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Scalarization approaches for set-valued vector optimization problems and vector variational inequalities ☆

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ABSTRACT

The scalarization approaches of Giannessi, Mastroeni and Pellegrini are extended to the study of set-valued vector optimization problems and set-valued weak vector optimization problems. Some equivalence results among set-valued (scalar) optimization problems, set-valued (scalar) quasi-optimization problems, set-valued vector optimization problems and set-valued weak vector optimization problems are established under the convexity assumption of objective functions. Some examples are provided to illustrate these results. The approaches are furthermore exploited to investigate the set-valued vector variational inequalities and set-valued weak vector variational inequalities, which are different from that suggested by Konnov. Some equivalence relations among set-valued (scalar) variational inequalities, set-valued (scalar) quasi-variational inequalities, set-valued vector variational inequalities and set-valued weak vector variational inequalities are also derived under some suitable conditions.

1. Introduction

Let R^m and R^n be the m and n dimensional Euclidean spaces, respectively, where m, n are given positive integers. Denote by $R^n_+ = \{x = (x_1, \dots, x_n)^\top \colon x_i \geqslant 0,\ i = 1, \dots, n\}$, where the symbol $\top$ denotes the transpose. A nonempty subset P of R^n is said to be a cone with apex at the origin iff $\lambda P \subseteq P$ for all $\lambda > 0$. P is said to be a convex cone iff P is a cone and $P + P = P$. P is called a pointed cone iff P is a cone and $P \cap (-P) = \{0\}$. The dual cone (or positive polar cone) of a convex cone P is given by

$$P^* = \left\{z \in R^n \colon \langle z, x\rangle \geqslant 0,\ \forall x \in P\right\},$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product.

Let $C \subseteq R^n$ be a closed, convex and pointed cone with the nonempty interior, i.e., $\operatorname{int} C \neq \emptyset$, and K be a nonempty convex subset of R^m. Let $F : K \to 2^{R^n}$ be a set-valued function with nonempty value. In this paper, we consider the following set-valued vector optimization problem:

$$\min_{C\setminus\{0\}} F(x), \qquad \text{subject to} \quad x \in K, \tag{1.1}$$

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where $\min_{C\backslash\{0\}}$ denotes vector minimum with respect to the cone $C\backslash\{0\}$. A pair (x^*, y^*) with $x^* \in K$ and $y^* \in F(x^*)$ is called a vector minimal solution of F on K iff

$$\big(y^* - F(K)\big) \cap \big(C\backslash\{0\}\big) = \emptyset,$$

where $F(K) = \bigcup_{x\in K} F(x)$.

We also consider the set-valued weak vector optimization problem as follows:

$$\min_{\operatorname{int} C} F(x), \qquad \text{subject to} \quad x \in K, \tag{1.2}$$

where $\min_{\operatorname{int} C}$ denotes vector minimum with respect to the cone $\operatorname{int} C$. A pair (x^*, y^*) with $x^* \in K$ and $y^* \in F(x^*)$ is called a weak vector minimal solution of F on K iff

$$\big(y^* - F(K)\big) \cap \operatorname{int} C = \emptyset.$$

Denote by $\operatorname{Min}_{C\backslash\{0\}} F(K)$ and $\operatorname{Min}_{\operatorname{int} C} F(K)$ the set of all vector minimal solutions of F on K and the set of all weak vector minimal solutions of F on K, respectively. It is clear that $\operatorname{Min}_{C\backslash\{0\}} F(K) \subseteq \operatorname{Min}_{\operatorname{int} C} F(K)$.

If $F : K \to R^n$ is a single-valued function, then (1.1) and (1.2) reduce to generalized vector Pareto problem (for short, GVPP) and generalized weak vector Pareto problem (for short, GWVPP), respectively (see, for example, [1,10,13] and the references therein). Furthermore, if $C = R^n_+$, then (1.1) and (1.2) become the classic vector Pareto problem (for short, VPP) and the classic weak vector Pareto problem (for short, WVPP), respectively.

Let $\Phi : K \to 2^{R^{n\times m}}$ be a set-valued function with matrix-values. We will consider the following set-valued vector variational inequalities: find $x^* \in K$ and $T^* \in \Phi(x^*)$ such that

$$T^*(y - x^*) \nleq_{C\backslash\{0\}} 0 \quad \forall y \in K, \tag{1.3}$$

where the inequality denotes $T^*(y - x^*) \notin -C\backslash\{0\}$. (1.3) can be called Stampacchia set-valued vector variational inequalities. A pair (x^*, T^*) with $x^* \in K$ and $T^* \in \Phi(x^*)$ is called a solution of (1.3). It is clear that (1.3) can be rewritten as

$$(T^*x^* - T^*K) \cap \big(C\backslash\{0\}\big) = \emptyset,$$

where $T^*K = \{T^*y\colon\ y \in K\}$.

We also consider the following set-valued weak vector variational inequalities: find $x^* \in K$ and $T^* \in \Phi(x^*)$ such that

$$T^*(y - x^*) \nleq_{\operatorname{int} C} 0 \quad \forall y \in K, \tag{1.4}$$

where the inequality means $T^*(y - x^*) \notin -\operatorname{int} C$. A pair (x^*, T^*) with $x^* \in K$ and $T^* \in \Phi(x^*)$ is called a solution of (1.4). Analogously, (1.4) can be rewritten as

$$(T^*x^* - T^*K) \cap \operatorname{int} C = \emptyset.$$

Denote by S_S and S_{SW} the set of all solutions of (1.3) and the set of all solutions of (1.4), respectively. Clearly, $S_S \subseteq S_{SW}$.

As it is well known, vector optimization problems (for short, VOP) are closely related to vector variational inequalities (for short, VVI) (see, for example, [1–3] and the references therein). Among solution approaches for VOP and VVI, scalarization is one of the most analyzed topics at least from the computational point of view (see, for example, [1,6–10,12,13]). In [4], Giannessi, Mastroeni and Pellegrini presented some scalarization approaches for GVPP and GWVPP. They set up scalar minimization problems for GVPP and GWVPP. Furthermore, they extended these methods to the study of VVI. Goh and Yang [5] also established a scalar variational inequality (for short, VI) for a weak vector variational inequality (for short, WVVI) and studied relationships between VI and WVVI. In [9], Konnov suggested a scalarized VI for (1.4) (in Banach spaces) and presented the equivalence between them under the assumption of each component of set-valued function Φ having convex and compact values.

Inspired by the work mentioned above, the purpose of this paper is to present some scalarization methods for (1.1)–(1.4). For this, we establish some equivalence relationships among set-valued (scalar) optimization problems (for short, SOP) and set-valued (scalar) quasi-optimization problems (for short, SQOP), (1.1) and (1.2) under convexity assumption of the objective functions. We then give some examples to illustrate the scalarization techniques. Furthermore, we exploit similar approaches to study (1.3) and (1.4). We also derive some equivalence relations among set-valued (scalar) variational inequalities (for short, SVI), set-valued (scalar) quasi-variational inequalities (for short, SQVI), (1.3) and (1.4) under suitable conditions. The scalarization approach for (1.4) presented in this paper is different from that of Konnov [9].

2. Preliminaries

In this section, we will recall some basic definitions and present several useful lemmas and propositions.

It is well known that the convexity plays an important role in the investigation of optimization problems and variational inequality problems. We recall first the following concepts for set-valued functions.

Definition 2.1. A set-valued function $F : K \to 2^{R^n}$ is said to be

(i) affine iff

$$\alpha F(x) + (1-\alpha)F(y) \subseteq F\big(\alpha x + (1-\alpha)y\big) \quad \forall x, y \in K,\ \alpha \in [0,1];$$

(ii) C-multifunction iff

$$\alpha F(x) + (1-\alpha)F(y) \subseteq F\big(\alpha x + (1-\alpha)y\big) + C \quad \forall x, y \in K,\ \alpha \in [0,1];$$

(iii) C-multifunctionlike iff $F(K) + C$ is convex.

Remark 2.1. In the case of (ii), if F is a single-valued function, then F is called C-function [2]. Furthermore, if $n = 1$ and $C = R_+$ ($C = -R_+$), then C-multifunction shrinks to convex (concave) function.

Remark 2.2. When $n = 1$ and $C = R_+$ ($C = -R_+$), then (ii) of Definition 2.1 characterizes convex (concave) multifunctions.

Remark 2.3. Under the assumption that C is a closed, convex and pointed cone, it is easy to see that if F is affine, then it is C-multifunction and has convex values. But the converse is not true in general.

Example 2.1. Let $m = n = 2$, $C = R^2_+$, $K = [0,1] \times [0,1]$ and

$$F(x) = \big[0, \max\{x_1, x_2\}\big] \times \big[0, \max\{x_1, x_2\}\big] \quad \forall x = (x_1, x_2)^\top \in K.$$

One can easily verify that F is C-multifunction with convex values. However, F is not affine. In fact, for $x_0 = (1,0)^\top$, $y_0 = (\frac{1}{4}, \frac{1}{2})^\top \in K$ and $\alpha_0 = \frac{1}{2}$, one has

$$\alpha_0 F(x_0) + (1-\alpha_0)F(y_0) = \frac{1}{2}\left\{[0,1] \times [0,1] + \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]\right\} = \left[0, \frac{3}{4}\right] \times \left[0, \frac{3}{4}\right],$$

$$F\big(\alpha_0 x_0 + (1-\alpha_0)y_0\big) = F\left(\left(\frac{5}{8}, \frac{1}{4}\right)^\top\right) = \left[0, \frac{5}{8}\right] \times \left[0, \frac{5}{8}\right]$$

and so

$$\alpha_0 F(x_0) + (1-\alpha_0)F(y_0) \supset F\big(\alpha_0 x_0 + (1-\alpha_0)y_0\big).$$

From the definition, it is easy to see the following lemma is true.

Lemma 2.1. *The following assertions hold*:

(i) *If F is affine, then $F(K)$ is convex and so F is C-multifunctionlike*;
(ii) *If F is C-multifunction, then F is C-multifunctionlike.*

The following result is useful for the study of (1.3) and (1.4) in Section 4.

Lemma 2.2. *Let $\Upsilon : K \to 2^{R^{n\times m}}$ be a set-valued function with matrix-values and $c_\diamond \in R^n$ be given. If Υ is affine, then $\Upsilon_{c_\diamond}$ and $\Upsilon^\top_{c_\diamond}$ are affine, where $\Upsilon_{c_\diamond}$ and $\Upsilon^\top_{c_\diamond}$ are defined by $\Upsilon_{c_\diamond}(x) = c^\top_\diamond \Upsilon(x) = \{c^\top_\diamond T\colon T \in \Upsilon(x)\}$ and $\Upsilon^\top_{c_\diamond}(x) = (\Upsilon_{c_\diamond}(x))^\top = (\Upsilon(x))^\top c_\diamond = \{T^\top c_\diamond\colon T \in \Upsilon(x)\}$ for all $x \in K$, respectively.*

Proof. Assume that Υ is affine. Let $x, y \in K$ and $\alpha \in [0,1]$ be any given. For any $u \in \Upsilon_{c_\diamond}(x)$ and $v \in \Upsilon_{c_\diamond}(y)$, there exist $T_1 \in \Upsilon(x)$ and $T_2 \in \Upsilon(y)$ such that $u = c^\top_\diamond T_1$ and $v = c^\top_\diamond T_2$. Note that the convexity of K implies that $\alpha x + (1-\alpha)y \in K$. Since Υ is affine, one has

$$\begin{aligned}
\alpha u + (1-\alpha)v &= \alpha c^\top_\diamond T_1 + (1-\alpha)c^\top_\diamond T_2 \\
&= c^\top_\diamond\big(\alpha T_1 + (1-\alpha)T_2\big) \\
&\in c^\top_\diamond\big(\alpha \Upsilon(x) + (1-\alpha)\Upsilon(y)\big) \\
&\subseteq c^\top_\diamond \Upsilon\big(\alpha x + (1-\alpha)y\big) \\
&= \Upsilon_{c_\diamond}\big(\alpha x + (1-\alpha)y\big).
\end{aligned}$$

Since $u \in \Upsilon_{c_\diamond}(x)$ and $v \in \Upsilon_{c_\diamond}(y)$ are arbitrary, it follows that

$$\alpha \Upsilon_{c_\diamond}(x) + (1-\alpha)\Upsilon_{c_\diamond}(y) \subseteq \Upsilon_{c_\diamond}\big(\alpha x + (1-\alpha)y\big),$$

which implies the affinity of $\Upsilon_{c_\diamond}$.

The affinity of $\Upsilon_{c_\diamond}^\top$ follows directly from that of $\Upsilon_{c_\diamond}$ by means of a transposition. The proof is complete. □

In order to apply the scalarization technique for the study of (1.1)–(1.4), we also need the following lemma.

Lemma 2.3. *The following arguments hold*:

(i) *If F is affine, then $\langle c^*, F(\cdot)\rangle$ is affine, where $c^* \in R^n$, $\langle c^*, \mathcal{A}\rangle = \{\langle c^*, a\rangle\colon a \in \mathcal{A}\}$ and $\mathcal{A} \subseteq R^n$*;
(ii) *Let $c^* \in C^*$. If F is C-multifunction, then $\langle c^*, F(\cdot)\rangle$ is R_+-multifunction.*

Proof. Let $c^* \in R^n$ be any given. Set $\Sigma(x) = \langle c^*, F(x)\rangle$ for all $x \in K$.

(i) Suppose that F is affine. Then for any given $x, y \in K$ and $\alpha \in [0,1]$, we have $\alpha x + (1-\alpha)y \in K$ since K is convex, and

$$\alpha F(x) + (1-\alpha)F(y) \subseteq F\big(\alpha x + (1-\alpha)y\big).$$

For any fixed $u \in \Sigma(x)$ and $v \in \Sigma(y)$, there are $a \in F(x)$ and $b \in F(y)$ such that $u = \langle c^*, a\rangle$ and $v = \langle c^*, b\rangle$. Thus,

$$\begin{aligned}\alpha u + (1-\alpha)v &= \alpha\langle c^*, a\rangle + (1-\alpha)\langle c^*, b\rangle\\ &= \big\langle c^*, \alpha a + (1-\alpha)b\big\rangle\\ &\in \big\langle c^*, \alpha F(x) + (1-\alpha)F(y)\big\rangle\\ &\subseteq \big\langle c^*, F\big(\alpha x + (1-\alpha)y\big)\big\rangle\\ &= \Sigma\big(\alpha x + (1-\alpha)y\big),\end{aligned}$$

which implies that

$$\alpha\Sigma(x) + (1-\alpha)\Sigma(y) \subseteq \Sigma\big(\alpha x + (1-\alpha)y\big).$$

(ii) Let $c^* \in C^*$. Assume that F is C-multifunction. Then for any given $x, y \in K$ and $\alpha \in [0,1]$, we obtain $\alpha x + (1-\alpha)y \in K$ since K is convex, and

$$\alpha F(x) + (1-\alpha)F(y) \subseteq F\big(\alpha x + (1-\alpha)y\big) + C.$$

For any $u \in \Sigma(x)$ and $v \in \Sigma(y)$, there are $a \in F(x)$ and $b \in F(y)$ such that $u = \langle c^*, a\rangle$ and $v = \langle c^*, b\rangle$, and thus

$$\begin{aligned}\alpha u + (1-\alpha)v &= \alpha\langle c^*, a\rangle + (1-\alpha)\langle c^*, b\rangle\\ &= \big\langle c^*, \alpha a + (1-\alpha)b\big\rangle\\ &\in \big\langle c^*, \alpha F(x) + (1-\alpha)F(y)\big\rangle\\ &\subseteq \big\langle c^*, F\big(\alpha x + (1-\alpha)y + C\big)\big\rangle\\ &\subseteq \big\langle c^*, F\big(\alpha x + (1-\alpha)y\big)\big\rangle + \big\langle c^*, C\big\rangle\\ &\subseteq \Sigma\big(\alpha x + (1-\alpha)y\big) + R_+.\end{aligned}$$

Consequently,

$$\alpha\Sigma(x) + (1-\alpha)\Sigma(y) \subseteq \Sigma\big(\alpha x + (1-\alpha)y\big) + R_+.$$

This completes the proof. □

As it is well known, monotonicity and C-operator are closely related to each other.

Definition 2.2. A set-valued function $M : K \to 2^{R^{n\times m}}$ with matrix-values (i.e., for any $x \in K$ and $T \in M(x)$, T is a $n \times m$-matrix) is called a C-operator iff, for any $x_1, x_2 \in K$,

$$(T_1 - T_2)(x_1 - x_2) \in C \quad \forall T_1 \in M(x_1),\ T_2 \in M(x_2).$$

Definition 2.3. A set-valued function $N : K \to 2^{R^m}$ is said to be monotone iff, for any $x_1, x_2 \in K$,

$$\langle u_1 - u_2, x_1 - x_2\rangle \geqslant 0 \quad \forall u_1 \in N(x_1),\ u_2 \in N(x_2).$$

We now present the following proposition.

Proposition 2.1. *Let $c_\diamond \in C^*$ be given and $\Psi : K \to 2^{R^{n\times m}}$ be a C-operator. Then $\Psi_{c_\diamond}^\top(x)$ is monotone, where $\Psi_{c_\diamond}^\top$ is given by $\Psi_{c_\diamond}^\top(x) = (c_\diamond^\top \Psi(x))^\top = (\Psi(x))^\top c_\diamond$ for all $x \in K$.*

Proof. Let $x_1, x_2 \in K$. Since Ψ is a C-operator,

$$(T_1 - T_2)(x_1 - x_2) \in C \quad \forall T_1 \in \Psi(x_1),\ T_2 \in \Psi(x_2).$$

Since $c_\diamond \in C^*$, it follows that

$$\big\langle c_\diamond, (T_1 - T_2)(x_1 - x_2)\big\rangle \geqslant 0 \quad \forall T_1 \in \Psi(x_1),\ T_2 \in \Psi(x_2)$$

and so

$$\langle u_1 - u_2, x_1 - x_2\rangle = \big\langle (T_1 - T_2)^\top c_\diamond, x_1 - x_2\big\rangle = \big\langle c_\diamond, (T_1 - T_2)(x_1 - x_2)\big\rangle \geqslant 0 \quad \forall u_1 \in \Psi_{c_\diamond}^\top(x_1),\ u_2 \in \Psi_{c_\diamond}^\top(x_2),$$

where $u_1 = T_1^\top c_\diamond$ and $u_2 = T_2^\top c_\diamond$. The proof is complete. □

3. Scalarization approaches for (1.1) and (1.2)

In [4], Giannessi, Mastroeni and Pellegrini presented a scalarization method to investigate GVPP and GWVPP. They assumed that the objective functions appeared in Propositions 7 and 13 are C-function and C-convexlike, respectively. In this section, we extend the scalarization method of Giannessi, Mastroeni and Pellegrini to study (1.1) and (1.2). For this, we define optimization problems that are scalarization of (1.1) and (1.2) and establish the equivalence between them under the assumption that objective functions are C-multifunctionlike.

Define the set-valued functions $S, H, G_{c^*} : K \to 2^K$, respectively, by

$$S(x) = \big\{y \in K:\ F(y) \subseteq F(x) - C\big\} \quad \forall x \in K,$$

$$H(x) = \big\{y \in K:\ F(y) \cap \big(F(x) - C\big) \neq \emptyset\big\} \quad \forall x \in K,$$

and

$$G_{c^*}(x) = \big\{y \in K:\ \big\langle c^*, F(y)\big\rangle \cap \big(\big\langle c^*, F(x)\big\rangle - R_+\big) \neq \emptyset\big\} \quad \forall x \in K,$$

where $c^* \in C^*$ is a given point.

Remark 3.1. If F collapses to a single-valued function, then

$$S(x) = H(x) = \big\{y \in K:\ F(y) \in F(x) - C\big\} \quad \forall x \in K$$

and

$$G_{c^*}(x) = \big\{y \in K:\ \big\langle c^*, F(y)\big\rangle \leqslant \big\langle c^*, F(x)\big\rangle\big\} \quad \forall x \in K,$$

which were considered by Giannessi, Mastroeni and Pellegrini [4].

Now we investigate some properties of set-valued functions S, H and G_{c^*} given above.

Proposition 3.1. *For any given $c^* \in C^*$, the following inclusions hold*:

$$x \in S(x) \subseteq H(x) \subseteq G_{c^*}(x) \quad \forall x \in K.$$

Proof. Clearly, $x \in S(x) \subseteq H(x)$ for all $x \in K$. It suffices to show that $H(x) \subseteq G_{c^*}(x)$ for all $x \in K$. For any $x \in K$ and $z \in H(x)$, we know that $z \in K$ and

$$F(z) \cap \big(F(x) - C\big) \neq \emptyset.$$

Consequently, there are $u \in F(z)$, $v \in F(x)$ and $c \in C$ such that $u = v - c$. Since $c^* \in C^*$, we have

$$\langle c^*, u\rangle = \langle c^*, v - c\rangle = \langle c^*, v\rangle - \langle c^*, c\rangle \leqslant \langle c^*, v\rangle,$$

or equivalently,

$$\langle c^*, F(z)\rangle \cap \left(\langle c^*, F(x)\rangle - R_+\right) \neq \emptyset,$$

which yields that

$$z \in \left\{y \in K\colon\ \langle c^*, F(y)\rangle \cap \left(\langle c^*, F(x)\rangle - R_+\right) \neq \emptyset\right\} = G_{c^*}(x).$$

The proof is complete. □

Proposition 3.2. *Assume that $x \in S(y)$ for some $y \in K$. For any given $c^* \in C^*$, the following inclusions hold*:

(i) $H(x) \subseteq H(y)$;
(ii) $G_{c^*}(x) \subseteq G_{c^*}(y)$.

Proof. Let $y \in K$ and $x \in S(y)$. Then

$$F(x) \subseteq F(y) - C. \tag{3.1}$$

(i) Assume that $u \in H(x)$. Then,

$$F(u) \cap \left(F(x) - C\right) \neq \emptyset. \tag{3.2}$$

Since C is a convex cone, (3.1) and (3.2) imply that

$$\emptyset \neq F(u) \cap \left(F(x) - C\right) \subseteq F(u) \cap \left(F(y) - C - C\right) \subseteq F(u) \cap \left(F(y) - C\right)$$

and so $u \in H(y)$. This yields that $H(x) \subseteq H(y)$.

(ii) Since $c^* \in C^*$, it follows from (3.1) that

$$\langle c^*, F(x)\rangle \subseteq \langle c^*, F(y) - C\rangle \subseteq \langle c^*, F(y)\rangle - \langle c^*, C\rangle \subseteq \langle c^*, F(y)\rangle - R_+. \tag{3.3}$$

Let $v \in G_{c^*}(x)$. Then

$$\langle c^*, F(v)\rangle \cap \left(\langle c^*, F(x)\rangle - R_+\right) \neq \emptyset. \tag{3.4}$$

From (3.3) and (3.4), we have

$$\emptyset \neq \langle c^*, F(v)\rangle \cap \left(\langle c^*, F(x)\rangle - R_+\right) \subseteq \langle c^*, F(v)\rangle \cap \left(\langle c^*, F(y)\rangle - R_+ - R_+\right) \subseteq \langle c^*, F(v)\rangle \cap \left(\langle c^*, F(y)\rangle - R_+\right)$$

and so

$$\langle c^*, F(v)\rangle \cap \left(\langle c^*, F(y)\rangle - R_+\right) \neq \emptyset.$$

By the definition of G_{c^*}, we have $v \in G_{c^*}(y)$. Thus, $G_{c^*}(x) \subseteq G_{c^*}(y)$ as $v \in G_{c^*}(x)$ is arbitrary. The proof is complete. □

The following proposition shows that the affinity of H and G_{c^*} is related closely to that of F, where $c^* \in C^*$ is given.

Proposition 3.3. *If F is affine, then*

(i) *H is affine*;
(ii) *G_{c^*} is affine for any $c^* \in R^n$.*

Proof. (i) We first prove that H is affine. For any $x, y \in K$ and $\alpha \in [0, 1]$, let $u \in H(x)$ and $v \in H(y)$. Then, there are $u, v \in K$ such that

$$F(u) \cap \left(F(x) - C\right) \neq \emptyset$$

and

$$F(v) \cap \left(F(y) - C\right) \neq \emptyset.$$

That is, there exist $a \in F(u)$, $b \in F(v)$, $w \in F(x)$ and $z \in F(y)$ such that

$$a \in w - C \tag{3.5}$$

and

$$b \in z - C. \tag{3.6}$$

The convexity of K implies that $\alpha u + (1-\alpha)v, \alpha x + (1-\alpha)y \in K$. Since F is affine and C is a convex cone, it follows from (3.5) and (3.6) that

$$\begin{aligned}\alpha a + (1-\alpha)b &\in \{\alpha F(u) + (1-\alpha)F(v)\} \cap \{\alpha w + (1-\alpha)z - C\}\\ &\subseteq F\big(\alpha u + (1-\alpha)v\big) \cap \{\alpha F(x) + (1-\alpha)F(y) - C\}\\ &\subseteq F\big(\alpha u + (1-\alpha)v\big) \cap \{F\big(\alpha x + (1-\alpha)y\big) - C\},\end{aligned}$$

or equivalently,

$$F\big(\alpha u + (1-\alpha)v\big) \cap \{F\big(\alpha x + (1-\alpha)y\big) - C\} \neq \emptyset.$$

This implies that

$$\alpha u + (1-\alpha)v \in H\big(\alpha x + (1-\alpha)y\big). \tag{3.7}$$

Since $u \in H(x)$ and $v \in H(y)$ are arbitrary, from (3.7), we have

$$\alpha H(x) + (1-\alpha)H(y) \subseteq H\big(\alpha x + (1-\alpha)y\big),$$

that is, H is affine.

(ii) Since F is affine, from Lemma 2.3(i), $\langle c^*, F(\cdot)\rangle$ is affine for any $c^* \in R^n$. Similarly, we can show that G_{c^*} is affine. The proof is complete. □

Let $c_0 \in C^*$ be given. Now, we consider the following set-valued (scalar) optimization problems:

Set-valued (scalar) optimization problem (for short, SOP):

$$\min_{R_+\setminus\{0\}} \langle c_0, F(x)\rangle, \qquad \text{subject to} \quad x \in K,$$

where $\min_{R_+\setminus\{0\}}$ denotes minimum with respect to the cone $R_+\setminus\{0\}$;

Set-valued (scalar) quasi-optimization problem (for short, SQOP):

$$\min_{R_+\setminus\{0\}} \langle c_0, F(x)\rangle, \qquad \text{subject to} \quad x \in \Xi(y),$$

which depends on the parameter $y \in K$, and where $\Xi = H, G_{c_0}$.

Remark 3.2. We would like to point out that SOP is still a problem of type of (1.1). In fact, if $n = 1$ and $C = R_+$, then (1.1) collapses to SOP.

A pair $(x^*, \langle c_0, y^*\rangle)$ with $x^* \in K$ and $y^* \in F(x^*)$ is called a minimal solution of $\langle c_0, F(\cdot)\rangle$ on K iff

$$\big(\langle c_0, y^*\rangle - \langle c_0, F(K)\rangle\big) \cap \big(R_+\setminus\{0\}\big) = \emptyset.$$

A pair $(x^*, \langle c_0, y^*\rangle)$ with $x^* \in \Xi(y)$ and $y^* \in F(x^*)$ is called a minimal solution of $\langle c_0, F(\cdot)\rangle$ on $\Xi(y)$ iff

$$\big(\langle c_0, y^*\rangle - \langle c_0, F\big(\Xi(y)\big)\rangle\big) \cap \big(R_+\setminus\{0\}\big) = \emptyset.$$

Denote by $\mathrm{Min}_{R_+\setminus\{0\}}\langle c_0, F(K)\rangle$ and $\mathrm{Min}_{R_+\setminus\{0\}}\langle c_0, F(\Xi(y))\rangle$ the set of all minimal solutions of $\langle c_0, F(\cdot)\rangle$ on K and the set of all minimal solutions of $\langle c_0, F(\cdot)\rangle$ on $\Xi(y)$, respectively.

If F is affine, then from Lemma 2.3 and Proposition 3.3, we know that $\langle c_0, F(\cdot)\rangle$, H and G_{c_0} are affine. Thus, the following result holds immediately.

Proposition 3.4. *Suppose that F is affine. Then, the objective functions SOP and SQOP are affine and the feasible sets of SOP and SQOP are convex.*

If $x^* \in H(y)$, then it follows from Proposition 3.1 that $H(y) \subseteq G_{c_0}(y) \subseteq K$ and so the following proposition is true.

Proposition 3.5. *Let $y \in K$ and $x^* \in H(y)$. Then,*

(i) $(x^*, \langle c_0, y^*\rangle) \in \mathrm{Min}_{R_+\setminus\{0\}}\langle c_0, F(K)\rangle$ *implies* $(x^*, \langle c_0, y^*\rangle) \in \mathrm{Min}_{R_+\setminus\{0\}}\langle c_0, F(G_{c_0}(y))\rangle$;
(ii) $(x^*, \langle c_0, y^*\rangle) \in \mathrm{Min}_{R_+\setminus\{0\}}\langle c_0, F(G_{c_0}(y))\rangle$ *implies* $(x^*, \langle c_0, y^*\rangle) \in \mathrm{Min}_{R_+\setminus\{0\}}\langle c_0, F(H(y))\rangle$.

Proposition 3.6. *Let $y^0 \in K$ and $x^0 \in S(y^0)$ be given. If $(x^0, \langle c_0, z^0\rangle) \in \mathrm{Min}_{R_+\setminus\{0\}}\langle c_0, F(H(y^0))\rangle$, then $(x^0, \langle c_0, z^0\rangle) \in \mathrm{Min}_{R_+\setminus\{0\}}\langle c_0, F(H(x^0))\rangle$.*

Proof. By using Proposition 3.2(i), one can easily prove the conclusion. □

Under some suitable conditions, we prove the following equivalence among SOP, (1.1) and (1.2).

Theorem 3.1. *The following conclusions hold*:

(i) *Let* $c_0 \in \operatorname{int} C^*$. *If* $(x^*, \langle c_0, y^*\rangle) \in \mathrm{Min}_{R_+\backslash\{0\}}\langle c_0, F(K)\rangle$, *then* $(x^*, y^*) \in \mathrm{Min}_{C\backslash\{0\}} F(K)$ *and so* $(x^*, y^*) \in \mathrm{Min}_{\operatorname{int} C} F(K)$;
(ii) *Suppose that* F *is* C*-multifunctionlike. If* $(x^*, y^*) \in \mathrm{Min}_{C\backslash\{0\}} F(K)$, *then there is* $c_{y^*} \in C^*\backslash\{0\}$ *such that* $(x^*, \langle c_{y^*}, y^*\rangle) \in \mathrm{Min}_{R_+\backslash\{0\}}\langle c_{y^*}, F(K)\rangle$;
(iii) *Suppose that* F *is* C*-multifunctionlike. If* $(x^*, y^*) \in \mathrm{Min}_{\operatorname{int} C} F(K)$, *then there is* $c_{y^*} \in C^*\backslash\{0\}$ *such that* $(x^*, \langle c_{y^*}, y^*\rangle) \in \mathrm{Min}_{R_+\backslash\{0\}}\langle c_{y^*}, F(K)\rangle$.

Proof. (i) Let $(x^*, \langle c_0, y^*\rangle) \in \mathrm{Min}_{R_+\backslash\{0\}}\langle c_0, F(K)\rangle$. Then, there exist $x^* \in K$ and $y^* \in F(x^*)$ such that

$$\big\{\langle c_0, y^*\rangle - \big\langle c_0, F(K)\big\rangle\big\} \cap \big(R_+\backslash\{0\}\big) = \emptyset. \tag{3.8}$$

Suppose to the contrary that $(x^*, y^*) \notin \mathrm{Min}_{C\backslash\{0\}} F(K)$, that is,

$$\big\{y^* - F(K)\big\} \cap \big(C\backslash\{0\}\big) \neq \emptyset.$$

Then, there are $u \in K$ and $z \in F(u)$ such that

$$y^* - z \in C\backslash\{0\}. \tag{3.9}$$

Since $c_0 \in \operatorname{int} C^*$, (3.9) implies that

$$\langle c_0, y^*\rangle - \langle c_0, z\rangle = \langle c_0, y^* - z\rangle > 0,$$

or equivalently,

$$\big\{\langle c_0, y^*\rangle - \big\langle c_0, F(K)\big\rangle\big\} \cap \big(R_+\backslash\{0\}\big) \neq \emptyset,$$

which is a contradiction with (3.8). Since $\mathrm{Min}_{C\backslash\{0\}} F(K) \subseteq \mathrm{Min}_{\operatorname{int} C} F(K)$, one has $(x^*, y^*) \in \mathrm{Min}_{\operatorname{int} C} F(K)$.

(ii) Since $\mathrm{Min}_{C\backslash\{0\}} F(K) \subseteq \mathrm{Min}_{\operatorname{int} C} F(K)$, the conclusion follows from (iii).

(iii) Let $(x^*, y^*) \in \mathrm{Min}_{\operatorname{int} C} F(K)$. Then, there exist $x^* \in K$ and $y^* \in F(x^*)$ such that

$$\big\{y^* - F(K)\big\} \cap (\operatorname{int} C) = \emptyset. \tag{3.10}$$

Since

$$\big\{y^* - F(K)\big\} \cap (\operatorname{int} C) = \big\{y^* - F(K)\big\} \cap (C + \operatorname{int} C) = \big\{y^* - \big(F(K) + C\big)\big\} \cap (\operatorname{int} C), \tag{3.11}$$

it follows from (3.10) and (3.11) that

$$\big\{y^* - \big(F(K) + C\big)\big\} \cap (\operatorname{int} C) = \emptyset. \tag{3.12}$$

Since F is C-multifunctionlike, we know that $F(K)+C$ is convex, and so is $y^* - (F(K)+C)$. By using the separation theorem (see, for example, [11]), it follows from (3.12) that there is $c_{y^*} \in R^n\backslash\{0\}$ such that

$$\big\langle c_{y^*}, y^* - (u + c)\big\rangle \leqslant 0 \leqslant \langle c_{y^*}, x\rangle \quad \forall u \in F(K),\ \forall c \in C,\ \forall x \in \operatorname{int} C. \tag{3.13}$$

Since $\operatorname{int} C + C = \operatorname{int} C$, from the second inequality in (3.13) one can easily check that $c_{y^*} \in C^*\backslash\{0\}$. Setting $c = 0$ in (3.13), one has

$$\langle c_{y^*}, y^* - u\rangle \leqslant 0 \quad \forall u \in F(K),$$

or equivalently,

$$\big\{\langle c_{y^*}, y^*\rangle - \big\langle c_{y^*}, F(K)\big\rangle\big\} \cap \big(R_+\backslash\{0\}\big) = \big\{\big\langle c_{y^*}, y^* - F(K)\big\rangle\big\} \cap \big(R_+\backslash\{0\}\big) = \emptyset,$$

which yields $(x^*, \langle c_{y^*}, y^*\rangle) \in \mathrm{Min}_{R_+\backslash\{0\}}\langle c_{y^*}, F(K)\rangle$. □

Now we give an example to illustrate the scalarization approaches.

Example 3.1. Let $m = n = 2$, $C = R^2_+$, $K = [0,1] \times [0,1]$ and $F(x) = [0, x_1] \times [0, x_2]$ for any $x = (x_1, x_2)^\top \in K$. One can easily verify that F is affine and so is C-multifunctionlike, $C^* = R^2_+$, $F(K) = K$.

Let $c^* = (\frac{1}{2}, \frac{1}{2})^\top \in C^*$. Then

$$S(x) = \{y \in K\colon\ y \in x - C\} = [0, x_1] \times [0, x_2] \quad \forall x \in K,$$

$$H(x) = K \quad \forall x \in K,$$

and

$$G_{c^*}(x) = K \quad \forall x \in K.$$

It is clear that Propositions 3.1–3.6 hold. We now check that all the conclusions (i)–(iii) of Theorem 3.1 hold.

(i) Let $x^* = (\frac{1}{2}, \frac{1}{2})^\top \in K$, $c_0 = (1,1)^\top \in \operatorname{int} C^*$ and $y^* = (0,0)^\top \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] = F(x^*)$. Then, we have

$$\langle c_0, y^* \rangle = 0$$

and

$$\big\langle c_0, F(K)\big\rangle = \langle c_0, K \rangle = \big\{x_1 + x_2\colon\ (x_1, x_2)^\top \in K\big\} = [0, 2].$$

Thus,

$$\langle c_0, y^* \rangle - \big\langle c_0, F(K)\big\rangle = 0 - [0, 2] = [-2, 0]$$

and so

$$\big\{\langle c_0, y^* \rangle - \big\langle c_0, F(K)\big\rangle\big\} \cap \big(R_+ \backslash \{0\}\big) = \emptyset,$$

that is, $(x^*, \langle c_0, y^* \rangle) \in \operatorname{Min}_{R_+ \backslash \{0\}} \langle c_0, F(K) \rangle$. One can easily verify that $(x^*, y^*) \in \operatorname{Min}_{C \backslash \{0\}} F(K)$ and $(x^*, y^*) \in \operatorname{Min}_{\operatorname{int} C} F(K)$. In fact, since

$$y^* - F(K) = (0,0)^\top - K = [-1, 0] \times [-1, 0],$$

we have

$$\big\{y^* - F(K)\big\} \cap \big(C \backslash \{0\}\big) = \emptyset,$$

and

$$\big\{y^* - F(K)\big\} \cap \operatorname{int} C = \emptyset.$$

(ii) Let $x^* = (\frac{1}{2}, \frac{1}{2})^\top \in K$ and $y^* = (0,0)^\top \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] = F(x^*)$. Then,

$$y^* - F(K) = (0,0)^\top - K = [-1, 0] \times [-1, 0]$$

and so

$$\big\{y^* - F(K)\big\} \cap \big(C \backslash \{0\}\big) = \emptyset,$$

that is, $(x^*, y^*) \in \operatorname{Min}_{C \backslash \{0\}} F(K)$. For $c_{y^*} = (1, 0)^\top \in C^* \backslash \{0\}$, we have

$$\langle c_{y^*}, y^* \rangle = 0$$

and

$$\big\langle c_{y^*}, F(K)\big\rangle = \langle c_{y^*}, K \rangle = \big\{x_1\colon\ (x_1, x_2)^\top \in K\big\} = [0, 1].$$

Consequently,

$$\langle c_{y^*}, y^* \rangle - \big\langle c_{y^*}, F(K)\big\rangle = 0 - [0, 1] = [-1, 0]$$

and hence

$$\big\{\langle c_{y^*}, y^* \rangle - \big\langle c_{y^*}, F(K)\big\rangle\big\} \cap \big(R_+ \backslash \{0\}\big) = \emptyset,$$

i.e., $(x^*, \langle c_{y^*}, y^* \rangle) \in \operatorname{Min}_{R_+ \backslash \{0\}} \langle c_{y^*}, F(K) \rangle$.

(iii) Let $x^* = (\frac{1}{2}, \frac{1}{2})^\top \in K$, $y^* = (0,0)^\top \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] = F(x^*)$ and $c_{y^*} = (1,1)^\top \in C^* \backslash \{0\}$. As shown in (i) and (ii), we can obtain $(x^*, y^*) \in \operatorname{Min}_{\operatorname{int} C} F(K)$ and

$$\begin{aligned}\{\langle c_{y^*}, y^*\rangle - \langle c_{y^*}, F(K)\rangle\} \cap (R_+\backslash\{0\}) &= \{\langle c_{y^*}, y^*\rangle - \langle c_{y^*}, K\rangle\} \cap (R_+\backslash\{0\}) \\ &= \{0 - \{x_1 + x_2\colon\ (x_1, x_2)^\top \in K\}\} \cap (R_+\backslash\{0\}) \\ &= \{0 - [0, 2]\} \cap (R_+\backslash\{0\}) \\ &= [-2, 0] \cap (R_+\backslash\{0\}) \\ &= \emptyset,\end{aligned}$$

that is, $(x^*, \langle c_{y^*}, y^*\rangle) \in \mathrm{Min}_{R_+\backslash\{0\}}\langle c_{y^*}, F(K)\rangle$.

Remark 3.3.

(a) If F is a single-valued function, then Propositions 3.1 and 3.2 reduce to Propositions 9 and 11 in [4], respectively.
(b) If F is a single-valued function, then (i) and (iii) of Theorem 3.1 reduce to (i) and (ii) of Proposition 13 in [4], respectively.

4. Scalarization approaches for (1.3) and (1.4)

By exploiting the similar scalarization techniques presented in Section 3, we turn to the investigation of (1.3) and (1.4) in this section.

Define set-valued functions $\triangle, \Theta$ and $\Theta_{c_\diamond} : K \to 2^K$, respectively, by

$$\triangle(x) = \{y \in K\colon\ \Phi(x)y \subseteq \Phi(x)x - C\} \quad \forall x \in K,$$

$$\Theta(x) = \{y \in K\colon\ \Phi(x)y \cap (\Phi(x)x - C) \neq \emptyset\} \quad \forall x \in K,$$

and

$$\Theta_{c_\diamond}(x) = \{y \in K\colon\ \langle\Phi_{c_\diamond}(x), y\rangle \cap (\langle\Phi_{c_\diamond}(x), x\rangle - R_+) \neq \emptyset\} \quad \forall x \in K,$$

where

$$\Phi_{c_\diamond}(x) = (\Phi(x))^\top c_\diamond = \{T^\top c_\diamond : T \in \Phi(x)\}$$

and $c_\diamond \in C^*$ is a given point.

Remark 4.1. If Φ collapses to a single-valued function and $K = R^m$, then

$$\triangle(x) = \Theta(x) = \{y \in K\colon\ \Phi(x)y \in \Phi(x)x - C\} \quad \forall x \in K,$$

and

$$\Theta_{c_\diamond}(x) = \{y \in K\colon\ \langle\Phi_{c_\diamond}(x), y\rangle \leqslant \langle\Phi_{c_\diamond}(x), x\rangle\} \quad \forall x \in K,$$

which were considered by Giannessi, Mastroeni and Pellegrini [4].

The following proposition gives some interesting relations among set-valued functions $\triangle$, Θ and $\Theta_{c_\diamond}$ defined above.

Proposition 4.1. *Let $c_\diamond \in C^*$ be given. Then, the following relations hold*:

$$x \in \triangle(x) \subseteq \Theta(x) \subseteq \Theta_{c_\diamond}(x) \quad \forall x \in K.$$

Proof. It is clear that $x \in \triangle(x) \subseteq \Theta(x)$ for all $x \in K$. It suffices to show that $\Theta(x) \subseteq \Theta_{c_\diamond}(x)$ for all $x \in K$. Let $x \in K$ and $y \in \Theta(x)$. Then,

$$\Phi(x)y \cap (\Phi(x)x - C) \neq \emptyset,$$

and so there are $T_1, T_2 \in \Phi(x)$ such that

$$T_1 y - T_2 x \in -C.$$

Since $c_\diamond \in C^*$,

$$\langle(T_1)^\top c_\diamond, y\rangle - \langle(T_2)^\top c_\diamond, x\rangle = \langle c_\diamond, T_1 y - T_2 x\rangle \leqslant 0$$

and so

$$\big\langle \Phi_{c_\diamond}(x), y\big\rangle \cap \big(\big\langle \Phi_{c_\diamond}(x), x\big\rangle - R_+\big) \neq \emptyset,$$

that is,

$$y \in \Theta_{c_\diamond}(x).$$

This completes the proof. □

Under the affinity of Φ, one can prove that Θ and $\Theta_{c_\diamond}$ have convex values. However, the affinity of Θ and $\Theta_{c_\diamond}$ cannot be derived.

Proposition 4.2. *The following statements hold*:

(i) *If Φ has convex values, then $\triangle$ has convex values.*
(ii) *Let $c_\diamond \in C^*$ be given. If Φ is affine, then Θ and $\Theta_{c_\diamond}$ have convex values.*

Proof. (i) Suppose that Φ has convex values. For any given $x \in K$, let $y_1, y_2 \in \triangle(x)$, $\alpha \in [0,1]$ and set $y = \alpha y_1 + (1-\alpha)y_2$. Then, $\Phi(x)y_1 \subseteq \Phi(x)x - C$ and $\Phi(x)y_2 \subseteq \Phi(x)x - C$. Since $\Phi(x)$ is convex, one can easily see that $\Phi(x)x$ is convex and so is $\Phi(x)x - C$. As a consequence,

$$\begin{aligned}\Phi(x)y &= \Phi(x)\big(\alpha y_1 + (1-\alpha)y_2\big)\\ &\subseteq \alpha\Phi(x)y_1 + (1-\alpha)\Phi(x)y_2\\ &\subseteq \alpha\big(\Phi(x)x - C\big) + (1-\alpha)\big(\Phi(x)x - C\big)\\ &\subseteq \Phi(x)x - C,\end{aligned}$$

which shows that $\triangle(x)$ is convex, that is, $\triangle$ has convex values.

(ii) Suppose that Φ is affine. From Lemma 2.2, we know that $\Phi_{c_\diamond}$ is affine. Now we prove that Θ has convex values. For any given $x \in K$, let $u, v \in \Theta(x)$ and $\alpha \in [0,1]$. Then, we know that $u, v \in K$,

$$\Phi(x)u \cap \big(\Phi(x)x - C\big) \neq \emptyset \tag{4.1}$$

and

$$\Phi(x)v \cap \big(\Phi(x)x - C\big) \neq \emptyset. \tag{4.2}$$

Since K is convex, $\alpha u + (1-\alpha)v \in K$. From (4.1) and (4.2), there are $T_i \in \Phi(x)$ $(i = 1, 2, 3, 4)$ such that

$$T_1u \in T_2x - C$$

and

$$T_3v \in T_4x - C.$$

Notice that C is a convex cone and Φ is affine. It follows that

$$\begin{aligned}\alpha T_1u + (1-\alpha)T_3v &\in \big(\alpha\Phi(x)u + (1-\alpha)\Phi(x)v\big) \cap \big(\alpha T_2x + (1-\alpha)T_4x - C - C\big)\\ &\subseteq \big(\Phi(x)\big(\alpha u + (1-\alpha)v\big)\big) \cap \big(\big[\alpha\Phi(x) + (1-\alpha)\Phi(x)\big]x - C\big)\\ &\subseteq \big(\Phi(x)\big(\alpha u + (1-\alpha)v\big)\big) \cap \big(\Phi(x)x - C\big),\end{aligned}$$

or equivalently,

$$\big(\Phi(x)\big(\alpha u + (1-\alpha)v\big)\big) \cap \big(\Phi(x)x - C\big) \neq \emptyset$$

and so

$$\alpha u + (1-\alpha)v \in \big\{y \in K:\ \Phi(x)y \cap \big(\Phi(x)x - C\big) \neq \emptyset\big\} = \Theta(x),$$

which yields that $\Theta(x)$ is convex. Similarly, we can show that $\Theta_{c_\diamond}$ has convex values. This completes the proof. □

Remark 4.2.

(a) If Φ is a single-valued function and $K = R^m$, then Proposition 4.1 collapses to Proposition 16 in [4].
(b) If Φ is a single-valued function and $K = R^m$, then Proposition 4.2(i) becomes Proposition 14 in [4].

We now consider the following set-valued (scalar) variational inequality (for short, SVI): find $x^\diamond \in K$ and $t^\diamond \in \Phi_{c_\diamond}(x^\diamond) = (\Phi(x^\diamond))^\top c_\diamond$ such that

$$\langle t^\diamond, y - x^\diamond \rangle \geqslant 0 \quad \forall y \in K.$$

A pair $(x^\diamond, t^\diamond)$ with $x^\diamond \in K$ and $t^\diamond \in \Phi_{c_\diamond}(x^\diamond) = (\Phi(x^\diamond))^\top c_\diamond$ is called a solution of SVI. We can rewrite SVI as

$$\big(\langle t^\diamond, x^\diamond \rangle - \langle t^\diamond, K \rangle\big) \cap \big(R_+ \backslash \{0\}\big) = \emptyset.$$

We also consider the set-valued (scalar) quasi-variational inequality (for short, SQVI): find $x^\diamond \in \Lambda(x)$ and $t^\diamond \in \Phi_{c_\diamond}(x^\diamond) = (\Phi(x^\diamond))^\top c_\diamond$ such that

$$\langle t^\diamond, y - x^\diamond \rangle \geqslant 0 \quad \forall y \in \Lambda(x),$$

where $c_\diamond \in C^*$, $x \in K$ is a parameter and $\Lambda = \Theta, \Theta_{c_\diamond}$. A pair $(x^\diamond, t^\diamond)$ with $x^\diamond \in \Lambda(x)$ and $t^\diamond \in \Phi_{c_\diamond}(x^\diamond) = (\Phi(x^\diamond))^\top c_\diamond$ is called a solution of SQVI. It is clear that SQVI can be rewritten as

$$\big(\langle t^\diamond, x^\diamond \rangle - \langle t^\diamond, \Lambda(x) \rangle\big) \cap \big(R_+ \backslash \{0\}\big) = \emptyset.$$

Denote by S_{SVI} and S_Λ the set of all solutions of SVI and the set of all solutions of SQVI, respectively.

Let $x^* \in \Theta(x)$. Since $\Theta(x) \subseteq \Theta_{c_\diamond}(x) \subseteq K$ (by Proposition 4.1), by the definition of solutions, it is easy to see the following conclusions hold.

Proposition 4.3. *Let $x \in K$ and $x^* \in \Theta(x)$. Then,*

(i) $(x^*, t^*) \in S_{SVI}$ *implies* $(x^*, t^*) \in S_{\Theta_{c_\diamond}}$;
(ii) $(x^*, t^*) \in S_{\Theta_{c_\diamond}}$ *implies* $(x^*, t^*) \in S_\Theta$.

The following conclusion is devoted to the equivalence among SVI, (1.3) and (1.4).

Theorem 4.1. *The following conclusions hold:*

(i) *Let $c_\diamond \in \operatorname{int} C^*$ be given. If $(x^\diamond, t^\diamond) \in S_{SVI}$, then there is $T^* \in \Phi(x^\diamond)$ with $t^\diamond = (T^*)^\top c_\diamond$ such that $(x^\diamond, T^*) \in S_S$ and so $(x^\diamond, T^*) \in S_{SW}$;*
(ii) *If $(x^\diamond, T^*) \in S_S$, then there is $c_* \in C^* \backslash \{0\}$ such that $(x^\diamond, (T^*)^\top c_*) \in S_{SVI}$;*
(iii) *If $(x^\diamond, T^*) \in S_{SW}$, then there is $c_* \in C^* \backslash \{0\}$ such that $(x^\diamond, (T^*)^\top c_*) \in S_{SVI}$.*

Proof. The proofs are similar to that of Theorem 3.1. For the completeness, we conclude them.

(i) Let $(x^\diamond, t^\diamond) \in S_{SVI}$. Then, there are $x^\diamond \in K$ and $t^\diamond \in \Phi_{c_\diamond}(x^\diamond) = (\Phi(x^\diamond))^\top c_\diamond$ such that

$$\big(\langle t^\diamond, x^\diamond \rangle - \langle t^\diamond, K \rangle\big) \cap \big(R_+ \backslash \{0\}\big) = \emptyset. \tag{4.3}$$

Thus, there exists $T^* \in \Phi(x^\diamond)$ with $t^\diamond = (T^*)^\top c_\diamond$ such that (4.3) holds. Suppose to the contrary that $(x^\diamond, T^*)$ does not solve (1.3), that is,

$$\big(T^* x^\diamond - T^* K\big) \cap \big(C \backslash \{0\}\big) \neq \emptyset.$$

Then, there is $u \in K$ such that

$$T^* x^\diamond - T^* u \in C \backslash \{0\}. \tag{4.4}$$

Since $c_\diamond \in \operatorname{int} C^*$, (4.4) implies that

$$\langle c_\diamond, T^* x^\diamond \rangle - \langle c_\diamond, T^* u \rangle = \langle c_\diamond, T^* x^\diamond - T^* u \rangle > 0,$$

or equivalently,

$$\big(\langle t^\diamond, x^\diamond \rangle - \langle t^\diamond, K \rangle\big) \cap \big(R_+ \backslash \{0\}\big) \neq \emptyset,$$

which is a contradiction with (4.3). Since every solution of (1.3) solves (1.4), we know that $(x^\diamond, T^*) \in S_{SW}$.

(ii) Since $S_S \subseteq S_{SW}$, the conclusion follows immediately from (iii).
(iii) Assume that $(x^\diamond, T^*)$ solves (1.3). Then, there are $x^\diamond \in K$ and $T^* \in \Phi(x^\diamond)$ such that

$$\big(T^* x^\diamond - T^* K\big) \cap (\operatorname{int} C) = \emptyset. \tag{4.5}$$

Noticing

$$\big(T^* x^\diamond - T^* K\big) \cap (\operatorname{int} C) = \big(T^* x^\diamond - T^* K\big) \cap (C + \operatorname{int} C) = \big(T^* x^\diamond - (T^* K + C)\big) \cap (\operatorname{int} C), \tag{4.6}$$

it follows from (4.5) and (4.6) that

$$\big(T^* x^\diamond - (T^* K + C)\big) \cap (\operatorname{int} C) = \emptyset. \tag{4.7}$$

Since K is convex and C is a convex cone, it is easy to see that T^*K is convex and so is $T^*K + C$. As the arguments in the proof of Theorem 3.1(iii), it follows from (4.7) that there is $c_* \in C^* \backslash \{0\}$ such that

$$\big\langle c_*, T^* x^\diamond - (T^* y + c)\big\rangle \leqslant 0 \quad \forall y \in K,\ \forall c \in C. \tag{4.8}$$

Setting $c = 0$ in (4.8), one has

$$\big\langle (T^*)^\top c_*, x^\diamond - y\big\rangle = \big\langle c_*, T^* x^\diamond - T^* y\big\rangle \leqslant 0 \quad \forall y \in K,$$

or equivalently,

$$\big(\big\langle (T^*)^\top c_*, x^\diamond\big\rangle - \big\langle (T^*)^\top c_*, K\big\rangle\big) \cap \big(R_+ \backslash \{0\}\big) = \big(\big\langle (T^*)^\top c_*, x^\diamond - K\big\rangle\big) \cap \big(R_+ \backslash \{0\}\big) = \emptyset,$$

which implies that $(x^\diamond, (T^*)^\top c_*) \in S_{SVI}$. This completes the proof. □

Remark 4.3. In [9], Konnov presented the scalarization approach for set-valued weak vector variational inequalities in Banach space X. Clearly, the problem considered by Konnov [9] collapses to (1.4) when $X = R^m$. However, we would like to point out the following differences between Theorem 4.1 and the results due to Konnov [9]:

(i) The scalar variational inequalities are given differently. In fact, the set-valued (scalar) variational inequalities for (1.4) in Theorem 4.1 are defined by using $c_\diamond \in C^*$. Nevertheless, in [9], Konnov defined the set-valued (scalar) variational inequalities for (1.4) by utilizing the convex hull of n components of the set-valued function $\Phi = (\Phi_1, \ldots, \Phi_n)^\top$.
(ii) The assumption conditions are different. In fact, in the results of Konnov [9], each component of the set-valued function $\Phi = (\Phi_1, \ldots, \Phi_n)^\top$ was assumed to have convex and compact values. However, Theorem 4.1 does not require such assumptions.

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