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \cdot J_1 = \frac{1}{4} \varepsilon_N^{-4} |l|^4 + O\left(\frac{(\log N)^2}{N}\right). \quad (3.13)$$

It remains to estimate J_2 in (3.12). Similarly,

$$J_2 = \frac{2}{N^2} \sum_{1 \leq r \neq s \leq N} \mathbb{E}[(e_k e_{k'} e_l^2)(X_0^r)] \mathbb{E}[(e_k e_{k'} e_l^2)(X_0^s)] + \frac{2}{N^2} \sum_{r=1}^N \mathbb{E}[(e_k^2 e_{k'}^2 e_l^4)(X_0^r)].$$

We write $J_{2,1}$ and $J_{2,2}$ for the two terms. We still have

$$J_{2,2} = \frac{2}{N} \int e_k^2 e_{k'}^2 e_l^4 dx \leq \frac{32}{N}.$$

Next,

$$J_{2,1} = 2 \left(1 - \frac{1}{N}\right) \left(\int e_k e_{k'} e_l^2 dx \right)^2.$$

As a result,

$$\sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \cdot J_2 = 2 \left(1 - \frac{1}{N}\right) \sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \left(\int e_k e_{k'} e_l^2 dx \right)^2 + O\left(\frac{(\log N)^2}{N}\right).$$

Note that the sum in the first quantity is equal to I_N defined in (3.6). Therefore,

$$\left| \sum_{k,k' \in \Lambda_N} C_{k,l}^2 C_{k',l}^2 \cdot J_2 \right| \leq C_4 + O\left(\frac{(\log N)^2}{N}\right).$$

Combining this estimate with (3.11)–(3.13), we finally get

$$\mathbb{E} R_N^2 \leq C_4 + O\left(\frac{(\log N)^2}{N}\right).$$

The proof is complete.

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