

Existence for nonoscillatory solutions of second-order nonlinear differential equations[☆]

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Abstract

In this paper, the existence of nonoscillatory solutions of the second-order nonlinear neutral differential equation

$$\left[r(t)(x(t) + P(t)x(t - \tau))' \right]' + \sum_{i=1}^m Q_i(t)f_i(x(t - \sigma_i)) = 0, \quad t \geq t_0,$$

where $m \geq 1$ is an integer, $\tau > 0$, $\sigma_i \geq 0$, $r, P, Q_i \in C([t_0, \infty), \mathbf{R})$, $f_i \in C(\mathbf{R}, \mathbf{R})$ ($i = 1, 2, \dots, m$), are studied. Some new sufficient conditions for the existence of a nonoscillatory solution of above equation are obtained for general $P(t)$ and $Q_i(t)$ ($i = 1, 2, \dots, m$) which means that we allow oscillatory $P(t)$ and $Q_i(t)$ ($i = 1, 2, \dots, m$). In particular, our results improve essentially and extend some known results in the recent references.

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1. Introduction

Consider the second-order nonlinear neutral differential equation

$$\left[r(t)(x(t) + P(t)x(t - \tau))' \right]' + \sum_{i=1}^m Q_i(t)f_i(x(t - \sigma_i)) = 0, \quad t \geq t_0. \quad (1)$$

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With respect to Eq. (1), throughout we shall assume the following:

- (i) $m \geq 1$ is an integer, $\tau > 0$, $\sigma_i \geq 0$;
- (ii) $r, P, Q_i \in C([t_0, \infty), \mathbf{R})$, $r(t) > 0$, $f_i \in C(\mathbf{R}, \mathbf{R})$, $i = 1, 2, \dots, m$.

Let $\varphi \in C([t_0 - \rho, \infty), \mathbf{R})$, where $\rho = \max_{1 \leq i \leq m} \{\tau, \sigma_i\}$, be a given function and let y_0 be a given constant. Using the method of steps, Eq. (1) has a unique solution $x \in C([t_0 - \rho, \infty), \mathbf{R})$ in the sense that both $x(t) + P(t)x(t - \tau)$ and $r(t)(x(t) + P(t)x(t - \tau))'$ are continuously differentiable for $t \geq t_0$, $x(t)$ satisfies Eq. (1) and

$$\begin{aligned} x(s) &= \varphi(s) \quad \text{for } s \in [t_0 - \rho, t_0], \\ [x(t) + P(t)x(t - \tau)]'_{t=t_0} &= y_0. \end{aligned}$$

A solution of Eq. (1) is called oscillatory if it has arbitrarily large zeros, and otherwise it is nonoscillatory.

Oscillation and nonoscillation of second-order neutral differential equations have been studied in recent years. We refer the reader to [1–10] and the references cited therein. The existence of nonoscillatory solution of second-order neutral differential equations received much less attention, which is due mainly to the technical difficulties arising in its analysis.

In 1998, Kulenovic and Hadziomerspahic [5] investigated the existence of nonoscillatory solutions of the second-order linear neutral differential equation

$$(x(t) + cx(t - \tau))'' + Q_1(t)x(t - \sigma_1) - Q_2(t)x(t - \sigma_2) = 0, \quad t \geq t_0, \quad (\text{E}_0)$$

where c is a constant.

In 2005, Yu and Wang [9] studies the existence of nonoscillatory solutions of the following second-order nonlinear neutral differential equation:

$$[r(t)(x(t) + P(t)x(t - \tau))']' + Q_1(t)f(x(t - \sigma_1)) - Q_2(t)g(x(t - \sigma_2)) = 0, \quad t \geq t_0, \quad (\text{E})$$

where $f, g \in C(\mathbf{R}, \mathbf{R})$. By using Banach contraction mapping principle, they proved the following theorems which extend results in [5].

Theorem A. [9, Theorem 1] *Assume that*

- (H₁) f and g satisfy local Lipschitz condition and $xf(x) > 0$, $xg(x) > 0$ for $x \neq 0$;
- (H₂) $Q_i(t) \geq 0$, $i = 1, 2$, $aQ_1(t) - Q_2(t)$ is eventually nonnegative for every $a > 0$;
- (H₃) there exists a constant P_0 such that $|P(t)| \leq P_0 < 1/2$, eventually;
- (H₄) $\int_{t_0}^{\infty} \int_{t_0}^t \frac{Q_i(s)}{r(s)} ds dt < \infty$.

Then Eq. (E) has a nonoscillatory solution.

Theorem B. [9, Theorem 2] *Suppose that conditions (H₁), (H₂) and (H₄) hold, and if one of the following two conditions is satisfied:*

- (H₅) $P(t) \geq 0$ eventually, and $0 < P_1 < 1$,
- (H₆) $P(t) \leq 0$ eventually, and $-1 < P_2 < 0$,

where $P_1 = \limsup_{t \rightarrow \infty} P(t)$, $P_2 = \liminf_{t \rightarrow \infty} P(t)$, then Eq. (E) has a nonoscillatory solution.

In this paper, by using Krasnoselskii's fixed point theorems and some new techniques, we obtain some sufficient conditions for the existence of a nonoscillatory solution of (1) for general $Q_i(t)$ ($i = 1, 2, \dots, m$) and $P(t)$ which means that we allow oscillatory $Q_i(t)$ ($i = 1, 2, \dots, m$) and $P(t)$. In particular, our results improve essentially Theorems A and B by removing the restrictive conditions (H_1) and (H_2) and relaxing the hypothesis (H_3) , (H_5) and (H_6) .

2. Main results

The following some fixed point theorems will be used to prove the main results in this section.

Lemma 1 (Krasnoselskii's Fixed Point Theorem). [3] *Let X be a Banach space, let Ω be a bounded closed convex subset of X and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then the equation*

$$S_1x + S_2x = x$$

has a solution in Ω .

Theorem 1. *Assume that there exist nonnegative constants c_1 and c_2 such that $c_1 + c_2 < 1$, $-c_2 \leq P(t) \leq c_1$. Further, assume that*

$$\int_{t_0}^{\infty} \int_{t_0}^t \frac{|Q_i(s)|}{r(s)} ds dt < \infty, \quad i = 1, 2, \dots, m. \quad (2)$$

Then Eq. (1) has a bounded nonoscillatory solution.

Proof. By interchanging the order of integral, we note that (2) is equivalent to

$$\int_{t_0}^{\infty} \int_s^{\infty} \frac{|Q_i(t)|}{r(s)} dt ds < \infty, \quad i = 1, 2, \dots, m. \quad (3)$$

By (3), we choose $T > t_0$ sufficiently large such that

$$\int_T^{\infty} \int_s^{\infty} \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds < \frac{1 - c_1 - c_2}{4},$$

where $M = \max_{(1-c_1-c_2)/2 \leq x \leq 1} \{|f_i(x)| : 1 \leq i \leq m\}$.

Let $C([t_0, \infty), \mathbf{R})$ be the set of all continuous functions with the norm $\|x\| = \sup_{t \geq t_0} |x(t)| < \infty$. Then $C([t_0, \infty), \mathbf{R})$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0, \infty), \mathbf{R})$ as follows:

$$\Omega = \left\{ x = x(t) \in C([t_0, \infty), \mathbf{R}) : \frac{1 - c_1 - c_2}{2} \leq x(t) \leq 1, t \geq t_0 \right\}.$$

Define two maps S_1 and $S_2 : \Omega \rightarrow C([t_0, \infty), \mathbf{R})$ as follows:

$$(S_1x)(t) = \begin{cases} \frac{3+c_1-3c_2}{4} - P(t)x(t-\tau), & t \geq T, \\ (S_1x)(T), & t_0 \leq t \leq T, \end{cases}$$

$$(S_2x)(t) = \begin{cases} \int_t^{\infty} \int_s^{\infty} \frac{1}{r(s)} \left(\sum_{i=1}^m Q_i(u) f_i(x(u-\sigma_i)) \right) du ds, & t \geq T, \\ (S_2x)(T), & t_0 \leq t \leq T. \end{cases}$$

(i) We shall show that for any $x, y \in \Omega$, $S_1x + S_2y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $t \geq T$, we get

$$\begin{aligned} & (S_1x)(t) + (S_2y)(t) \\ & \leq \frac{3+c_1-3c_2}{4} - P(t)x(t-\tau) + \int_t^\infty \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| |f_i(y(u-\sigma_i))| \right) du ds \\ & \leq \frac{3+c_1-3c_2}{4} + c_2 + \int_T^\infty \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds \\ & \leq \frac{3+c_1-3c_2}{4} + c_2 + \frac{1-c_1-c_2}{4} = 1. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & (S_1x)(t) + (S_2y)(t) \\ & \geq \frac{3+c_1-3c_2}{4} - P(t)x(t-\tau) - \int_t^\infty \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| |f_i(y(u-\sigma_i))| \right) du ds \\ & \geq \frac{3+c_1-3c_2}{4} - c_1 - \int_T^\infty \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds \\ & \geq \frac{3+c_1-3c_2}{4} - c_1 - \frac{1-c_1-c_2}{4} = \frac{1-c_1-c_2}{2}. \end{aligned}$$

Hence,

$$\frac{1-c_1-c_2}{2} \leq (S_1x)(t) + (S_2y)(t) \leq 1 \quad \text{for } t \geq t_0.$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

(ii) We shall show that S_1 is a contraction mapping on Ω .

In fact, for $x, y \in \Omega$ and $t \geq T$, we have

$$|(S_1x)(t) - (S_1y)(t)| \leq |P(t)| |x(t-\tau) - y(t-\tau)| \leq c_0 \|x - y\|,$$

where $c_0 = \max\{c_1, c_2\}$. This implies that

$$\|S_1x - S_1y\| \leq c_0 \|x - y\|.$$

Since $0 < c_0 < 1$, we conclude that S_1 is a contraction mapping on Ω .

(iii) We now show that S_2 is completely continuous.

First, we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \rightarrow x(t)$ as $k \rightarrow \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq T$, we have

$$\begin{aligned} & |(S_2x_k)(t) - (S_2x)(t)| \\ & \leq \int_t^\infty \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| |f_i(x_k(u-\sigma_i)) - f_i(x(u-\sigma_i))| \right) du ds \end{aligned}$$

$$\leq \int_T^\infty \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| |f_i(x_k(u - \sigma_i)) - f_i(x(u - \sigma_i))| \right) du ds.$$

Since $|f_i(x_k(t - \sigma_i)) - f_i(x(t - \sigma_i))| \rightarrow 0$ as $k \rightarrow \infty$ for $i = 1, 2, \dots, m$, by applying the Lebesgue dominated convergence theorem, we conclude that $\lim_{k \rightarrow \infty} \|(S_2 x_k)(t) - (S_2 x)(t)\| = 0$. This means that S_2 is continuous.

Next, we show $S_2 \Omega$ is relatively compact. It suffices to show that the family of functions $\{S_2 x: x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness is obvious. For the equicontinuity, according to Levitan's result [6], we only need to show that, for any given $\varepsilon > 0$, $[T, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε . By (3), for any $\varepsilon > 0$, take $T^* \geq T$ large enough so that

$$\int_{T^*}^\infty \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds < \frac{\varepsilon}{2}.$$

Then, for $x \in \Omega$, $t_2 > t_1 \geq T^*$,

$$\begin{aligned} & |(S_2 x)(t_2) - (S_2 x)(t_1)| \\ & \leq \int_{t_2}^\infty \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| |f_i(x(u - \sigma_i))| \right) du ds \\ & \quad + \int_{t_1}^\infty \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| |f_i(x(u - \sigma_i))| \right) du ds \\ & \leq \int_{t_2}^\infty \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds + \int_{t_1}^\infty \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds \\ & < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

For $x \in \Omega$ and $T \leq t_1 < t_2 \leq T^* + 1$,

$$\begin{aligned} & |(S_2 x)(t_2) - (S_2 x)(t_1)| \\ & \leq \int_{t_1}^{t_2} \int_s^\infty \frac{1}{r(s)} \left(\sum_{i=1}^m |Q_i(u)| |f_i(x(u - \sigma_i))| \right) du ds \\ & \leq \int_{t_1}^{t_2} \int_s^\infty \frac{M}{r(s)} \sum_{i=1}^m |Q_i(u)| du ds. \end{aligned}$$

Then there exists $\delta > 0$ such that

$$|(S_2 x)(t_2) - (S_2 x)(t_1)| < \varepsilon, \quad \text{if } 0 < t_2 - t_1 < \delta.$$

For any $x \in \Omega$, $t_0 \leq t_1 < t_2 \leq T$, it is easy to see that

$$|(S_2 x)(t_2) - (S_2 x)(t_1)| = 0 < \varepsilon.$$

Therefore $\{S_2x: x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact. By Lemma 1, there is $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. It is easy to see that $x_0(t)$ is a nonoscillatory solution of Eq. (1). The proof is complete. \square

Remark 1. Theorem 1 improves essentially Theorems 1 and 2 in [9] by removing the restrictive conditions (H_1) and (H_2) . In the case where $r(t) \equiv 1$, Theorem 1 improves essentially Theorem 2.2 in [7].

Remark 2. If condition (H_3) holds, we choose $c_1 = c_2 = P_0$, then $c_1 + c_2 = 2P_0 < 1$ and $-c_2 \leq P(t) \leq c_1$. If condition (H_5) or (H_6) holds, then there exists $\varepsilon > 0$ such that

$$0 \leq P(t) \leq P_1 + \varepsilon < 1 \quad \text{or} \quad -1 \leq P_2 - \varepsilon \leq P(t) \leq 0.$$

We choose $c_1 = P_1 + \varepsilon$, $c_2 = 0$ or $c_1 = 0$, $c_2 = -P_2 + \varepsilon$. Clearly, $c_1 + c_2 < 1$ and $-c_2 \leq P(t) \leq c_1$. Hence, the conditions of Theorem 1 relaxing the hypotheses (H_3) , (H_5) and (H_6) .

Remark 3. Minor adjustments are only necessary to discuss the neutral differential equation

$$\left[r(t)(x(t) + P(t)x(t - \tau))' \right]' + F(t, x(\sigma_1(t)), \dots, x(\sigma_n(t))) = 0, \quad t \geq t_0,$$

where $F: [t_0, \infty) \times \mathbf{R} \times \dots \times \mathbf{R} \rightarrow \mathbf{R}$ is continuous, $\sigma_i(t) \rightarrow \infty$ ($i = 1, 2, \dots, n$), as $t \rightarrow \infty$, $n \geq 1$ is an integer. We omit the details.

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