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Asymptotic Stability for Tracking Control of Nonlinear Uncertain Dynamical Systems Described by Differential Inclusions

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In this paper, we will study the tracking control problem for nonlinear uncertain dynamical systems. Two generalized feedback control inputs have been proposed such that the feedback-controlled systems satisfy the complete tracking control property with exponential asymptotic stability and the trajectories of the systems are steered to the pre-specified observation map with an exponential convergence rate. Moreover, an estimate of the tracking time of the trajectories attaining the observation map has also been given. An example inspired from a guided missile problem illustrates the use of our main results. © 2001 Academic Press

Key Words: tracking control property; differential inclusion; feedback-controlled system; uncertain dynamical systems; exponential asymptotic stability.

1. INTRODUCTION

In this paper, we will study the tracking control problem for a class of nonlinear uncertain dynamical systems described by differential inclusions. The tracking control problem for a class of uncertain dynamical systems without the feedback-controlled observer under a single-valued differentiable observation map has been studied by Chen *et al*. [5]. The authors [5] designed a generalized feedback control such that the nonlinear uncertain dynamical system satisfies the tracking property under a single-valued observation map. Here, we will investigate the tracking control problem for the nonlinear uncertain dynamical system with the feedback-controlled

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observer and tracker, and prove that the nonlinear uncertain dynamical system satisfies the complete tracking control property with exponential asymptotic stability. These results play important roles in the theory of uncertain dynamical systems about tracking control missiles (see Example 5.1 in Section 5).

In most earlier work on tracking control for nonlinear dynamical systems, the dynamics of the systems are described by usual ordinary differential equations (see [4, 6, 10]). Note, however, that if control synthesis is an objective, then discontinuous feedback is a natural candidate in many problems of stabilization and optimization. These make the traditional theory of ordinary differential equations unapplicable for both analysis and synthesis purposes, i.e., the traditional Carathéodory concepts become useless (see [8, Sect. 1]), and uncertainty may be an intrinsic feature (see [7–9]). In this paper, the approach is in the spirit of [8, 9] but with a fundamental distinction: in [8, 9], functional properties of the uncertain systems are assumed that ensure, for any control and any admissible realization of uncertainty, the classical (Carathéodory) concept of solution of the differential equation is adequate. In the present paper, nonlinear uncertain dynamical systems are more generally defined via differential inclusions, the right-hand side of which takes the form of two set-valued maps as

$$\begin{cases} \dot{x}(t) \in F(x(t), y(t), u_1(t)) \\ \dot{y}(t) \in G(x(t), y(t), u_2(t)), \end{cases} \tag{1.1a}$$

$$F(x, y, u_1) :\equiv f(x, y) + P(x, y)u_1 + F_\alpha(x, y), \tag{1.1b}$$

$$G(x, y, u_2) :\equiv g(x, y) + Q(x, y)u_2 + Q(x, y)\left[F_\beta(x, y) + F_\gamma(u_2)\right], \tag{1.1c}$$

where $t \in [0, \infty)$ is the time variable, $u_1(t) \in \Re^q$ and $u_2(t) \in \Re^p$ are the control inputs, and $x(t) \in \Re^n$, $y(t) \in \Re^m$ denote the states of the system. The set-valued maps $F_\alpha(x, y) \subseteq \Re^n$, $F_\beta(x, y) \subseteq \Re^p$, and $F_\gamma(u) \subseteq \Re^p$ model the system uncertainty. The functions: f: $\Re^n \times \Re^m \to \Re^n$, g: $\Re^n \times \Re^m \to \Re^m$, P: $\Re^n \times \Re^m \to \Re^{n\times q}$, and Q: $\Re^n \times \Re^m \to \Re^{m\times p}$ are single-valued continuous on $\Re^n \times \Re^m$ with linear growth. With the state feedback inputs $u_1(t) = u_1(x(t), y(t))$ and $u_2(t) = u_2(x(t), y(t))$, two (tracker-observer) feedback-controlled systems (1.1) become

$$\begin{cases} \dot{x}(t) \in F_c(x(t), y(t)) :\equiv F(x(t), y(t), u_1(x(t), y(t))) \\ \dot{y}(t) \in G_c(x(t), y(t)) :\equiv G(x(t), y(t), u_2(x(t), y(t))). \end{cases} \tag{1.2}$$

When $P(x, y) = \{0\}$ for all $x \in \Re^n$ and $y \in \Re^m$, observe that the system (1.2) may be regarded as the model of the feedback-controlled uncertain dynamical system (1.3) without feedback-controlled input $u_1(x(t), y(t))$ described as

$$\begin{cases} \dot{x}(t) \in F(x(t), y(t)) \\ \dot{y}(t) \in G(x(t), y(t), u_2(x(t), y(t))). \end{cases} \tag{1.3}$$

This implies that the system (1.3) is a special case of the feedback-controlled system (1.1).

When $F_\alpha(x, y) = F_\beta(x, y) = F_\gamma(u_2) = \{0\}$ for all $x \in \Re^n$, $y \in \Re^m$, and $u \in \Re^p$, observe that the original system (1.1) may be regarded as the model of the nominal system (1.3) without uncertainty described as

$$\begin{cases} \dot{x}(t) = f(x(t), y(t)) + P(x(t), y(t))u_1(t) \\ \dot{y}(t) = g(x(t), y(t)) + Q(x(t), y(t))u_2(t). \end{cases} \tag{1.4}$$

This implies that the nominal system (1.4) is a special case of the feedback-controlled system (1.1) subject to uncertainty.

Throughout this paper, let $H(\cdot)$ be a single-valued continuously differentiable observation map, where $H: \Re^n \to \Re^m$ is Lipschitz of rank K_H; that is, there exists a constant $K_H \geq 0$ such that

$$\|H(x) - H(y)\| \leq K_H \|x - y\| \qquad \text{for all } x, y \in \Re^n.$$

We will consider the complete tracking control problem for nonlinear uncertain dynamical systems with exponential asymptotic stability. The goal is to find a pair of generalized feedback control inputs $u_1 = u_1(x, y)$ and $u_2 = u_2(x, y)$ such that for any initial state $(x_0, y_0) \in Graph(H)$, all solutions $(x(\cdot), y(\cdot))$ of the system (1.1), starting from (x_0, y_0), satisfy $y(t) = H(x(t))$ for all $t \geq 0$ and $\|J(x(t))\| \leq \alpha_0 \exp(-\beta_0 \cdot t) \to 0$ as $t \to \infty$, where $J: \Re^n \to \Re^q$ is a single-valued continuously differentiable Lipschitz function on $\Re^n$; α_0, β_0 are positive constants; and $\|\cdot\|$ denotes the Euclidean norm or the corresponding induced norm of a matrix. Furthermore, if $(x_0, y_0) \notin Graph(H)$, namely y_0 is not traced by $H(x_0)$ at initial state, to construct a pair of generalized feedback control inputs, there exists a constant $T \geq 0$ such that the nonlinear uncertain dynamical systems (1.2) enjoy the complete tracking control property with exponential asymptotic stability along $J(x(t))$ after a finite time T. Moreover, an estimate of the tracking time T of all trajectories $y(t)$ attaining the observation map $H(x(t))$ is given.

2. DEFINITIONS OF TRACKING CONTROL

For convenience, the norm $\|F(x)\|$ of a set-valued map $F: X \mapsto Y$ is defined by $\|F(x)\| := \sup_{y \in F(x)} \|y\|$ for all $x \in Dom(F)$, where $(X, \| \ \|)$ and $(Y, \| \ \|)$ are normed linear spaces. Throughout the paper, let $H: \Re^n \to \Re^m$ and $J: \Re^n \to \Re^q$ be single-valued continuously differentiable Lipschitz functions. We define the nonlinear uncertain dynamical system described by differential inclusions satisfying the complete tracking control property with exponential asymptotic stability as follows.

DEFINITION 2.1. We say that the system (1.1) under H satisfies the complete tracking control property with asymptotic stability along $J(x(t))$ to $0 \in \Re^q$ if for any initial state $(x_0, y_0) \in Graph(H)$, there exists a pair of feedback control inputs $u_1 = u_1(x, y)$ and $u_2 = u_2(x, y)$ such that all solutions $(x(\cdot), y(\cdot))$ of the differential inclusions (1.1) starting at (x_0, y_0) defined on $[0, \infty)$ satisfy $y(t) = H(x(t))$ for all $t \geq 0$ and $\|J(x(t))\| \to 0$ as $t \to \infty$. Moreover, if $J(x(t)) = Z(x(t)) - z_0 \in \Re^q$, then we say that the systems (1.1) under H satisfy the tracking control property with asymptotic stability along $Z(x(t))$ to $z_0 \in \Re^q$.

DEFINITION 2.2. We say that the system (1.1) under H satisfies the complete tracking control property with exponential asymptotic stability along $J(x(t))$ if for any initial state $(x_0, y_0) \in Graph(H)$, there exists a pair of feedback control inputs $u_1 = u_1(x, y)$ and $u_2 = u_2(x, y)$ such that all solutions $(x(\cdot), y(\cdot))$ of the differential inclusions (1.1) starting at (x_0, y_0) defined on $[0, \infty)$ satisfy $y(t) = H(x(t))$ for all $t \geq 0$ and $\|J(x(t))\| \leq \alpha_0 \exp(-\beta_0 \cdot t)$, where α_0, β_0 are positive constants.

DEFINITION 2.3. We say that the system (1.1) under H satisfies the complete tracking control property with exponential asymptotic stability along $J(x(t))$ after a finite time if there exists a constant $T \geq 0$ such that for any initial state $(x_0, y_0) \notin Graph(H)$, all solutions $(x(\cdot), y(\cdot))$ of the differential inclusions (1.1) starting at (x_0, y_0) defined on $[0, \infty)$ satisfy $y(t) = H(x(t))$ for all $t \geq T$ and $\|J(x(t))\| \leq \alpha_0 \exp(-\beta_0 \cdot t)$.

Remark 2.1. Clearly, by Definition 2.2 and Definition 2.3, we obtain that the complete tracking control property with exponential asymptotic stability along $J(x(t))$ implies the complete tracking control property with asymptotic stability along $J(x(t))$ to 0.

3. ASSUMPTIONS AND DESIGNS OF CONTROL INPUTS FOR UNCERTAIN DYNAMICAL SYSTEMS

In this paper, we consider the nonlinear uncertain dynamical system (1.1) described by differential inclusions,

$$\begin{cases} \dot{x}(t) \in F(x(t), y(t), u_1(t)) \\ \dot{y}(t) \in G(x(t), y(t), u_2(t)), \end{cases}$$

$$F(x, y, u_1) :\equiv f(x, y) + P(x, y)u_1 + F_\alpha(x, y),$$

$$G(x, y, u_2) :\equiv g(x, y) + Q(x, y)u_2 + Q(x, y)\left[F_\beta(x, y) + F_\gamma(u_2)\right],$$

satisfying the following conditions.

3.1. *Assumptions*

Throughout the paper, the following assumptions are made.

(A1) The single-valued functions $f(x, y)$, $g(x, y)$, $P(x, y)$, and $Q(x, y)$ are continuous on $\Re^n \times \Re^m$ with linear growth;

(A2) $F_\alpha(x, y)$, $F_\beta(x, y)$, and $F_\gamma(u_2)$ are upper semicontinuous with convex and compact values for all $x \in \Re^n$, $y \in \Re^m$, and $u_2 \in \Re^p$;

(A3) $\|F_\alpha(x, y)\| \leq k_\alpha(x, y)$, $\|F_\beta(x, y)\| \leq k_\beta(x, y)$, and $\|F_\gamma(u_2)\| \leq \eta\|u_2\|$ for all $x \in \Re^n$, $y \in \Re^m$, $u_2 \in \Re^p$, where $k_\alpha(x, y)$ and $k_\beta(x, y)$ are nonnegative real-valued functions with linear growth, and $\eta < 1$;

(A4) $\|Q(x, y)F_\gamma(u_2)\| \leq r_1\|Q(x, y)u_2\|$ for all $x \in \Re^n$, $y \in \Re^m$, and $u_2 \in \Re^p$, where r_1 is a known positive constant;

(A5) $\|P(x, y)u_1\| \leq r_2\|u_1\|$ and $\|Q(x, y)u_2\| \leq r_3\|u_2\|$ for all $x \in \Re^n$, $y \in \Re^m$, $u_1 \in \Re^q$, and $u_2 \in \Re^p$, where r_2 and r_3 are known positive constants;

(A6) $\frac{\partial J(x)}{\partial x}P(x, y)$ is invertible for all $x \in \Re^n$, $y \in \Re^m$, and $0 < K_P = \sup_{x \in \Re^n,\, y \in \Re^m}\{\|\frac{\partial J(x)}{\partial x}P(x, y)\|^{-1}\} < \infty$, where J: $\Re^n \to \Re^q$ is a single-valued continuously differentiable Lipschitz function with the Lipschitz constant $K_J > 0$;

(A7) $\operatorname{rank}[Q(x, y)] = m \leq p$ for all $x \in \Re^n$, $y \in \Re^m$, and $0 < \|(QQ^T)^{-1}Q\|_\infty :\equiv \sup_{x \in \Re^n,\, y \in \Re^m}\{\|[Q(x, y)Q^T(x, y)]^{-1}Q(x, y)\|\} < \infty$.

Remark 3.1. Note that the existence of solutions $(x(\cdot), y(\cdot))$ defined on $[0, \infty)$ for the closed-loop system (1.2), satisfying assumptions (A1)–(A6), is guaranteed. More precisely, the assumptions (A1)–(A6) imply that $F_c(\cdot,\cdot)$ and $G_c(\cdot,\cdot)$ enjoy linear growth (see Theorem 3.1 in Section 3). The assumption (A7) is used for the tracking property and the estimation of tracking time (see Theorem 4.2 in Section 4).

Remark 3.2. For the nominal system (1.4), take $k_\alpha(x, y) = k_\beta(x, y) = \eta = r_1 = 0$ in Assumption 3.1. Then assumptions (A1)–(A4) always hold. The existence of solutions $(x(\cdot), y(\cdot))$ defined on $[0, \infty)$ for the system (1.4), satisfying assumptions (A5)–(A6), is also guaranteed.

3.2. *Design of Control Inputs for Uncertain Dynamical Systems*

Now, we consider the nonlinear uncertain dynamical system (1.2) described by differential inclusions with a pair of control inputs u_1 and u_2 as

$$\begin{cases} u_1 = u_{1n}(x, y) + u_{1c}(x, y) \\ u_2 = u_{2n}(x, y) + u_{2c}(x, y), \end{cases} \tag{3.1}$$

$$\begin{aligned} u_{1n}(x, y) = &-\left(\frac{\partial J(x)}{\partial x} P(x, y)\right)^{-1} \frac{\partial J(x)}{\partial x} f(x, y) \\ &- \sigma P^T(x, y)\left(\frac{\partial J(x)}{\partial x}\right)^T J(x) - \rho\left(\frac{\partial J(x)}{\partial x} P(x, y)\right)^{-1} J(x), \end{aligned} \tag{3.2}$$

$$u_{1c}(x, y) = -k_1(x, y)\Psi_1\left[\left(\frac{\partial J(x)}{\partial x} P(x, y)\right)^{-1} J(x)\right], \tag{3.3}$$

$$\begin{aligned} Q(x, y)u_{2n}(x, y) = &A(y - H(x)) - g(x, y) \\ &+ \frac{\partial H(x)}{\partial x}[f(x, y) + P(x, y)u_1], \end{aligned} \tag{3.4}$$

$$u_{2c}(x, y) = -k_2(x, y) \cdot \Psi_2[Q^T(x, y)M(y - H(x))], \tag{3.5}$$

where σ and ρ are positive constants; M is the positive definite symmetric $m \times m$ matrix satisfying the following Lyapunov equation,

$$A^T M + MA = -L, \tag{3.6}$$

L is an arbitrary positive definite symmetric $m \times m$ matrix and A is an Hurwitz $m \times m$ matrix; $k_1(x, y)$ and $k_2(x, y)$ are any positive rear-valued continuous functions with linear growth satisfying

$$
\begin{aligned}
& k_1(x, y) \geq K_P \cdot K_J \cdot k_\alpha(x, y), \qquad k_2(x, y) \geq k_0(x, y), \\
& \text{for all } x \in \Re^n, \ y \in \Re^m, \qquad\qquad (3.7) \\
& k_0(x, y) :\equiv (1 - \eta)^{-1} \cdot \left[\left\| Q^+(x, y) \frac{\partial H(x)}{\partial x} \right\| k_\alpha(x, y) \right. \\
& \qquad \left. + k_\beta(x, y) + \eta \| u_{2n}(x, y) \| + \delta \right],
\end{aligned}
$$

K_J is the Lipschitz constant of $J(\cdot)$, Q^+ is the right inverse of Q, and δ is any positive constant; the extended sign multifunctions $\Psi_1\colon \Re^q \mapsto \Re^q$ and $\Psi_2\colon \Re^p \mapsto \Re^p$ are upper semicontinuous on $\Re^q$ and $\Re^p$, respectively,

$$
\Psi_1(\xi) :\equiv \begin{cases} \xi / \| \xi \| & \text{if } \xi \neq 0 \\ \{ \theta \in \Re^q \mid \| \theta \| \leq 1 \} & \text{if } \xi = 0. \end{cases}
$$

Remark 3.3. In (3.6), the design of Hurwitz matrix A and the positive definite symmetric matrix M depends on the exponential asymptotic rate of convergence about the trajectory $y(t)$ to the observation map $H(x(t))$ (see Theorem 4.4 in Section 4).

Remark 3.4. For the nominal system (1.4), take $k_\alpha(x, y) = k_\beta(x, y) = \eta = r_1 = 0$. Then we can obtain the $k_1(x, y) = 0$ and $k_2(x, y) = \delta$. This implies that the control input $u_{1c}(x, y) = 0$ in (3.1).

3.3. *The Existence of a Solution for the Control System*

For the existence of solutions of differential inclusions (1.2), in general, $F_c(\cdot,\cdot)$ and $G_c(\cdot,\cdot)$ need to satisfy the assumption of upper semicontinuity. More precisely, if $F_c(\cdot,\cdot)$ and $G_c(\cdot,\cdot)$ are upper semicontinuous with convex and compact values, for any initial state $(x_0, y_0) \in Graph(H)$, then there exists a positive T and a solution $(x(\cdot), y(\cdot))$ defined on $[0, T]$ for the system (1.2) such that either $T = \infty$ or $T < \infty$ and $\limsup_{t \to T^-} \|(x(t), y(t))\| = \infty$ (see [1, p. 98, Theorem 3; 2, p. 390, Theorem 10.1.3]). Further more adequate information—a priori estimates on the growth of $F_c(\cdot,\cdot)$ and $G_c(\cdot,\cdot)$—allow exclusion of the case when $\limsup_{t \to T^-} \|(x(t), y(t))\| = \infty$. This is the case for instance when both $F_c(\cdot,\cdot)$ and $G_c(\cdot,\cdot)$ are bounded. In general, we can take $T = \infty$ when $F_c(\cdot,\cdot)$ and $G_c(\cdot,\cdot)$ enjoy linear growth (see [3, p. 62]); that is, there exist positive constants c_1, c_2 such that

$$\|F_c(x, y)\| \le c_1(\|(x, y)\| + 1) \qquad \text{and}$$
$$\|G_c(x, y)\| \le c_2(\|(x, y)\| + 1) \text{ for all } (x, y) \in \Re^n \times \Re^m.$$

We say that F is a Marchaud map if it is nontrivial, upper semicontinuous, and has compact convex images and linear growth. Clearly, any single-valued Lipschitz map is a Marchaud map.

For the existence of a solution $x(\cdot)$ defined on $[0, \infty)$ for the closed-loop system (1.2), we only show that $F_c(x, y)$ and $G_c(x, y)$ are Marchaud maps as follows.

THEOREM 3.1. *The feedback-controlled systems of* (1.2) *satisfy the assumptions* (A1)–(A6), *subject to the controller* (3.1) *with* (3.2)–(3.7). *Then we have that $F_c(x, y)$ and $G_c(x, y)$ in* (1.2) *are Marchaud maps.*

Proof. By (A1) and (A2), $F_c(x, y)$ and $G_c(x, y)$ are upper semicontinuous with convex and compact values for all $x \in \Re^n$, $y \in \Re^m$. We only check $F_c(x, y)$ and $G_c(x, y)$ are dominated by any linear growth maps, which implies that $F_c(x, y)$ and $G_c(x, y)$ are Marchaud maps. By (A3), we have

$$\begin{aligned}\|F_c(x, y)\| :\equiv & \sup_{z \in F_c(x, y)} \|z\| \\ & \le \|f(x, y)\| + \|P(x, y)(u_{1n} + u_{1c})\| + \|F_\alpha(x, y)\| \\ & \le \|f(x, y)\| + \|P(x, y)u_{1n}\| + \|P(x, y)u_{1c}\| + k_\alpha(x, y) \\ & \qquad\qquad \text{for all } x \in \Re^n,\ y \in \Re^m.\end{aligned}$$

Note that by (3.2), (3.3), (A5), and (A6),

$$\|P(x, y)u_{1n}\| \le K_P \cdot K_J \cdot \|f(x, y)\| + (\sigma \cdot \|J(x)\|)/K_P + \rho \cdot K_P \cdot \|J(x)\|,$$
$$\|P(x, y)u_{1c}\| \le r_2 \cdot k_1(x, y) \qquad \text{for all } x \in \Re^n,\ y \in \Re^m.$$

Hence for all $x \in \Re^n$, $y \in \Re^m$,

$$\begin{aligned}\|F_c(x, y)\| \le & (1 + K_P \cdot K_J)\|f(x, y)\| + (\rho \cdot K_P + \sigma/K_P) \cdot \|J(x)\| \\ & + r_2 \cdot k_1(x, y) + k_\alpha(x, y).\end{aligned}$$

This shows that $F_c(x, y)$ is dominated by a linear growth map. So $F_c(x, y)$ has linear growth.

By (A3)–(A6), we have

$$
\begin{aligned}
&\|G_c(x,y)\| \\
&\quad :\equiv \sup_{z \in G_c(x,y)} \|z\| \\
&\quad \le \|g(x,y)\| + \|Q(x,y)u_2\| + \left\| Q(x,y)\left[F_\beta(x,y) + F_\gamma(u_2)\right] \right\| \\
&\quad \le \|g(x,y)\| + \|Q(x,y)u_2\| + \|Q(x,y)F_\beta(x,y)\| + \|Q(x,y)F_\gamma(u_2)\| \\
&\quad \le \|g(x,y)\| + \|Q(x,y)u_2\| + r_3 k_\beta(x,y) + r_1\|Q(x,y)u_2\| \\
&\quad \le \|g(x,y)\| + (1 + r_1)\|Q(x,y)u_2\| + r_3 k_\beta(x,y).
\end{aligned} \tag{3.8}
$$

Note that

$$
\begin{aligned}
&\|Q(x,y)u_2\| \\
&\qquad = \left\| Q(x,y)(u_{2n} + u_{2c}) \right\| \\
&\qquad \le \left\| A[y - H(x)] - g(x,y) + \frac{\partial H(x)}{\partial x}\left[f(x,y) + P(x,y)u_1\right] \right\| \\
&\qquad\quad + \|Q(x,y)u_{2c}\| \\
&\qquad \le \|A\|(\|y\| + \|H(x)\|) + \|g(x,y)\| \\
&\qquad\quad + K_H(\|f(x,y)\| + \|P(x,y)u_1\|) + r_3 \cdot k_2(x,y), \\
&\|P(x,y)u_1\| \\
&\qquad \le \|P(x,y)u_{1n}\| + \|P(x,y)u_{1c}\| \\
&\qquad \le K_P \cdot K_J \cdot \|f(x,y)\| + (\rho \cdot K_P + \sigma/K_P) \cdot \|J(x)\| + r_2 \cdot k_1(x,y).
\end{aligned} \tag{3.9}
$$

Combine (3.8) and (3.9). Then we obtain

$$
\begin{aligned}
\|G_c(x,y)\| \le (2 + r_1)\|g(x,y)\| &+ r_3 k_\beta(x,y) \\
&+ (1 + r_1)\Big[\|A\|(\|y\| + \|H(x)\|) \\
&\qquad + K_H \cdot (1 + K_P \cdot K_J)\|f(x,y)\| \\
&\qquad + K_H \cdot (\rho \cdot K_P + \sigma/K_P)\|J(x)\| \\
&\qquad + K_H \cdot r_2 \cdot k_1(x,y) + r_3 k_2(x,y)\Big],
\end{aligned}
$$

where $K_H \ge 0$ is the Lipschitz constant of H. Hence $G_c(x, y)$ is dominated by a linear growth map. This shows that $G_c(x, y)$ has linear growth. ∎

4. COMPLETE TRACKING CONTROL PROPERTY WITH EXPONENTIAL ASYMPTOTIC STABILITY

For convenience, denote $\lambda_m(W)$ and $\lambda_M(W)$ as the minimum and the maximum eigenvalues of the real symmetric matrix W, respectively. The Euclidean inner product is denoted by $\langle \cdot , \cdot \rangle$. We also define $\langle\langle x, S\rangle\rangle$ to be the subset $\{\langle x, s\rangle | s \in S\}$ of $\Re$ and define $\langle\langle x, S\rangle\rangle \leq r$ to denote $\langle x, s\rangle \leq r$ for all $s \in S$, where $r \in \Re$.

4.1. *The Asymptotic Stability of the Observation Map $H(\cdot)$*

THEOREM 4.1. *Let $(x(t), y(t))$ be any trajectory of the feedback-controlled system* (1.2) *satisfying* (A1)–(A7), *subject to the controller* (3.1) *with* (3.2)–(3.7). *Then the trajectory $y(t)$ of the system* (1.2) *is steered to the pre-specified observation map $H(\cdot)$ with an exponential convergence rate. Moreover, we have*

$$\|y(t) - H(x(t))\| \leq \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \cdot \|y(0) - H(x(0))\| \cdot e^{-\frac{\lambda_m(L)}{2\lambda_M(M)}t}$$

for all $t \geq 0$.

Proof. Let $e = y - H(x)$ be the deviation of the state y from the observation map $H(x)$. For simplicity in notation, we set

$$\begin{aligned}
f(x,e) &:\equiv f(x, e + H(x)), & g(x,e) &:\equiv g(x, e + H(x)),\\
F_\alpha(x,e) &:\equiv F_\alpha(x, e + H(x)), & F_\beta(x,e) &:\equiv F_\beta(x, e + H(x)),\\
k(x,e) &:\equiv k(x, e + H(x)), & k_\alpha(x,e) &:\equiv k_\alpha(x, e + H(x)),\\
k_\beta(x,e) &:\equiv k_\beta(x, e + H(x)), & k_2(x,e) &:\equiv k_2(x, e + H(x)),\\
P(x,e) &:\equiv P(x, e + H(x)), & Q(x,e) &:\equiv Q(x, e + H(x)),\\
Q^T(x,e) &:\equiv Q^T(x, e + H(x)), & Q^+(x,e) &:\equiv Q^+(x, e + H(x)).
\end{aligned}$$

In terms of state x and error e, the closed-loop system (1.2) becomes

$$\begin{aligned}
\dot{x}(t) &\in f(x(t), e(t)) + P(x(t), e(t))u_1 + F_\alpha(x(t), e(t)),\\
\dot{e}(t) &\in Ae(t) + Q(x(t), e(t))u_{2c}(t) + Q(x(t), e(t))\overline{F}(x(t), e(t)),
\end{aligned}$$

where

$$\begin{aligned}\overline{F}(x(t), e(t)) &= F_\beta(x(t), e(t)) + F_\gamma(u_2(t)) \\ &\qquad - Q^+(x(t), e(t))\frac{\partial H(x)}{\partial x}F_\alpha(x(t), e(t)).\end{aligned}$$

Let $V(e) = (1/2)e^T Me$ for all $e \in \Re^m$. For all $e \in \Re^m$, we obtain

$$\begin{aligned}\dot{V}(e) &= \frac{1}{2}(\dot{e}^T Me + e^T M\dot{e}) \\ &= e^T M\dot{e} \in \langle Me, Ae\rangle + \langle\langle Me, Q(x,e)u_{2c}\rangle\rangle \\ &\qquad + \left\langle\left\langle Me, Q(x,e)\left[F_\beta(x,e) + F_\gamma(u_{2n} + u_{2c}) \right.\right.\right.\\ &\qquad\qquad \left.\left.\left. -Q^+(x,e)\frac{\partial H(x)}{\partial x}F_\alpha(x,e)\right]\right\rangle\right\rangle \\ &= e^T MAe + \langle\langle Q^T(x,e)Me, u_{2c}\rangle\rangle \\ &\qquad + \left\langle\left\langle Q^T(x,e)Me, F_\beta(x,e) + F_\gamma(u_{2n} + u_{2c}) \right.\right.\\ &\qquad\qquad \left.\left. -Q^+(x,e)\frac{\partial H(x)}{\partial x}F_\alpha(x,e)\right\rangle\right\rangle \\ &\le -\frac{1}{2}e^T Le - k_2(x,e)\|Q^T(x,e)Me\| \\ &\qquad + \left[k_\beta(x,e) + \eta\|u_{2n}\| + \eta k_2(x,e) \right.\\ &\qquad\qquad \left. + \left\|Q^+(x,e)\frac{\partial H(x)}{\partial x}\right\|k_\alpha(x,e)\right]\|Q^T(x,e)Me\| \\ &= -\frac{1}{2}e^T Le - (1-\eta)k_2(x,e)\|Q^T(x,e)Me\| \\ &\qquad + \left[k_\beta(x,e) + \eta\|u_{2n}\| + \left\|Q^+(x,e)\frac{\partial H(x)}{\partial x}\right\|k_\alpha(x,e)\right] \\ &\qquad \times \|Q^T(x,e)Me\| \\ &= -\frac{1}{2}e^T Le - \delta\|Q^T(x,e)Me\| \le 0. \qquad (4.1)\end{aligned}$$

This shows that $V(e(t))$ is a decreasing function in t and $\dot{V}(e) \leq (-1/2)e^T Le$. Since $V(e) \leq (1/2)\lambda_M(M)\|e\|^2$ and $(1/2)\lambda_m(L)\|e\|^2 \leq (1/2)e^T Le$, we obtain

$$\dot{V}(e) \leq -\frac{1}{2}e^T Le \leq -\frac{\lambda_m(L)}{\lambda_M(M)}V(e).$$

Hence for all $t \geq 0$,

$$V(e(t)) \leq V(e(0))e^{-\frac{\lambda_m(L)}{\lambda_M(M)}t}. \tag{4.2}$$

Since $(1/2)\lambda_m(M)\|e\|^2 \leq V(e) \leq (1/2)\lambda_M(M)\|e\|^2$, by (4.2), we obtain

$$\|y(t) - H(x(t))\| \leq \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \cdot \|y(0) - H(x(0))\| \cdot e^{-\frac{\lambda_m(L)}{2\lambda_M(M)}t}$$

for all $t \geq 0$.

This shows that the trajectory $y(t)$ of the feedback-controlled system (1.2) is steered to the observation map $y(t) = H(x(t))$ with an exponential convergence rate. ∎

Remark 4.1. In the preceding Theorem 4.1, we easily obtain that if for any initial state $(x_0, y_0) \in Graph(H)$, all solutions $(x(\cdot), y(\cdot))$ of the system (1.2), starting from (x_0, y_0), satisfy $y(t) = H(x(t))$ for all $t \geq 0$; that is, $Graph(H)$ is invariant for the controlled system (1.2).

For the nominal system (1.4), all trajectories $y(\cdot)$ are also steered to the observation map $H(\cdot)$ with an exponential convergence rate as follows.

COROLLARY 4.1. *Let $(x(t), y(t))$ be any trajectory of the nominal system* (1.4) *satisfying* (A1) *and* (A5)–(A7), *subject to the controller* (3.1) *with* (3.2)–(3.7), *where $u_{1c}(x, y) :\equiv 0$, $k_2(x, y) :\equiv \delta$ (see Remark* 3.2 *and Remark* 3.4). *Then the trajectory $y(t)$ of the system* (1.4) *is steered to the pre-specified observation map $H(\cdot)$ with an exponential convergence rate.*

4.2. *An Estimate of the Tracking Time*

THEOREM 4.2. *Let $(x(t), y(t))$ be any trajectory of the feedback-controlled system* (1.2) *satisfying* (A1)–(A7), *subject to the controller* (3.1) *with* (3.2)–(3.7). *If for any initial state $(x(0), y(0)) \notin Graph(H)$, then an estimate of the tracking time T of all trajectories attaining $H(\cdot)$ is bounded by*

$$\frac{\|(QQ^T)^{-1}Q\|_\infty}{\delta}\sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}}\,\|e(0)\|,$$

where $\|e(0)\| = \|y(0) - H(x(0))\|$ *denotes the distance from the initial state* $y(0)$ *to the observation map* $H(x(0))$.

Proof. Since $(1/2)\lambda_m(M)\|e\|^2 \le (1/2)e^TMe = V(e)$, we have

$$\begin{aligned} V(e) :\equiv \frac{1}{2}\langle e, Me\rangle &= \frac{1}{2}\langle e, (QQ^T)^{-1}QQ^TMe\rangle \\ &\le \frac{1}{2}\|(QQ^T)^{-1}Q\|\,\|e\|\,\|Q^TMe\| \\ &\le \frac{1}{2}\|(QQ^T)^{-1}Q\|_\infty\left(\frac{2V(e)}{\lambda_m(M)}\right)^{\frac{1}{2}}\|Q^TMe\|, \end{aligned} \tag{4.3}$$

$$\|Q^TMe\| \ge \left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2}\right)^{\frac{1}{2}}(V(e))^{\frac{1}{2}}.$$

By (4.1) and (4.3), we have

$$\dot{V}(e) \le -\delta\|Q^TMe\| \le -\delta\left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2}\right)^{\frac{1}{2}}(V(e))^{\frac{1}{2}}. \tag{4.4}$$

Without loss of generality, we assume that $V(e(0)) \ne 0$; otherwise the trajectory y attains $H(x)$ at $t = 0$. Let T be the smallest time of the trajectory y attaining $H(x)$, i.e., $V(e(T)) = 0$ and $V(e(t)) \ne 0$ for all $t \in [0, T)$, where $T > 0$. First, we show that T is finite. Suppose that T is infinite. Then $V(e(t)) \ne 0$ for all $t > 0$, and by (4.4), we have, for all $t > 0$,

$$\int_{V(e(0))}^{V(e(t))}(V)^{-\frac{1}{2}}dV \le -\int_0^t \delta\left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2}\right)^{\frac{1}{2}}dt,$$

$$2\left[(V(e(t)))^{\frac{1}{2}} - (V(e(0)))^{\frac{1}{2}}\right] \le -\delta\left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2}\right)^{\frac{1}{2}}t, \tag{4.5}$$

$$2\left[(V(e(0)))^{\frac{1}{2}} - (V(e(t)))^{\frac{1}{2}}\right] \ge \delta\left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2}\right)^{\frac{1}{2}}t.$$

Since $V(e(t)) \to 0$ as $t \to \infty$, by (4.5), we obtain

$$\begin{aligned} 2(V(e(0)))^{\frac{1}{2}} &= \lim_{t\to\infty}\left(2\left[(V(e(0)))^{\frac{1}{2}} - (V(e(t)))^{\frac{1}{2}}\right]\right) \\ &\ge \lim_{t\to\infty}\left(\delta\left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2}\right)^{\frac{1}{2}}t\right) = \infty. \end{aligned}$$

This contradicts the fact that $2(V(e(0)))^{\frac{1}{2}} < \infty$. Thus T is finite. Note that by (4.4), we obtain

$$\int_{V(e(0))}^{V(e(T))} (V)^{-\frac{1}{2}} dV \leq -\int_0^T \delta \left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2} \right)^{\frac{1}{2}} dt.$$

This implies that

$$\begin{aligned} -2(V(e(0)))^{\frac{1}{2}} &= 2\left[(V(e(T)))^{\frac{1}{2}} - (V(e(0)))^{\frac{1}{2}} \right] \\ &\leq -\delta \left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2} \right)^{\frac{1}{2}} T. \end{aligned}$$

Since $(1/2)\lambda_m(M)\|e\|^2 \leq V(e) \leq (1/2)\lambda_M(M)\|e\|^2$, we obtain

$$\delta \left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2} \right)^{\frac{1}{2}} T \leq 2\sqrt{\frac{1}{2}\lambda_M(M)}\, \|e(0)\|.$$

Hence

$$T \leq \frac{\|(QQ^T)^{-1}Q\|_\infty}{\delta} \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}}\, \|e(0)\|.$$

∎

COROLLARY 4.2. *In the preceding Theorem* 4.2, *we have proved that the tracking time depends on both the distance from* $y(\cdot)$ *to the observation map* $H(\cdot)$ *at the initial state and the eigenvalues of the real symmetric matrix* M. *More precisely*, *combine Theorem* 4.1 *and Theorem* 4.2. *Then we obtain*

$$\|y(t) - H(x(t))\| \leq \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \cdot \|y(0) - H(x(0))\| \cdot e^{-\frac{\lambda_m(L)}{2\lambda_M(M)}t} \qquad \textit{for all } t \in [0, T)$$

and

$$y(t) = H(x(t)) \qquad \textit{for all } t \geq T.$$

For the nominal system (1.4), an estimate of the tracking time of all trajectories $y(\cdot)$ attaining the observation map $H(\cdot)$ is given as follows.

COROLLARY 4.3. *Let* $(x(t), y(t))$ *be any trajectory of the nominal system* (1.4) *satisfying* (A1) *and* (A5)–(A7), *subject to the controller* (3.1) *with* (3.2)–(3.7), *where* $u_{1c}(x, y) :\equiv 0$, $k_2(x, y) :\equiv \delta$. *Then an estimate of the tracking time* T *of the trajectory* $y(\cdot)$ *attaining* $H(\cdot)$ *is bounded by*

$$\frac{\|(QQ^T)^{-1}Q\|_\infty}{\delta}\sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}}\,\|e(0)\|.$$

4.3. *The Asymptotic Stability of the Guidance Map* $J(\cdot)$

THEOREM 4.3. *Let* $(x(t), y(t))$ *be any trajectory of the feedback-controlled system* (1.2) *satisfying* (A1)–(A7), *subject to the controller* (3.1) *with* (3.2)–(3.7). *Then the trajectory* $x(t)$ *of the systems* (1.2) *is asymptotically stable along* $J(x(t))$ *to* 0 *with an exponential convergence rate.*

Proof. Let $W(x(t)) = \frac{1}{2}(J(x(t)))^T J(x(t))$. Without loss of generality, we assume that $J(x) \neq 0$. Now we calculate the derivative of $W(x)$ as follows. For all $J(x) \neq 0$, we have

$$\begin{aligned}
\dot{W}(x(t)) &= (J(x(t)))^T \dot{J}(x(t)) \\
&\in \left\langle\!\!\left\langle J(x), \frac{\partial J(x)}{\partial x}[f(x,y) + P(x,y)(u_1) + F_\alpha(x,y)]\right\rangle\!\!\right\rangle \\
&= J^T(x)\frac{\partial J(x)}{\partial x}f(x,y) + J^T(x)\frac{\partial J(x)}{\partial x}P(x,y)(u_{1n} + u_{1c}) \\
&\quad + \left\langle\!\!\left\langle J(x), \frac{\partial J(x)}{\partial x}F_\alpha(x,y)\right\rangle\!\!\right\rangle \\
&= J^T(x)\frac{\partial J(x)}{\partial x}f(x,y) + J^T(x)\frac{\partial J(x)}{\partial x}P(x,y) \\
&\quad \times\left[-\left(\frac{\partial J(x)}{\partial x}P(x,y)\right)^{-1}\frac{\partial J(x)}{\partial x}f(x,y)\right] \\
&\quad + J^T(x)\frac{\partial J(x)}{\partial x}P(x,y)\left[-\sigma P^T(x,y)\left(\frac{\partial J(x)}{\partial x}\right)^T J(x)\right] \\
&\quad + J^T(x)\frac{\partial J(x)}{\partial x}P(x,y)\left[-\rho\left(\frac{\partial J(x)}{\partial x}P(x,y)\right)^{-1} J(x)\right] \\
&\quad + J^T(x)\frac{\partial J(x)}{\partial x}P(x,y)u_{1c}(x,y)
\end{aligned}$$

$$
\begin{aligned}
&+ \left\langle\left\langle J(x), \frac{\partial J(x)}{\partial x} F_\alpha(x,y) \right\rangle\right\rangle \\
&\le -\sigma \left\| P^T(x,y) \left(\frac{\partial J(x)}{\partial x} \right)^T J(x) \right\|^2 - \rho \|J(x)\|^2 \\
&\quad - \frac{k_1(x,y)\|J(x)\|^2}{\left\| \left(\frac{\partial J(x)}{\partial x} P(x,y) \right)^{-1} J(x) \right\|} + k_\alpha(x,y)\|J(x)\| \left\| \frac{\partial J(x)}{\partial x} \right\| \\
&\le -\sigma \left\| P^T(x,y) \left(\frac{\partial J(x)}{\partial x} \right)^T J(x) \right\|^2 - \rho \cdot \|J(x)\|^2 \\
&\quad - \frac{k_1(x,y)\|J(x)\|^2}{\left\| \frac{\partial J(x)}{\partial x} P(x,y) \right\|^{-1} \|J(x)\|} + k_\alpha(x,y) \cdot K_J \cdot \|J(x)\| \\
&\le -\sigma \left\| P^T(x,y) \left(\frac{\partial J(x)}{\partial x} \right)^T J(x) \right\|^2 - \rho \|J(x)\|^2 \\
&\quad - \frac{k_1(x,y)\|J(x)\|}{K_P} + k_\alpha(x,y) \cdot K_J \cdot \|J(x)\| \\
&\le -\sigma \left\| P^T(x,y) \left(\frac{\partial J(x)}{\partial x} \right)^T J(x) \right\|^2 - \rho \|J(x)\|^2 \\
&\le -\sigma \lambda_m \left[\left(\frac{\partial J(x)}{\partial x} P(x,y) \right)^T \left(\frac{\partial J(x)}{\partial x} P(x,y) \right) \right] \\
&\quad \times \|J(x)\|^2 - \rho \|J(x)\|^2 \\
&\le -(\sigma \cdot \lambda_0 + \rho)\|J(x)\|^2 = -2 \cdot (\sigma \cdot \lambda_0 + \rho) \cdot W(x(t)),
\end{aligned}
$$

where

$$
\lambda_0 :\equiv \inf_{x \in \Re^n,\ y \in \Re^m} \left\{ \lambda_m \left[\left(\frac{\partial J(x)}{\partial x} P(x,y) \right)^T \left(\frac{\partial J(x)}{\partial x} P(x,y) \right) \right] \right\}.
$$

Hence $\frac{d(w(x(t)))}{dt} \le -2 \cdot (\sigma \cdot \lambda_0 + \rho) \cdot W(x(t))$.

Thus we obtain that for all $t \geq 0$, $\frac{1}{2}\|J(x(t))\|^2 :\equiv W(x(t)) \leq W(x(0))e^{-2(\sigma\cdot\lambda_0+\rho)t}$. This shows that

$$\|J(x(t))\| \leq \sqrt{J(x(0))}\, e^{-(\sigma\cdot\lambda_0+\rho)t} \to 0, \qquad \text{as } t \to \infty,$$

that is, the trajectory $x(t)$ of the system (1.2) is asymptotically stable along $J(x(t))$ to 0 with an exponential convergence rate. ∎

For the nominal system (1.4), all trajectories $x(t)$ of the system are also asymptotically stable along $J(x(t))$ to 0 with an exponential convergence rate as follows.

COROLLARY 4.4. *Let $(x(t), y(t))$ be any trajectory of the nominal system* (1.4) *satisfying* (A1) *and* (A5)–(A7), *subject to the controller* (3.1) *with* (3.2)–(3.7), *where $u_{1c}(x, y) := 0$, $k_2(x, y) := \delta$. Then the trajectory $x(t)$ of the system* (1.4) *is asymptotically stable along $J(x(t))$ to* 0 *with an exponential convergence rate.*

Combine Theorem 4.1, Theorem 4.2, and Theorem 4.3. We obtain the main theorem as follows.

THEOREM 4.4. *Let $(x(t), y(t))$ be any trajectory of the feedback-controlled system* (1.2) *satisfying* (A1)–(A7). *If for any initial state $(x_0, y_0) \notin$ Graph(H), then the controller* (3.1) *with* (3.2)–(3.7) *such that the system* (1.2) *under H satisfies the complete tracking control property with exponential asymptotic stability along $J(x(t))$ after a finite time, i.e.*,

$$\|y(t) - H(x(t))\| \leq \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \cdot \|y(0) - H(x(0))\| \cdot e^{-\frac{\lambda_m(L)}{2\lambda_M(M)}t} \qquad \textit{for all } t \in [0, T),$$

$$y(t) = H(x(t)) \qquad \textit{for all } t \geq T,$$

and

$$\|J(x(t))\| \leq \sqrt{J(x(0))}\, e^{-(\sigma\cdot\lambda_0+\rho)t} \to 0 \qquad \textit{as } t \to \infty,$$

where the tracking time T of all trajectories $y(\cdot)$ attaining $H(\cdot)$ is bounded by

$$\frac{\|(QQ^T)^{-1}Q\|_\infty}{\delta}\sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}}\,\|e(0)\|.$$

5. AN ILLUSTRATIVE EXAMPLE

An example has been provided to illustrate the use of our main result about the tracking control problem for the guided missile as follows.

In Example 5.1, the trajectory of the guided missile satisfying the uncertain dynamical system (5.1) described by differential inclusions is traced by an observation function $H(\cdot)$, where y is the state of the guided missile and x is the state of the infrared laser beam transmitted by the guided plane or satellite. Here, let the curve $Z(x) = x$ be a guideline of x in the infrared laser guidance system. Note if $H(x) = x$ and $Z(x) = H(x)$, then the guided missile y and the laser guided beam x touch each other, that is, the missile y can be guided to the guideline (see Fig. 1). The goal is to find a pair of generalized feedback control inputs $u_1(x, y)$ and $u_2(x, y)$ such that the missile y can be guided by the infrared laser beam x to the guideline after a finite time T, and the guided missile y is asymptotically stable along the guideline $y = Z(x) = x$ to the target $z_0 = 3$ of an attack, and so take $J(x) = x - 3$. This implies that the nonlinear uncertain dynamical systems (5.1) enjoy the complete tracking control property with exponential asymptotic stability along $J(x(t))$ after a finite time T.

EXAMPLE 5.1. Consider the tracking control problem for the following uncertain dynamical system described by differential inclusions,

$$\begin{cases} \dot{x}(t) \in F(x(t), y(t), u_1(t)) \\ \dot{y}(t) \in G(x(t), y(t), u_2(t)), \end{cases} \tag{5.1}$$

$$F(x, y, u_1) :\equiv f(x, y) + P(x, y)u_1 + F_\alpha(x, y),$$

$$G(x, y, u_2) :\equiv g(x, y) + Q(x, y)u_2 + Q(x, y)\left[F_\beta(x, y) + F_\gamma(u_2)\right],$$

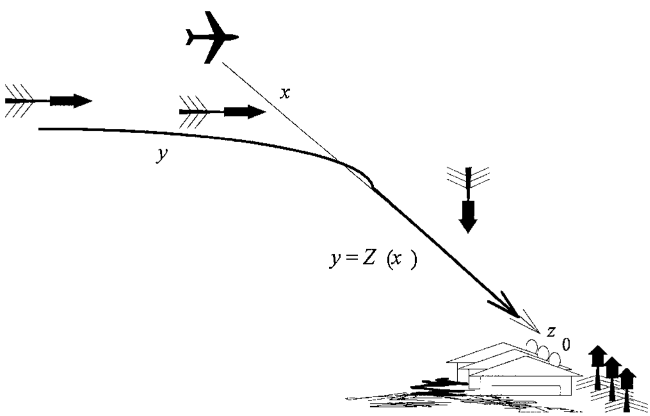

FIG. 1. The guided missile along the guideline to the target of an attack.

where

$$f(x, y) = x + y\cos(xy) + 2, \qquad g(x, y) = y + x\sin(xy) + 2,$$

$$P(x, y) = 1 + (\sin x)^2 + (\cos y)^2,$$

$$Q(x, y) = 1 + (\sin(xy))^2 + (\cos x)^2,$$

$$F_\alpha(x, y) = \{a(1 + x\cos y + y\sin x) + a\,SIGN(xy) \mid a \in [-1, 1]\},$$

$$SIGN(xy) = \begin{cases} -1, & xy < 0, \\ [-1, 1], & xy = 0, \\ 1, & xy > 0, \end{cases}$$

$$F_\beta(x, y) = \{b|x - y|\cos(x) + 4 \mid b \in [-1, 1]\},$$

$$F_\gamma(u_2) = \{cu_2\sin(u_2) \mid c \in [-0.5, 0.5]\}.$$

From (A2)–(A3), we have

$$k_\alpha(x, y) = 2 + |x| + |y|, \qquad k_\beta(x, y) = 4 + |x| + |y|,\ \eta = 0.5.$$

For example, for $a = 1$, $b = 1$, and $c = 0.5$, by the modified Runge–Kutta method, some typical phase trajectories of the uncontrolled system are depicted in Fig. 2.

If we choose $A = -1$ and $L = 2$, then, by (3.6), we have $M = 1$. Furthermore, let $H(x) = x$, $J(x) = x - 3$, $\sigma = 1$, $\rho = 1$, and $\delta = 0.5$. Then we can calculate the explicit form of the controllers $u_1(t)$ and $u_2(t)$ given by (3.1) with (3.2)–(3.7). They are shown as

$$u_1(t) = u_{1n}(x(t), y(t)) + u_{1c}(x(t), y(t)),$$

$$u_2(t) = u_{2n}(x(t), y(t)) + u_{2c}(x(t), y(t)),$$

$$\begin{aligned} u_{1n}&(x, y) \\ &= -\frac{f(x, y) - [P(x, y)]^2(x - 3) - (x - 3)}{P(x, y)} \\ &= -\frac{x + y\cos(xy) + 2 + (x - 3)\cdot\left[\left(1 + (\sin x)^2 + (\cos y)^2\right)^2 + 1\right]}{1 + (\sin x)^2 + (\cos y)^2}, \end{aligned}$$

$$u_{1c}(t) = -(2 + |x| + |y|)\Psi(\zeta),\ \zeta = \frac{x - 3}{1 + (\sin x)^2 + (\cos y)^2},$$

$$u_{2n}(x, y) = \frac{A(y - H(x)) - g(x, y) + f(x, y) + P(x, y)u_1}{Q(x, y)},$$

$$= -\frac{2(y - x) + x\sin(xy) - y\cos(xy) - P(x, y)u_1}{1 + (\sin(xy))^2 + (\cos x)^2},$$

$$u_{2c}(t) = -k_2(x, y)\Psi(\xi),$$

$$k_2(x, y) = 2\big[(2 + |x| + |y|) + (4 + |x| + |y|) + 0.5|u_{2n}| + 0.5\big],$$

$$\xi = (y + x)\big(1 + (\sin(xy))^2 + (\cos x)^2\big).$$

From the simulation results, all trajectories of the feedback-controlled system reach the observation map $H(x)$ in a finite time and remain on $H(x)$ thereafter. Moreover, all trajectories $x(t)$ of the system (1.2) satisfy the tracking control property with asymptotic stability along the line $Z(x) = x$ to 3, that is, Theorem 4.4 holds. By the modified Runge–Kutta method, some typical phase trajectories of the feedback-controlled system are depicted in Fig. 3.

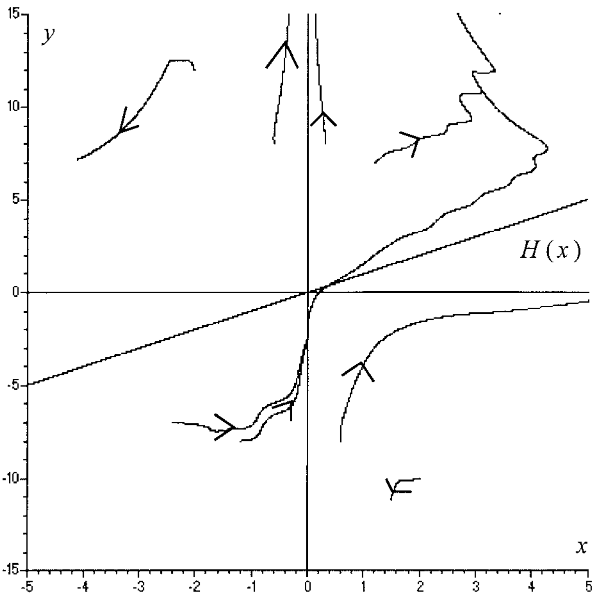

FIG. 2. Typical phase trajectories of the uncontrolled system.

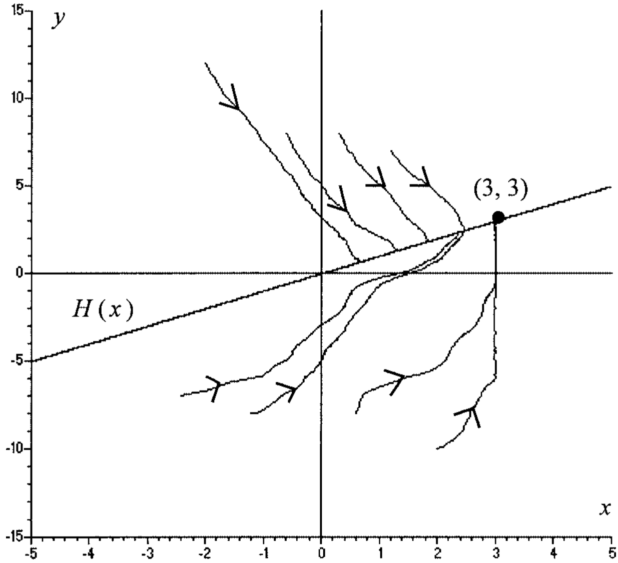

FIG. 3. Typical phase trajectories of the feedback-controlled system.

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