J. Math. Anal. Appl. 350 (2009) 252–255

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Weak* continuous states on Banach algebras ☆

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ARTICLE INFO

Article history:
Received 11 August 2008
Available online 2 October 2008
Submitted by D. Blecher

Keywords:
Banach algebra
Dissipative elements
Normal states

ABSTRACT

We prove that if a unital Banach algebra A is the dual of a Banach space $A_\sharp$ then the set of normal states is weak* dense in the set of all states on A. Further, normal states linearly span $A_\sharp$.

1. Introduction

An important tool in functional analysis is Goldstine's theorem [5, 3.27], which says that for a dual Banach space A the unit ball of its predual $A_\sharp$ is weak* dense in the unit ball $B_{A^\sharp}$ of the dual $A^\sharp$ of A. Given a norm one element $x \in A$, we may consider the set of 'states' $S^x(A) = \{\rho \in B_{A^\sharp}\colon\ \rho(x) = 1\}$ and the subset of 'normal states' $S^x_n(A) = S^x \cap A_\sharp$. In general $S^x_n(A)$ need not be weak* dense in $S^x(A)$, for $S^x_n(A)$ may even be empty if x does not achieve its norm as a functional on $A_\sharp$. In this note we show that if A is a unital Banach algebra and $x = 1$ is the unit of A (with $\|1\| = 1$), then the set of normal states $S_n(A) := S^1_n(A)$ is weak* dense in the set $S(A) := S^1(A)$ of all states. Using this, we also show that $S_n(A)$ spans the predual of A. Of course, all this is well known for von Neumann algebras. That $S(A)$ spans $A^\sharp$ for any unital Banach algebra A was proved by Moore [7] ([1,10] contain simpler proofs).

Our method is based on a consideration of dissipative elements. Recall that an element $a \in A$ is *dissipative* if its numerical range $W(a) := \{\rho(a)\colon\ \rho \in S(A)\}$ is contained in the left half-plane $\operatorname{Re} z \leqslant 0$. To show that the set D_A of all such elements is weak* closed, if A is a dual space, we will need a suitable metric characterization of dissipative elements (Lemma 2.1 below). A similar, but not the same, characterization was observed in [2] for C*-algebras; however, the argument from [2] does not apply to Banach algebras. For each $a \in D_A$ the element $1 - a$ is invertible since its numerical range (hence also its spectrum) is contained in the half-plane $\operatorname{Re} z \geqslant 1$. We will only need the estimate

$$\left\|(1-a)^{-1}\right\| \leqslant 1 \quad (a \in D_A), \tag{1.1}$$

which is known from the Hille–Yosida theorem on generators of operator semigroups [6]. In our present context it can easily be derived by applying the well-known estimate $\|e^{ta}\| \leqslant 1$, $t \geqslant 0$ (see [4, p. 55] or [11, A13(4)]) to the integral representation $(1-a)^{-1} = \int_0^\infty e^{-t(1-a)}\,dt$, which in turn can be verified directly.

☆ Partially supported by the Ministry of Science and Education of Slovenia.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.09.072

2. Dissipative elements and normal states

Lemma 2.1. *An element a of a unital Banach algebra A is dissipative if and only if*

$$\|1 + ta\| \leqslant 1 + t^2\|a\|^2 \quad \text{for all } t \geqslant 0. \tag{2.1}$$

In particular, if $\|a\| \leqslant 1$, then $a \in D_A$ if and only if $\|1 + ta\| \leqslant 1 + t^2$ for all $t \geqslant 0$.

Proof. If a satisfies (2.1), then for every state $\rho \in S(A)$ and $t > 0$ we have that $|1 + t\rho(a)|^2 = |\rho(1 + ta)|^2 \leqslant \|1 + ta\|^2 \leqslant (1 + t^2\|a\|^2)^2$. This implies that $2\operatorname{Re}\rho(a) \leqslant t(2\|a\|^2 - |\rho(a)|^2) + t^3\|a\|^4$, hence (letting $t \to 0$) $\operatorname{Re}\rho(a) \leqslant 0$.

For a proof of the reverse direction, note that by (1.1) each $a \in D_A$ satisfies

$$\|1 + a\| = \big\|(1 - a)^{-1}\big(1 - a^2\big)\big\| \leqslant \big\|1 - a^2\big\| \leqslant 1 + \|a\|^2.$$

But, since ta is also dissipative if $t \geqslant 0$, we may replace a by ta in the last inequality, which yields (2.1). The last sentence of the lemma follows now easily. □

In C*-algebras the estimate (2.1) can be improved to $\|1 + ta\|^2 \leqslant 1 + t^2\|a\|^2$ ($a \in D_A$, $t \geqslant 0$), a consequence of the C*-identity [2]. This sharper estimate holds also in some other natural examples of Banach algebras, but the author does not know if it holds in general. Since this topic is not essential for our purposes here, we will postpone further discussion on it to the end of the paper.

Theorem 2.2. *If a unital Banach algebra A is a dual Banach space, then D_A is a weak* closed subset of A. Moreover, $S_n(A)$ is weak* dense in $S(A)$.*

Proof. The proof is the same as for operator spaces [2]. Since it is very short, we will sketch it here for completeness. Since D_A is convex and $tD_A \subseteq D_A$ if $t \geqslant 0$, to prove that D_A is weak* closed, it suffices to show that the intersection of D_A with the closed unit ball of A is weak* closed (see e.g. [5, 4.44]). But this follows immediately from Lemma 2.1.

Denote by $A^{\sharp+}$ the set of all nonnegative multiples of states on A and by $(D_A)^\circ$ the set of all $\rho \in A^\sharp$ such that $\operatorname{Re}\rho(a) \leqslant 0$ for all $a \in D_A$. Clearly $A^{\sharp+} \subseteq (D_A)^\circ$. To prove that $A^{\sharp+} = (D_A)^\circ$, let $\rho \in (D_A)^\circ$. Since $it1 \in D_A$ for all $t \in \mathbb{R}$ and $-1 \in D_A$, it follows that $\rho(1) \geqslant 0$. Since $a - \|a\|1 \in D_A$ for each $a \in A$, we have that $\operatorname{Re}\rho(a) \leqslant \|a\|\rho(1)$. Replacing in this inequality a by zx for all $z \in \mathbb{C}$ with $|z| = 1$, it follows that $|\rho(a)| \leqslant \|a\|\rho(1)$, hence $\rho \in A^{\sharp+}$.

Now put $A_\sharp^+ = A_\sharp \cap A^{\sharp+}$ and $(D_A)_\circ = (D_A)^\circ \cap A_\sharp$. Then $A_\sharp^+ = (D_A)_\circ$. Since D_A is weak* closed, a bipolar type argument shows that $D_A = ((D_A)_\circ)^\circ$ and that $(D_A)_\circ$ is weak* dense in $(D_A)^\circ$. This means that $A_\sharp^+$ is weak* dense in $A^{\sharp+}$. Now it follows easily that $S_n(A)$ is weak* dense in $S(A)$. □

Corollary 2.3. *Let A be as in Theorem 2.2. For every closed convex subset C of $\mathbb{C}$ the set $A_C = \{a \in A\colon\ W(a) \subseteq C\}$ is weak* closed in A.*

Proof. Since C is the intersection of half-planes containing it, this follows from the fact that $A_{\{\operatorname{Re} z \leqslant 0\}} = D_A$ is weak* closed. □

Theorem 2.4. *If A is as in Theorem 2.2, then $S_n(A)$ spans $A_\sharp$. Each $\omega \in A_\sharp$ with $\|\omega\| < (e\sqrt{2})^{-1}$ (where $\log e = 1$) can be written as $\omega = t_1\omega_1 - t_2\omega_2 + i(t_3\omega_3 - t_4\omega_4)$, where $\omega_j \in S_n(A)$ and $t_j \in [0, 1]$.*

Proof. Put $S = S(A)$ and $S_n = S_n(A)$. For a subset V of A define the polar $V^\diamond$ by $V^\diamond = \{\rho \in A^\sharp\colon\ |\operatorname{Re}\rho(a)| \leqslant 1\ \forall a \in V\}$. In the same way define also polars of subsets of $A_\sharp$ and 'prepolars' $V_\diamond$ of subsets of A or $A^\sharp$. Let $U = S_\diamond$. Then U is the set of all elements $a \in A$ with the numerical range contained in the strip $|\operatorname{Re} z| \leqslant 1$, hence U is weak* closed by Corollary 2.3. Since S_n is weak* dense in S_n by Theorem 2.2 and $U = S_\diamond$, it follows that $U = S_n^\diamond$, hence by the bipolar theorem $U_\diamond = \overline{\mathrm{co}}(-S_n \cup S_n)$. Let $V = iU \cap U$. Then $V_\diamond$ is equal to the norm closure of the convex hull of $(iU)_\diamond \cup U_\diamond$, hence (since $(iU)_\diamond = -iU_\diamond = -i\,\overline{\mathrm{co}}(-S_n \cup S_n)$) $V_\diamond$ is the closure of the set $S_0 := \mathrm{co}(S_n \cup (-S_n) \cup (iS_n) \cup (-iS_n))$. On the other hand, by the definition of U, V is just the set of all $a \in A$ with the numerical range $W(a)$ contained in the square $[-1, 1] \times [-i, i]$. Since for every $a \in A$ the inequality $\|a\| \leqslant ew(a)$ holds, where $w(a)$ is the numerical radius of a (see [4] or [8, 2.6.4]), it follows that V is contained in the closed ball $(\sqrt{2}e)B_A$ of A with the center 0 and radius $\sqrt{2}e$. Consequently $\overline{S_0} = V_\diamond \supseteq (\sqrt{2}e)^{-1}B_{A_\sharp}$.

Let $T = \{t\omega\colon\ \omega \in S_n,\ t \in [0, 1]\}$. Since S_n is norm closed and bounded and the interval $[0, 1]$ is compact, it is not hard to verify that T is norm closed. Further, since $S_n \subseteq T$ is convex and $tT \subseteq T$ for all $t \in [0, 1]$, it follows from the definition of S_0 that $S_0 \subseteq T_0 := T - T + iT - iT$. Therefore we conclude from the previous paragraph that $B_{A_\sharp} \subseteq \sqrt{2}e\overline{T_0}$. Thus, given $\omega \in A_\sharp$ and $\delta \in (0, 1)$, there exists $\omega_0 \in \|\omega\|\sqrt{2}eT_0$ such that $\|\omega - \omega_0\| < \delta$. Applying the same to $\omega - \omega_0$, we find $\omega_1 \in \delta\sqrt{2}eT_0$ such that $\|\omega - \omega_0 - \omega_1\| \leqslant \delta^2$. Continuing, we find a sequence of functionals $\omega_n \in \delta^n\sqrt{2}eT_0$ such that

$$\|\omega - \omega_0 - \cdots - \omega_n\| \leqslant \delta^{n+1}.$$

Thus $\omega = \omega_0 + \sum_{n=1}^{\infty} \omega_n$ is of the form

$$\omega = \sqrt{2}e\left(\|\omega\|\rho_0 + \sum_{n=1}^{\infty} \delta^n \rho_n\right), \tag{2.2}$$

where $\rho_n \in T_0$. By the definition of T_0 we have that $\rho_n = \rho_{n,1} - \rho_{n,2} + i(\rho_{n,3} - \rho_{n,4})$, where $\rho_{n,j} \in T = \mathrm{co}(\{0\} \cup S_n)$. Put $\gamma = \|\omega\| + \sum_{n=1}^{\infty} \delta^n = \|\omega\| + \delta(1-\delta)^{-1}$. Since T is closed and convex, $\psi_j := \gamma^{-1}(\|\omega\|\rho_{0,j} + \sum_{n=1}^{\infty} \delta^n \rho_{n,j}) \in T$ for each j. From (2.2) we have now

$$\omega = (\sqrt{2}e)\gamma\big(\psi_1 - \psi_2 + i(\psi_3 - \psi_4)\big), \tag{2.3}$$

a linear combination of normal states. If $\|\omega\| < (e\sqrt{2})^{-1}$, we may choose δ so small that $\gamma \leqslant (e\sqrt{2})^{-1}$, and we then conclude from (2.3) that ω is of the form $\omega = \sum_{j=0}^{3} t_j i^j \omega_j$, where $\omega_j \in S_n$ and $t_j \in [0,1]$. □

The well-known characterizations of hermitian elements in a Banach algebra [4,8,12] do not seem to imply easily that the real subspace A^h of all such elements is weak* closed, if A is a dual Banach space. For this reason we state here another simple characterization. In the case of C*-algebras this characterization has been observed earlier by others and demonstrated to be useful [3].

Proposition 2.5. *An element h in a unital Banach algebra A is hermitian if and only if*

$$\|h + it1\|^2 \leqslant \|h\|^2 + t^2 \quad \text{for all } t \in \mathbb{R}. \tag{2.4}$$

Thus, if A is a dual Banach space, then A^h is a weak closed subset of A.*

Proof. If (2.4) holds, then for each $\rho \in S(A)$,

$$\left|\rho(h) + it\right|^2 = \left|\rho(h + it1)\right|^2 \leqslant \|h\|^2 + t^2 \quad (t \in \mathbb{R}), \tag{2.5}$$

which implies (by letting $t \to \infty$) that $\rho(h) \in \mathbb{R}$, hence h is hermitian. Conversely, if h is hermitian then (2.5) holds. But, by a result of Sinclair [9] the norm of an element of the form $a = h + \lambda 1$, where h is hermitian and $\lambda \in \mathbb{C}$, is equal to the spectral radius $r(a)$, hence also to the numerical radius $w(a)$ (since in general $r(a) \leqslant w(a) \leqslant \|a\|$). Thus, taking in (2.5) the supremum over all states $\rho \in S(A)$, we get (2.4). □

The estimate

$$\|1 + a\|^2 \leqslant 1 + \|a\|^2, \tag{2.6}$$

which holds for all dissipative elements in C*-algebras, holds in general Banach algebras at least for dissipative elements of a special form. For example, using the fact that for hermitian elements the norm is equal to the spectral radius, it can be shown that (2.6) holds for dissipative hermitian elements.

Proposition 2.6. *In any unital Banach algebra A each dissipative element of the form $a = -tp$, where p is an idempotent and $t \in (0,\infty)$, satisfies the estimate* (2.6)*.*

Proof. By the well-known criterion [4, p. 55] $-p$ is dissipative if and only if $\|e^{-tp}\| \leqslant 1$ for all $t \geqslant 0$. From the Taylor series expansion of e^{-tp} we compute that $e^{-tp} = (1-s)q + s1$, where $q = 1 - p$ and $s = e^{-t}$. So, $-p$ is dissipative if and only if $\|(1-s)q + s1\| \leqslant 1$ for all $s \in [0,1]$, which is equivalent to $\|q\| \leqslant 1$. Since the norm of any nonzero idempotent is at least 1, we conclude that $-p$ (hence also $-sp$, if $s > 0$) is dissipative if and only if $\|q\| = 1$.

To prove (2.6), where $a = -tp$ is assumed dissipative (thus $\|q\| = 1$), consider first the case when $t \in (0,1]$. Put $s = 1 - t$ and note that

$$\|1 - tp\|^2 = \left\|s1 + (1-s)q\right\|^2 \leqslant \big(s + (1-s)\|q\|\big)^2 = 1 \leqslant 1 + t^2\|p\|^2.$$

On the other hand, if $t > 1$, then

$$\|1 - tp\|^2 = \left\|q + (1-t)p\right\|^2 \leqslant \big(1 + (t-1)\|p\|\big)^2 = 1 + t^2\|p\|^2 - 2t\|p\|\big(\|p\| - 1\big) - \|p\|\big(2 - \|p\|\big) \leqslant 1 + t^2\|p\|^2,$$

since $1 \leqslant \|p\| = \|1 - q\| \leqslant 2$. □

Question. Do all dissipative elements in each unital Banach algebra satisfy (2.6)?

Acknowledgment

The author is grateful to David Blecher for a question which prompt the investigation presented here.

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