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On normal families and differential polynomials for meromorphic functions ☆

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Abstract

We consider the normality criterion for a families $\mathcal{F}$ meromorphic in the unit disc Δ, and show that if there exist functions $a(z)$ holomorphic in Δ, $a(z) \neq 1$, for each $z \in \Delta$, such that there not only exists a positive number ε_0 such that $|a_n(a(z)-1)-1| \geqslant \varepsilon_0$ for arbitrary sequence of integers $a_n (n \in \mathbb{N})$ and for any $z \in \Delta$, but also exists a positive number $B > 0$ such that for every $f(z) \in \mathcal{F}$, $B|f'(z)| \leqslant |f(z)|$ whenever $f(z)f''(z) - a(z)(f'(z))^2 = 0$ in Δ. Then $\{\frac{f'(z)}{f(z)}\colon f(z) \in \mathcal{F}\}$ is normal in Δ.

Keywords: Differential polynomials; Meromorphic functions; Zeros; Normality

1. Introduction and the main result

Hayman[5] proved in 1959 that if f is meromorphic in the complex plane $\mathbb{C}$ and if $f(z) \neq 0$ and $f' \neq 1$ for all $z \in \mathbb{C}$, then f is constant.The corresponding normality criterion is due to Gu [4]: the family of all functions f meromorphic in a domain D and having the property that $f(z) \neq 0$ and $f' \neq 1$ for all $z \in D$ is normal. In 2000, W. Bergweiler [1] generalized Gu's results above by allowing f to have zeros, and obtained the following result.

Theorem 1.1. *(See Theorem 1 in [1].) Let A, ε be positive real numbers. Let $\mathcal{F}$ be the family of all functions $f(z)$ meromorphic in D which satisfy the following conditions*:

(i) *If $f(z) = 0$, then $0 < |f'(z)| \leqslant A$.*
(ii) *If $z \in D$, then $f'(z) \neq 1$.*
(iii) *If Δ is a disk in D and if f has $m \geqslant 2$ zeros $z_1, z_2, \ldots, z_m \in \Delta$, then for $m > 2$ there exists $k \in \{-1\} \cup \{1, 2, \ldots, m-2\}$ such that*

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$$\left|\sum_{j=1}^{m} f'(z_j)^k - m^{k+1}\right| \geqslant \varepsilon; \tag{1.1}$$

for $m = 2$, z_1, z_2 satisfy the following inequality

$$\left|\frac{1}{f'(z_1)} + \frac{1}{f'(z_2)} - 1\right| \geqslant \varepsilon. \tag{1.2}$$

Then $\mathcal{F}$ is normal in D.

If $m = 2$ in (iii), then the only possible choice for k is $k = -1$, and (1.1) reduces to (1.2). The choice $k = 0$ has been excluded in (iii) because (1.1) is never satisfied in this case. In 2005, W.C. Lin, H.X. Yi [8] obtained one result corresponding to the case $m = 2$ in (iii) of Theorem 1.1 as follows.

Theorem 1.2. *(See Proposition in [8].) Let A, B, ε be positive real numbers. Let $\mathcal{F}$ be the family of all functions $f(z)$ meromorphic in D which satisfy the following conditions*:

(i) *If $f(z) = 0$, then $0 < |f'(z)| \leqslant A$.*
(ii) *If $f'(z) = 1$, then $|f(z)| \geqslant B$.*
(iii) *If Δ is a disk in D and if f has $m \geqslant 2$ zeros $z_1, z_2, \ldots, z_m \in \Delta$, then*

$$\left|\sum_{j=1}^{m} f'(z_j)^{-1} - 1\right| \geqslant \varepsilon. \tag{1.3}$$

Then $\mathcal{F}$ is normal in D.

In this paper, we obtain the following result.

Theorem 1.3. *Let A, B, ε be positive real numbers and $b(z)$ be a non-vanishing holomorphic function in a domain D. Let $\mathcal{F}$ be the family of all functions $f(z)$ meromorphic in D which satisfy the following conditions*:

(i) *If $f(z) = 0$, then $0 < |f'(z)| \leqslant A$.*
(ii) *If $f'(z) = b(z)$, then $|f(z)| \geqslant B$.*
(iii) *If Δ is a disk in D and if $f(z)$ has $m \geqslant 2$ zeros $z_1, z_2, \ldots, z_m \in \Delta$, the following formula always holds*

$$\left|\sum_{j=1}^{m} f'(z_j)^{-1} b(z) - 1\right| \geqslant \varepsilon. \tag{1.4}$$

Then $\mathcal{F}$ is normal in D.

For families $\mathcal{F}$ of meromorphic functions in D, W. Bergweiler [1] applied a family $\{f/f'\colon f(z) \in \mathcal{F}\}$ to Theorem 1.1 and obtained the following result, whose corresponding result for families of holomorphic functions is due to Schwick [6].

Theorem 1.4. *(See Theorem 3 in [1].) Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be the family of all functions f meromorphic in D such that f and f'' do not have zeros. Then $\{f/f'\colon f \in \mathcal{F}\}$ is normal.*

In 2005, W.C. Lin, H.X. Yi [8] applied a family $\{\frac{f}{(a-1)f'}\colon f \in \mathcal{F}\}$ to Theorem 1.2 and obtained the following result.

Theorem 1.5. *(See Theorem 2 in [8].) Let $\mathcal{F}$ be the family of all functions f meromorphic in the unit disc Δ and let constants $a \neq 1$, $\frac{n\pm 1}{n}$ for positive integers $n \in \mathbb{N}$. If for every $f \in \mathcal{F}$, $f(z)f''(z) - a(f'(z))^2 \neq 0$ in Δ. Then $\{\frac{f'}{f}\colon f \in \mathcal{F}\}$ is normal in Δ.*

In the present paper, for families $\mathcal{F}$ of meromorphic functions, and for a function $a(z)$ holomorphic in a domain D, we also consider that the normality of the family $\{\frac{f'}{f}\colon f \in \mathcal{F}\}$ and obtain the following theorem as an application of Theorem 1.3.

Theorem 1.6. *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disc Δ. Suppose that there exist holomorphic functions $a(z)$ in Δ, $a(z) \neq 1$, for each $z \in \Delta$, such that*

(i) *there exists a positive number ε_0 such that for any integer a_n, and any $z \in \Delta$, the following inequality holds*

$$\left|a_n\big(a(z)-1\big)-1\right| \geqslant \varepsilon_0, \tag{1.5}$$

(ii) *there exists a positive number $B > 0$ such that for every $f(z) \in \mathcal{F}$, $B|f'(z)| \leqslant |f(z)|$ whenever $f(z)f''(z) - a(z)(f'(z))^2 = 0$ in Δ.*

Then $\{\frac{f'(z)}{f(z)}\colon f(z) \in \mathcal{F}\}$ is normal in Δ.

For the restriction (i) of $a(z)$ in Theorem 1.6, we have the following notes,

Remark 1. If $a(z)$ is a constant a in Δ, then from Lemma 3.1 in [8], it follows that $a(z)$ satisfies the condition (i) in Theorem 1.6 if and only if $a \neq 1 \pm \frac{1}{n}$, for every positive integer $n \in \mathbb{N}$.

Remark 2. If $a(z)$ is not identically equal to a constant a in Δ, and satisfies the condition (i) in Theorem 1.6, then we immediately deduce that

$$a(z) \neq 1 \pm \frac{1}{n} \tag{1.6}$$

for every positive integer $n \in \mathbb{N}$, and every $z \in \Delta$.

Remark 3. Taking $a(z) = 1 + 3e^z$, $z \in \Delta = \{z\colon |z| < 1\}$, and $\mathcal{F} = \{f_n(z) \mid f_n(z) = e^{nz}, z \in \Delta,\ n \in \mathbb{N}\}$, so we immediately have that

(a) $a(z) \neq 1$, for every $z \in \Delta$,
(b) there exists a positive number $\varepsilon_0 = \frac{3}{e} - 1 > 0$ such that $|a_n(a(z)-1)-1| \geqslant \varepsilon_0$ for arbitrary sequences of integers a_n and any $z \in \Delta$, and
(c) $f_n(z)f_n''(z) - a(z)(f_n'(z))^2 \neq 0$, for every $f_n(z) \in \mathcal{F}$, and any $z \in \Delta$.

Then from Theorem 1.6, we deduce that $\{\frac{f_n'(z)}{f_n(z)}\colon f_n(z) \in \mathcal{F}\}$ is normal in Δ. In fact, it is clear that $\{\frac{f_n'(z)}{f_n(z)}\colon f_n(z) \in \mathcal{F}\} = \{n\colon f_n(z) \in \mathcal{F}\}$ is normal in Δ.

This example $f_n(z) = e^{nz}$ only implies that there indeed exists function $a(z)$, which does not identically equal to a constant a in Δ, such that $a(z)$ satisfies condition (i) and a decadent case of (ii) in Theorem 1.6, in which $f_n(z)$ satisfies $f_n(z)f_n''(z) - a(z)(f_n'(z))^2 \neq 0$ for any $z \in \Delta$.

In particular, from Remark 1 above we have that if holomorphic function $a(z)$ in Theorem 1.6 is a constant a, then we have a corollary of Theorem 1.6 as follows.

Corollary 1.7. *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disc Δ. Suppose that there exists a constant a, $a \neq 1$, $\frac{n\pm 1}{n}$, such that there exists a positive number $B > 0$ such that for every $f(z) \in \mathcal{F}$, $B|f'(z)| \leqslant |f(z)|$ whenever $f(z)f''(z) - a(f'(z))^2 = 0$ in Δ. Then $\{\frac{f'(z)}{f(z)}\colon f(z) \in \mathcal{F}\}$ is normal in Δ, where $n \in \mathbb{N}$.*

For the case that $f(z)f''(z) - a\cdot(f'(z))^2 \neq 0$, for $z \in \Delta$, Corollary 1.7 properly is Theorem 1.5 proved by W.C. Lin, H.X. Yi (see Theorem 2 in [8]). In fact, we also have the following corollary from Theorem 1.6.

Corollary 1.8. *Let $\mathcal{F}$ be a family of meromorphic functions and let $\mathcal{F}$ be a family of functions meromorphic in the unit disc Δ. Suppose that there exists a function $a(z)$ holomorphic in Δ, $a(z) \neq 1$, for each $z \in \Delta$, such that*

(i) *there exists a positive number ε_0 such that*

$$\left|a_n\big(a(z)-1\big)-1\right| \geqslant \varepsilon_0 \tag{1.7}$$

for any integer a_n, and any $z \in \Delta$.

(ii) *For every $f(z) \in \mathcal{F}$, and each $z \in \Delta$, $f(z)f''(z) - a(z)(f'(z))^2 \neq 0$.*

Then $\{\frac{f'(z)}{f(z)}\colon f(z) \in \mathcal{F}\}$ is normal in Δ.

If $a(z)$ is identically equal to a constant a in Δ, and $a \neq 1$, $\frac{n\pm 1}{n}$ $(n \in \mathbb{N})$, then from $|a_n(a-1)-1| \geqslant \varepsilon_0$ of Lemma 3.1 of [8], it follows that $a(z)$ satisfies (1.7). Also in this case the condition $ff^{(2)} - af^2 \neq 0$ is the same as condition (ii) in Corollary 1.8. Therefore, from Corollary 1.8 we immediately deduce that $\{\frac{f'(z)}{f(z)}\colon f(z) \in \mathcal{F}\}$ is normal in Δ. This shows that Corollary 1.8 is a generalizations of Theorem 1.5. From this meaning, Theorem 1.6 generalizes Theorem 1.5 due to Lin and Yi [8].

2. Some lemmas

To prove the above theorems, we need some lemmas as follow:

Lemma 2.1. *(See [2].) Let $g(z)$ be a transcendental meromorphic function with finite order. If g(z) has only finitely many critical values, then $g(z)$ has only finitely many asymptotic values.*

Lemma 2.2. *(See [1,7].) Let $g(z)$ be a transcendental meromorphic function and suppose that the set of all finite critical and asymptotic values of $g(z)$ is bounded. Then there exists $R > 0$ such that if $|z| > R$ and $|g(z)| > R$, then*

$$\frac{|g'(z)|}{|g(z)|} \geqslant \frac{\log|g(z)|}{16\pi|z|}.$$

Lemma 2.3. *(See [3].) Let $f(z) = a_n z_n + a_{n-1} z_{n-1} + \cdots + a_0 + \frac{p(z)}{q(z)}$, where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$, $p(z)$ and $q(z)$ are two co-prime polynomials with* $\deg p(z) < \deg q(z)$, *let k be a positive integer. If $f^{(k)}(z) \neq 1$, then*

$$f(z) = \frac{z^k}{k!} + \cdots + a_0 + \frac{1}{(az+b)^m}$$

where a $(\neq 0)$, b are constants, m is a positive integer.

Lemma 2.4. *(See [9].) Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ, all of whose zeros have multiplicity at least k, and suppose there exists $A \geqslant 1$ such that $|f^{(k)}(z)| \leqslant A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leqslant \alpha \leqslant k$:*

(a) *a number r, $0 < r < 1$,*
(b) *points z_n, $|z_n| < r$,*
(c) *functions $f_n \in \mathcal{F}$, and*
(d) *positive numbers $\rho_n \to 0$ such that*

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} = g_n(\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where $g(\xi)$ is a meromorphic function on $\mathbb{C}$ such that

$$g^{\#}(\xi) \leqslant g^{\#}(0) = kA + 1.$$

From Lemma 2.2,we have

Lemma 2.5. *(See [8].) Let $f(z)$ be a meromorphic function with finite order, all of whose zeros are of multiplicity (at least) k, and let A be a positive real number. If $|f^{(k)}(z)| \leqslant A$ when $f(z)=0$, then for each l, $1 \leqslant l \leqslant k$, $f^{(l)}(z)$ assumes any finite non-zero value infinitely often.*

Lemma 2.6. *(See [1].) Let $f(z)=z+a+\frac{b}{(z+c)^l}$ with $a,b,c \in C$, $b \neq 0$, $l \in \mathbb{N}$ and let $p \in \{0,1,2,\dots,l\}$, then*

$$\operatorname{Res}\left[\frac{(f')^p}{f}, -c\right] = 1-(l+1)^p.$$

3. Proofs of theorems

3.1. Proof of Theorem 1.3

Suppose that $\mathcal{F}$ is not normal in D, then there exists point $z_0 \in D$ such that $\mathcal{F}$ is not normal at z_0. From Lemma 2.4, there exist function family $f_n \subseteq \mathcal{F}$, points z_n, $z_n \to z_0$, positive numbers $\rho_n \to 0$ such that

$$\frac{f_n(z_n+\rho_n\xi)}{\rho_n} = g_n(\xi) \to g(\xi) \tag{3.1}$$

locally uniformly with respect to the spherical metric, where $g(\xi)$ is a meromorphic function on $\mathbb{C}$ such that

$$g^{\#}(\xi) \leqslant g^{\#}(0) = A+1, \tag{3.2}$$

$$f_n'(z_n+\rho_n\xi) = g_n'(\xi) \to g'(\xi). \tag{3.3}$$

We may claim that the following conclusions are true.

(I) $|g'(\xi)| \leqslant A$ whenever $g(\xi)=0$.

In fact, suppose that there exists point ξ_0 such that $g(\xi_0)=0$, by Hurwitz's Theorem, there exists point sequence $\xi_n \to \xi_0$ such that $g_n(\xi_n)=0$, so $f_n(z_n+\rho_n\xi_n)=0$. It follows that $|g'(\xi_0)| \leqslant A$ from the conditions that $0<|f'(z)| \leqslant A$, whenever $f(z)=0$.

(II) $g'(\xi) \neq b(z_0)$.

Suppose that $g'(\xi_0)=b(z_0) \neq 0$, if $g'(\xi) \equiv b(z_0)$, then $g(\xi) \equiv b(z_0)\xi + b_0$ and $|b(z_0)| \leqslant A$. We may deduce that

$$g^{\#}(0) = \frac{|b(z_0)|}{1+|b_0|^2} \leqslant A < A+1$$

which is a contradiction to formula (3.2), thus $g'(\xi) \not\equiv b(z_0)$. Now again by Hurwitz's Theorem, there exists point sequence $\xi_n \to \xi_0$ such that $g_n'(\xi_n)=b(z_n+\rho_n\xi_n)$, so $f_n'(z_n+\rho_n\xi_n)=b(z_n+\rho_n\xi_n)$. It follows that $g(\xi_0)=\infty$ from the conditions that $|f(z)| \geqslant B>0$ whenever $f'(z)=b(z)$. This is also a contradiction.

(III) $g(\xi)$ is non-polynomials rational function.

Suppose not, then $g(\xi)$ is either polynomials function or meromorphic and transcendental function with order 2 at most. Suppose that $g(\xi)$ is polynomials function, we distinguish two cases.

Case 1. $\deg g(\xi) \geqslant 2$, then there exists a point ξ such that $g'(\xi)=b(z_0)$, this is a contradiction to the conclusion (II).

Case 2. $\deg g(\xi)=1$, then $g(\xi)=a\xi+b$ where $|a| \leqslant A$ and $g^{\#}(0)=\frac{|a|}{1+|b|^2} \leqslant A < A+1$, a contradiction.

If $g(\xi)$ is meromorphic and transcendental function with order 2 at most, we can deduce a contradiction from Lemma 2.5 above. Thereby, the conclusion (III) also holds.

Now again from Lemma 2.3, we have the expression of $g(\xi)$

$$g(\xi) = b(z_0)\xi + a + \frac{b}{(\xi + c)^k} \tag{3.4}$$

where $a, b, c \in \mathbb{C}$, $b \neq 0$, $k \in \mathbb{N}$.

We write $m := k + 1$ and $R > \max_{1 \leqslant j \leqslant m} |\xi_j|$, where $\xi_j (1 \leqslant j \leqslant m)$ are the zeros of $g(\xi)$. For sufficiently large n we find m distinct zeros $\xi_{j,n} \in D(0, R)$ $(1 \leqslant j \leqslant m)$ such that $g_n(\xi_{j,n}) = 0$ for $1 \leqslant j \leqslant m$. Denoting $\zeta_{j,n} := z_n + \rho_n \xi_{j,n}$, $1 \leqslant j \leqslant m$, then $\zeta_{j,n}$ $(1 \leqslant j \leqslant m)$ are the zeros of f_n. Moreover, $\zeta_{j,n} \in \Delta_n := D(z_n, \rho_n R)$ for $1 \leqslant j \leqslant m$, and for sufficiently large n, $\Delta_n \subset D$ and f_n has no further zeros in Δ_n. Therefore, by (3.3), we deduce the next limit as follows:

$$\sum_{j=1}^{m} f_n'(\zeta_{j,n})^{-1} = \sum_{j=1}^{m} g_n'(\xi_{j,n})^{-1} = \sum_{j=1}^{m} \mathrm{Res}\left(\frac{1}{g_n}, \xi_{j,n}\right) \to \sum_{\xi \in g^{-1}(0)} \mathrm{Res}\left(\frac{1}{g}, \xi\right). \tag{3.5}$$

On the other hand, from (3.4) we have that

$$\frac{1}{g(\xi)} = \frac{1}{b(z_0)\xi} + O\left(\frac{1}{\xi^2}\right)$$

as $\xi \to \infty$, and hence by (3.4), (3.5) we obtain the following limits

$$\sum_{j=1}^{m} f_n'(\zeta_{j,n})^{-1} \longrightarrow \frac{1}{b(z_0)}.$$

This is a contradiction to (1.3). Therefore, the conclusion of Theorem 1.3 holds.

3.2. *Proof of Theorem 1.6*

In the sequel, we shall give the complete proof of Theorem 1.6 by Theorem 1.3.

Setting

$$h(z) := -\frac{f(z)}{f'(z)}, \quad f(z) \in \mathcal{F}. \tag{3.6}$$

Then we only have to prove that the family $\mathcal{H} := \{h(z), f \in \mathcal{F}\}$ is normal in Δ.

Now we counter the first derivative $h'(z)$ of $h(z)$ and we have

$$h'(z) = \frac{f(z)f''(z) - (f'(z))^2}{(f'(z))^2} = \frac{ff'' - a(z)(f')^2}{(f')^2} + b(z) \tag{3.7}$$

where $b(z) \equiv a(z) - 1$. It is clear to see that the following inequality from the condition (i) in Theorem 1.6. is

$$\left|a_n b(z) - 1\right| \geqslant \varepsilon_0 \tag{3.8}$$

for any integer a_n and for every $z \in \Delta$. So, we have that

$$b(z) \neq \pm\frac{1}{n}, \quad z \in \Delta. \tag{3.9}$$

Firstly, by a simple computations, we have that if ξ is a zero or a pole of $f(z)$ with an order n, $n \in \mathbb{N}$, then $h'(\xi) = \mp\frac{1}{n}$.

We may claim that

(i) If $h(\xi) = 0$, $\xi \in \Delta$, then $0 < |h'(\xi)| \leqslant 1$.
(ii) If there exist points $z \in \Delta$ such that $h'(z) = b(z)$, then $|h(z)| \geqslant B$.
(iii) If Δ_1 is a disk in Δ and if $h(z)$ has $m \geqslant 2$ zeros $z_1, z_2, \ldots, z_m \in \Delta_1$, then

$$\left|b(z) \cdot \left(\sum_{j=1}^{m} h'(z_j)^{-1}\right) - 1\right| \geqslant \varepsilon_0. \tag{3.10}$$

In fact, if ξ is a zero of $h(z)$, from (3.6) we know that ξ is either $f(\xi)=0$ or $f(\xi)=\infty$, thus $h'(\xi)=\mp\frac{1}{n}$. Then the claim (i) above holds.

Now suppose that there exists a point z_0, $z_0\in\Delta$, such that $h'(z_0)=b(z_0)$, then from (3.7) we may deduce similarly that

$$f(z_0)f''(z_0)-a(z_0)\big(f'(z_0)\big)^2=0. \tag{3.11}$$

So $B|f'(z_0)|\leqslant|f(z_0)|$. On the other hand, from (3.11) we have that $f'(z_0)\neq 0$. Otherwise, if $f'(z_0)=0$, we know that $f(z_0)=0$ or $f''(z_0)=0$ from (3.11). If $f(z_0)=0$, then we obtain that $h'(z_0)=-\frac{1}{n}$ which contradicts (3.9). Thus, we have $f(z_0)\neq 0$. Combining with $f'(z_0)=0$, we deduce that $h(z_0)=\infty$ which contradicts $h'(z_0)=b(z_0)$. Hence, we deduce that $f'(z_0)\neq 0$ and arrive at $|h(z_0)|\geqslant B$. That is, the claim (ii) above holds.

Finally, suppose that $\Delta_1\subset\Delta$ is a disk and $h(z)$ has $m\geqslant 2$ zeros $z_1,z_2,\ldots,z_m$ in Δ_1. Denoting

$$a_m=\sum_{j=1}^{m}h'(z_j)^{-1}.$$

It is not difficult to see that a_m is a sequence of integers. From (3.8), we have that there exists a positive ε_0 such that

$$\left|b(z)\cdot\left(\sum_{j=1}^{m}h'(z_j)^{-1}\right)-1\right|\geqslant\varepsilon_0. \tag{3.12}$$

Thereby, the claim (iii) above is true also.

Above all, by Theorem 1.3 and from the claim (i)–(iii), we deduce immediately that $\mathcal{H}$ is normal in Δ. Thereby, the conclusion of Theorem 1.6 holds. This gives the complete proof of Theorem 1.6.

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