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ABSTRACT

In this paper, we estimate the size of ρ_n 's in the famous L. Zalcman's lemma. With it, we obtain some uniqueness theorems for meromorphic functions f and f' when they share two transcendental meromorphic functions.

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1. Introduction and main results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$ and let $R(z)$ be a meromorphic function or a finite complex number. If $g(z) - R = 0$ whenever $f(z) - R = 0$, we write $f(z) = R \Rightarrow g(z) = R$. If $f(z) = R \Rightarrow g(z) = R$ and $g(z) = R \Rightarrow f(z) = R$, we write $f(z) = R \Leftrightarrow g(z) = R$ and say that f and g share R IM (ignoring multiplicity). If $f - R$ and $g - R$ have the same zeros with the same multiplicities, we write $f(z) = R \Rrightarrow g(z) = R$ and say that f and g share R CM (counting multiplicity) (see [9]). It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [5,9].

Rubel and Yang appear to be the first to study the entire functions that share values with their derivatives. In 1977, they proved the following well-known theorem.

Theorem A. Let a and b be complex numbers such that $b \neq a$, and let $f(z)$ be a nonconstant entire function. If $f(z)$ and $f'(z)$ share values a and b CM, then $f = f'$.

Since then, shared value problems, especially, the case of f and f' sharing values, have been studied by many authors and a number of profound results have been obtained (see, e.g. [1,4], etc.).

Recently, Li and Yi [7] proved the following related result.

Theorem B. Let a and b be complex numbers such that $b \neq a$, 0, and let $f(z)$ be a nonconstant entire function. If $f(z) = a \Rightarrow f'(z) = a$ and $f(z) = b \Rightarrow f'(z) = b$, then one of the following cases must occur:

- (1) $f \equiv f'$.
- (2) $f = Ce^{\frac{b}{b-a}z} + a$, where C is a nonzero constant.

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In this paper, the order of f is defined by

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r},$$

where, and in the sequel, $M(r, f) = \max_{|z|=r} |f(z)|$.

The main purpose of this paper is to replace the values a, b in Theorem B by two meromorphic functions $R_1(z)$ and $R_2(z)$, here $R_1(z) = P_1(z) \cdot e^{Q(z)}$, $R_2(z) = P_2(z) \cdot e^{Q(z)}$, $Q(z)$ is a polynomial and $P_1(z), P_2(z)$ are rational functions. We prove the following results which are the improvements and complements of the above theorems.

Remark 1.1. In this paper we use R_1 and R_2 to denote $R_1(z)$ and $R_2(z)$ which are defined as above.

Theorem 1.1. Let R_1 and R_2 be two functions and $R_2 (\not\equiv R_1, 0)$. Let $f(z)$ be an entire function. If $f(z) = R_1 \Rightarrow f'(z) = R_1$, $f(z) = R_2 \Rightarrow f'(z) = R_2$, and the order of R_1 is less than the order of f , then one of the following cases must occur:

- (1) $f \equiv f'$.
- (2) $f = R_2 + Ce^{\lambda z}$ and $(\lambda - 1)R_1 = \lambda R_2 - R'_2$ where C, λ are two nonzero constants. In fact, R_1, R_2 are two polynomials.

Theorem 1.2. Let R_1 and R_2 be two functions and $R_2 (\not\equiv R_1, 0)$. Let $f(z)$ be a nonconstant meromorphic function with finitely many poles. If $f(z) = R_1 \Rightarrow f'(z) = R_1$, $f(z) = R_2 \Rightarrow f'(z) = R_2$. If f, R_1 have no common poles and the order of R_1 is less than the order of f , then one of the following cases must occur:

- (1) $f \equiv f'$.
- (2) $f = R_2 + Ce^{\lambda z}$ and $(\lambda - 1)R_1 = \lambda R_2 - R'_2$ where C, λ are two nonzero constants. In fact, R_1, R_2 are two polynomials.

Remark 1.2. We add an example here to point out that the case (2) in Theorem 1.2 cannot be deleted.

Example 1. Let $f = Ae^{\frac{z}{2}} + z$, $R_1 = 2 - z$ and $R_2 = z$. Then

$$\frac{f' - R_1}{f - R_1} = \frac{1}{2} \quad \text{and} \quad f \neq R_2.$$

Thus, it satisfies the assumption of Theorem 1.2.

Obviously, Theorem 1.2 is a generalization of Theorem 1.1.

It does not seem that the above theorems can be proved by using the methods in [7,8]. In order to prove our theorems, we need the following propositions which are interesting in their's own right.

Proposition 1.3. Let $f(z)$ be a nonconstant meromorphic function with finitely many poles, and let $R_1, R_2 (\not\equiv R_1)$ be two meromorphic functions, and either $P_1 \neq c_1$ or $P_2 \neq c_2$, where c_1, c_2 are two constants. If

$$f(z) = R_1 \Rightarrow f'(z) = R_1 \quad \text{and} \quad f(z) = R_2 \Rightarrow f'(z) = R_2,$$

then $f(z)$ is of finite order.

Proposition 1.4. Let $f(z)$ be a nonconstant meromorphic function with finitely many poles and $f \neq 0$. If

$$f(z) = R_2 \Rightarrow f'(z) = R_2,$$

here R_2 is defined as in Theorem 1.2 and $R_2 \not\equiv 0$, then $f(z)$ is of finite order.

Proposition 1.5. Let $f(z)$ be a nonconstant meromorphic function with finitely many poles, and let $R_3 = c_1 e^z$, $R_4 = c_2 e^z$, where c_1, c_2 are two nonzero constants. If

$$f(z) = R_3 \Leftrightarrow f'(z) = R_3 \quad \text{and} \quad f(z) = R_4 \Rightarrow f'(z) = R_4,$$

then $f(z)$ is of finite order.

2. Some lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 2.1. Let $\{f_n\}$ be a family of functions meromorphic (analytic) on the unit disc Δ . If $a_n \rightarrow a$, $|a| < 1$, and $f_n^{\sharp}(a_n) \rightarrow \infty$, then there exist

1. a subsequence of f_n (which we still write as $\{f_n\}$);
2. points $z_n \rightarrow z_0$ $|z_0| < 1$;
3. a positive number $\rho_n \rightarrow 0$

such that $f_n(z_n + \rho_n \xi) = g_n(\xi) \rightarrow g(\xi)$ locally uniformly, where g is a nonconstant meromorphic (entire) function on $\mathbb{C}$, such that

$$\rho_n \leq \frac{M}{f_n^\sharp(a_n)},$$

where M is a constant which is independent of n .

Here, as usual, $g^\sharp(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative.

Comparing it with the lemma of [10], we only need to estimate the size of the ρ_n 's.

In fact, in the lemma of [10], let $z_n^* = a_n$. Then (for n large enough) $|a_n| \leq r^* < r$, $|z_n| < r$ and

$$M_n = \max_{|z| \leq r} \left(1 - \frac{|z|^2}{r^2}\right) f_n^\sharp(z) = \left(1 - \frac{|z_n|^2}{r^2}\right) f_n^\sharp(z_n) \quad \text{and} \quad \rho_n = \frac{1}{f_n^\sharp(z_n)}.$$

Thus, $(1 - \frac{|z_n|^2}{r^2}) f_n^\sharp(z_n) \geq (1 - \frac{|a_n|^2}{r^2}) f_n^\sharp(a_n)$ and

$$\rho_n = \frac{1}{f_n^\sharp(z_n)} \leq \frac{1 - \frac{|z_n|^2}{r^2}}{1 - \frac{|a_n|^2}{r^2}} \frac{1}{f_n^\sharp(a_n)} \leq \frac{1}{1 - \frac{|(r^*)|^2}{r^2}} \frac{1}{f_n^\sharp(a_n)} \leq \frac{M}{f_n^\sharp(a_n)},$$

where M is a constant which is independent of n . Thus, we obtain the lemma. In [6], Liu, Nevo and Pang proved a similar lemma.

Lemma 2.2. (See [6].) Let f be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n \rightarrow \infty$, such that for every $N > 0$, $f^\sharp(z_n) > |z_n|^N$ if n is sufficiently large.

Lemma 2.3. (See [10].) Let $\mathcal{F}$ be a family of functions in the unit disc Δ with the property that for each $f \in \mathcal{F}$, all zeros are of multiplicity at least k . Suppose that there exists a number $A \geq 1$ such that $|f^k(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f = 0$. If $\mathcal{F}$ is not normal in Δ , then for $0 \leq \alpha \leq k$, there exist

1. a number $r \in (0, 1)$;
2. a sequence of complex numbers z_n , $|z_n| < r$;
3. a sequence of function $f_n \in \mathcal{F}$ (which we still write as $\{f_n\}$);
4. a sequence of positive numbers $\rho_n \rightarrow 0$

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ locally uniformly (with respect to the spherical metric) to a nonconstant entire function $g(\xi)$ on $\mathbb{C}$, and moreover, the zeros of $g(\xi)$ are of multiplicity at least k , $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$.

Lemma 2.4. (See [3].) A normal function has order at most two. A normal entire function is of exponential type, and thus has order at most one.

Lemma 2.5. (See [2].) Let f and a be meromorphic functions of finite order such that both of f and a have finitely many poles, f and a have no common poles and the order of a is less than the order of f . If f and f' share a CM, then $f' - a \equiv c(f - a)$ for some nonzero constant c .

3. Proof of propositions

Proof of Proposition 1.3. Let $H = f - R_1$, then we have

$$H = 0 \quad \Rightarrow \quad H' = R_1 - R'_1, \quad H = R_2 - R_1 \quad \Rightarrow \quad H' = R_2 - R'_1.$$

Let $T = R_2 - R_1$ and $T_2 = R_2 - R'_1$. Then $T \not\equiv 0$, we consider the function $F = \frac{H}{T}$.

Case 1. F is of finite order. Hence $f = FT + R_1$ is of finite order as well.

Case 2. F is of infinite order. By Lemma 2.2, there exist $w_n \rightarrow \infty$, such that for every $N > 0$, if n is sufficiently large

$$F^\sharp(w_n) > |w_n|^N. \quad (3.1)$$

Note that $f(z)$ and $1/T$ only have finitely many poles, then there are an $r_1 > 0$ such that $f(z)$ is analytic in $|z| \geq r_1$ and an $r_2 > 0$ such that $1/T$ is analytic in $|z| \geq r_2$. Let $r = \max\{r_1, r_2\}$ and $D = \{z: |z| \geq r\}$, then F is analytic in D . Without loss of generality, we may assume $|w_n| \geq r + 1$ for all n . We define $D_1 = \{z: |z| < 1\}$ and

$$F_n(z) = F(w_n + z).$$

Then all $F_n(z)$ are analytic in D_1 and $F_n^\#(0) = F^\#(w_n) \rightarrow \infty$ as $n \rightarrow \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Therefore, we can apply Lemma 2.1. Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequence $(z_n)_n$ and $(\rho_n)_n$, such that $|z_n| < r < 1$ and $\rho_n \rightarrow 0$ such that the sequence $(g_n)_n$ defined by

$$g_n(\zeta) = F_n(z_n + \rho_n \zeta) = \frac{H(w_n + z_n + \rho_n \zeta)}{T(w_n + z_n + \rho_n \zeta)} \rightarrow g(\zeta) \quad (3.2)$$

locally uniformly in $\mathbb{C}$ where g is a nonconstant entire function and

$$\rho_n \leq \frac{M}{F_n^\#(0)} = \frac{M}{F^\#(w_n)}, \quad (3.3)$$

here, M is a positive number.

From (3.1) and (3.3), we deduce that: for every $N > 0$, if n sufficiently large

$$\rho_n \leq M|w_n|^{-N}. \quad (3.4)$$

Let $G_n(\zeta) = \rho_n \frac{H'(w_n + z_n + \rho_n \zeta)}{T(w_n + z_n + \rho_n \zeta)}$ and let N be sufficiently large, then from (3.2) we can get that

$$G_n(\zeta) = g_n'(\zeta) + \frac{\rho_n g_n(\zeta) T'(w_n + z_n + \rho_n \zeta)}{T(w_n + z_n + \rho_n \zeta)} \rightarrow g'(\zeta) \quad (3.5)$$

locally uniformly in $\mathbb{C}$.

Suppose that $g(\zeta_0) = 1$, then by Hurwitz's theorem, there exist $\zeta_n, \zeta_n \rightarrow \zeta_0$, such that (for n sufficiently large)

$$g_n(\zeta_n) = \frac{H(w_n + z_n + \rho_n \zeta_n)}{T(w_n + z_n + \rho_n \zeta_n)} = 1.$$

By the assumption we have

$$H'(w_n + z_n + \rho_n \zeta_n) = T_2(w_n + z_n + \rho_n \zeta_n). \quad (3.6)$$

Let $P_1(z) = \frac{Q_1}{Q_2}$, $P_2(z) = \frac{Q_3}{Q_4}$. Here Q_i are polynomials and let $\deg(Q_i) = l_i$ ($i = 1, 2, 3, 4$), $\deg(Q) = l$.

We have

$$\frac{T_2(w_n + z_n + \rho_n \zeta)}{T(w_n + z_n + \rho_n \zeta)} = \frac{A(w_n + z_n + \rho_n \zeta)}{B(w_n + z_n + \rho_n \zeta)} = O(|w_n|^L),$$

here, $A(z) = Q_4 Q_1^2 - Q_2 Q_3 Q_1' + Q_3 Q_1 Q_2' - Q_2 Q_1 Q_3 Q'$, $B(z) = Q_4 Q_1^2 - Q_1 Q_2 Q_3$, and $\deg(A(z)) = L$. Now, let $N > L$, with (3.4), we deduce that

$$\rho_n \frac{T_2(w_n + z_n + \rho_n \zeta)}{T(w_n + z_n + \rho_n \zeta)} = O(|w_n|^{L-N}) \rightarrow 0. \quad (3.7)$$

By (3.5), (3.6) and (3.7) we obtain that

$$g'(\zeta_0) = \lim_{n \rightarrow \infty} \rho_n \frac{H'(w_n + z_n + \rho_n \zeta_n)}{T(w_n + z_n + \rho_n \zeta_n)} = \lim_{n \rightarrow \infty} \rho_n \frac{T_2(w_n + z_n + \rho_n \zeta_n)}{T(w_n + z_n + \rho_n \zeta_n)} = 0.$$

Thus $g(\zeta) = 1 \Rightarrow g'(\zeta) = 0$, which yields that the zeros of $g - 1$ are of multiplicity at least 2. Similarly, we can prove that the zeros of g are of multiplicity at least 2.

Next, we shall distinguish two cases below.

Subcase 1. $R_2 \neq c_2 e^z$, here c_2 is a constant.

Suppose ξ_0 is a zero of $g(\zeta) - 1$ with multiplicity $m (\geq 2)$, then $g^{(m)}(\xi_0) \neq 0$. Thus there exists a positive number δ_1 , such that

$$g(\zeta) \neq 0, \quad g'(\zeta) \neq 0, \quad g^{(m)}(\zeta) \neq 0 \quad (3.8)$$

on $D_{\delta_1}^0 = \{z: 0 < |\zeta - \xi_0| < \delta_1\}$.

Noting that $g(\zeta) \neq 1$, by Rouché theorem, there exist $\zeta_{n,j}$ ($j = 1, 2, \dots, m$) on $D_{\delta_1/2} = \{\xi: |\zeta - \xi_0| < \delta_1/2\}$, such that

$$g_n(\zeta_{n,j}) = F(w_n + z_n + \rho_n \zeta_{n,j}) = 1 \quad (j = 1, \dots, m).$$

Note that, for n large enough

$$H(w_n + z_n + \rho_n \zeta) = T(w_n + z_n + \rho_n \zeta).$$

Let $V_n(\zeta_{n,j}) = H(w_n + z_n + \rho_n \zeta) - T(w_n + z_n + \rho_n \zeta)$ ($j = 1, 2, \dots, m$) then for n large enough,

$$\begin{aligned} V'_n(\zeta_{n,j}) &= \rho_n [H'(w_n + z_n + \rho_n \zeta_{n,j}) - T'(w_n + z_n + \rho_n \zeta_{n,j})] \\ &= \rho_n [T_2(w_n + z_n + \rho_n \zeta_{n,j}) - T'(w_n + z_n + \rho_n \zeta_{n,j})] \\ &= \rho_n e^Q [P_2 - P'_2 - P_2 Q']. \end{aligned} \quad (3.9)$$

If $P_2 - P'_2 - P_2 Q' = 0$, then we deduce $P'_2 = 0$, $Q' = 1$, thus we have $R_2 = c_2 e^z$, this is a contradiction. By (3.9), $V'_n(\zeta_{n,j}) \neq 0$. So each $\zeta_{n,j}$ is a simple zero of $g(\zeta) - 1$, that is $\zeta_{n,j} \neq \zeta_{n,i}$ ($1 \leq i \neq j \leq m$).

Similarly to (3.7), we have

$$\rho_n \frac{T_2(w_n + z_n + \rho_n \zeta) - T'_2(w_n + z_n + \rho_n \zeta)}{T(w_n + z_n + \rho_n \zeta)} \rightarrow 0. \quad (3.10)$$

Noting (3.5) and (3.10), we have

$$S_n(\zeta) = G_n(\zeta) - \rho_n \frac{T_2(w_n + z_n + \rho_n \zeta) - T'_2(w_n + z_n + \rho_n \zeta)}{T(w_n + z_n + \rho_n \zeta)} \rightarrow g'(\zeta), \quad (3.11)$$

and $S_n(\zeta_{n,j}) = 0$ ($j = 1, \dots, m$). From (3.8), we have

$$\lim_{n \rightarrow \infty} \zeta_{n,j} = \xi_0 \quad (j = 1, 2, \dots, m).$$

Noting (3.8), (3.11) and that $S_n(\zeta)$ has m zeros $\zeta_{n,j}$ ($j = 1, 2, \dots, m$) in $D_{\delta_1/2}$, then ξ_0 is a zero of $g'(\zeta)$ with multiplicity m , and thus $g^{(m)}(\xi_0) = 0$. This is a contradiction. Hence $g(\zeta) \neq 1$.

Subcase 2. $R_1 \neq c_1 e^z$, here c_1 is a constant.

Suppose ξ_1 is a zero of $g(\zeta)$ with multiplicity k (≥ 2), then $g^{(k)}(\xi_1) \neq 0$ and we can see that there exists a positive number δ_2 , such that

$$g(\zeta) \neq 0, \quad g'(\zeta) \neq 0, \quad g^{(k)}(\zeta) \neq 0 \quad (3.12)$$

on $D_{\delta_2}^0 = \{z: 0 < |z - \xi_1| < \delta_2\}$.

$g(\zeta) \neq 0$, by Rouché theorem again, there exist $\zeta_{n,i}$ ($i = 1, 2, \dots, k$) on $D_{\delta_2/2} = \{\xi: |\zeta - \xi_0| < \delta_2/2\}$ such that

$$g_n(\zeta_{n,i}) = H(w_n + z_n + \rho_n \zeta_{n,i}) = 0 \quad (i = 1, \dots, k).$$

Note that, for n large enough

$$\begin{aligned} H'(w_n + z_n + \rho_n \zeta_{n,i}) &= R_1(w_n + z_n + \rho_n \zeta_{n,i}) - R'_1(w_n + z_n + \rho_n \zeta_{n,i}) \\ &= e^Q [P_1 - P'_1 - P_1 Q']. \end{aligned} \quad (3.13)$$

Since $R_1 \neq c_1 e^z$, we have that (3.13) cannot be zero, so each $\zeta_{n,i}$ is a simple zero of $H(w_n + z_n + \rho_n \zeta)$, that is $\zeta_{n,i} \neq \zeta_{n,j}$ ($1 \leq i \neq j \leq k$).

Similarly discussing as in Subcase 1 we can get

$$\rho_n \frac{R_1(w_n + z_n + \rho_n \zeta_{n,i}) - R'_1(w_n + z_n + \rho_n \zeta_{n,i})}{T(w_n + z_n + \rho_n \zeta_{n,i})} \rightarrow 0$$

and

$$K_n(\zeta) = G_n(\zeta) - \rho_n \frac{R_1(w_n + z_n + \rho_n \zeta_{n,i}) - R'_1(w_n + z_n + \rho_n \zeta_{n,i})}{T(w_n + z_n + \rho_n \zeta_{n,i})} \rightarrow g'(\zeta) \quad (3.14)$$

holds and $K_n(\zeta_{n,i}) = 0$ ($i = 1, \dots, k$). From (3.12), we have

$$\lim_{n \rightarrow \infty} \zeta_{n,i} = \xi_1 \quad (i = 1, 2, \dots, k).$$

Noting (3.12), (3.14) and that $K_n(\zeta)$ has k zeros $\zeta_{n,i}$ ($i = 1, 2, \dots, k$) in $D_{\delta_2/2}$, then ξ_1 is a zero of $g'(\zeta)$ with multiplicity k , and thus $g^{(k)}(\xi_1) = 0$. This is a contradiction. Hence $g(\zeta) \neq 0$.

From the above discussion, no matter which case happens, by Nevanlinna's second fundamental theorem, we arrive at a contradiction. This completes the proof of Proposition 1.3. $\square$

Proof of Proposition 1.4. The proof of Proposition 1.4 is similar to the proof of Proposition 1.3. We here only need to assume $H = f$, $T = R_2$, $F = \frac{f}{R_2}$, and $g_n(\zeta) = F_n(z_n + \rho_n \zeta)$. Then we can still get that the zeros of $g - 1$ are of multiplicity at least 2.

Note that $g(\zeta) \neq 0$, then by Nevanlinna's second fundamental theorem, we can get a contradiction. This completes the proof of Proposition 1.4. $\square$

Proof of Proposition 1.5. Since $R_4 \neq 0$, we consider the function $F = \frac{f}{R_4} - 1$.

Case 1. F is of finite order. Hence $f = (F + 1)R_4$ is of finite order as well.

Case 2. F is of infinite order. By Lemma 2.2, there exist $w_n \rightarrow \infty$, such that for every $N > 0$, if n is sufficiently large

$$F^\sharp(w_n) > |w_n|^N.$$

Note that $f(z)$ only has finitely many poles, then there is an $r > 0$ such that $f(z)$ is analytic in $|z| \geq r$. Without loss of generality, we may assume $|w_n| \geq r + 1$ for all n . We define $D_1 = \{z: |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \left\{ \frac{f(w_n + z)}{R_4(w_n + z)} - 1 \right\}. \quad (3.15)$$

From (3.15), if $F(w_n + z) = 0$, thus $f(w_n + z) = R_4(w_n + z)$. Noting that $f = R_4 \Rightarrow f' = R_4$ and $R_4(z) = R_4'(z)$, then we obtain the following: if $n \rightarrow \infty$, then

$$|F_n'| = \left| \left(\frac{f(w_n + z)}{R_4(w_n + z)} \right)' \right| \leq \left| \frac{f'(w_n + z)}{R_4(w_n + z)} \right| + \left| \frac{f(w_n + z)}{R_4(w_n + z)} \right| \left| \frac{R_4'(w_n + z)}{R_4(w_n + z)} \right| < 2. \quad (3.16)$$

Obviously, $F_n(z)$ are analytic in D_1 and $F_n^\sharp(0) = F^\sharp(w_n) \rightarrow \infty$ as $n \rightarrow \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Therefore, we can apply Lemma 2.3. Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequence $(z_n)_n$ and $(\rho_n)_n$, such that $|z_n| < r < 1$ and $\rho_n \rightarrow 0$ such that the sequence $(g_n)_n$ defined by

$$g_n(\xi) = \rho_n^{-1} F_n(z_n + \rho_n \xi) = \rho_n^{-1} \left\{ \frac{f(w_n + z_n + \rho_n \xi)}{R_4(w_n + z_n + \rho_n \xi)} - 1 \right\} \rightarrow g(\xi) \quad (3.17)$$

locally uniformly on $\mathbb{C}$ where $g(\xi)$ is a nonconstant entire function and $g^\sharp(\xi) \leq g^\sharp(0) = 3$, and the order of $g(\xi)$ is at most 1.

First, we shall prove that $g = 0 \Rightarrow g' = 0$ on $\mathbb{C}$. Suppose that there exists a point ξ_0 such that $g(\xi_0) = 0$. Then by Hurwitz's theorem, there exist $\xi_n, \xi_n \rightarrow \xi_0$ as $n \rightarrow \infty$ such that (for n sufficiently large)

$$g_n(\xi_n) = \rho_n^{-1} \left\{ \frac{f_n(w_n + z_n + \rho_n \xi_n)}{R_4(w_n + z_n + \rho_n \xi_n)} - 1 \right\} = 0.$$

This implies $f_n(w_n + z_n + \rho_n \xi_n) = R_4(w_n + z_n + \rho_n \xi_n)$. Let $G_n(\xi) = \frac{f_n'(w_n + z_n + \rho_n \xi)}{R_4(w_n + z_n + \rho_n \xi)}$, $G_n(\xi) \rightarrow g'(\xi) + 1$ ($n \rightarrow \infty$). Noting that $f = R_4 \Rightarrow f' = R_4$, from (3.16), (3.17) we have

$$G_n'(\xi_n) = \frac{f_n'(w_n + z_n + \rho_n \xi_n) - f_n(w_n + z_n + \rho_n \xi_n)}{R_4(w_n + z_n + \rho_n \xi_n)} = 0 \quad (n \rightarrow \infty).$$

Thus

$$g'(\xi_0) = \lim_{n \rightarrow \infty} G_n'(\xi_n) = 0.$$

This shows that $g = 0 \Rightarrow g' = 0$.

Next we will prove that $g'(\xi) \neq \frac{c_1}{c_2} - 1$ on $\mathbb{C}$. Suppose that there exists a point ξ_0 such that $g'(\xi_0) = \frac{c_1}{c_2} - 1$. If $g'(\xi) \equiv \frac{c_1}{c_2}$, then $g(\xi) = (\frac{c_1}{c_2} - 1)\xi + c$, where c is a constant, which together with the fact $g = 0 \Rightarrow g' = 0$ gives $\frac{c_1}{c_2} = 1$, in contradiction to the assumptions. Thus $g'(\xi) \neq \frac{c_1}{c_2} - 1$. Since $G_n(\xi) - \frac{R_3(w_n + z_n + \rho_n \xi)}{R_4(w_n + z_n + \rho_n \xi)} \rightarrow g'(\xi) + 1 - \frac{c_1}{c_2}$ as $n \rightarrow \infty$ and $g'(\xi_0) = \frac{c_1}{c_2} - 1$, by Hurwitz's theorem, there exist $\xi_n \rightarrow \xi_0$ as $n \rightarrow \infty$, such that (for n sufficiently large)

$$\begin{aligned} G_n(\xi_n) - \frac{R_3(w_n + z_n + \rho_n \xi_n)}{R_4(w_n + z_n + \rho_n \xi_n)} &= 0 \\ \Rightarrow f_n'(w_n + z_n + \rho_n \xi_n) &= R_3(w_n + z_n + \rho_n \xi_n). \end{aligned} \quad (3.18)$$

Noting that $f = R_3 \Leftrightarrow f' = R_3$, from (3.17), (3.18) (for n sufficiently large) we have

$$g_n(\xi_n) = \rho_n^{-1} \left\{ \frac{f_n(w_n + z_n + \rho_n \xi_n)}{R_4(w_n + z_n + \rho_n \xi_n)} - 1 \right\} = \rho_n^{-1} \left(\frac{R_3(w_n + z_n + \rho_n \xi_n)}{R_4(w_n + z_n + \rho_n \xi_n)} - 1 \right).$$

Since $c_2 \neq c_1$ and $\rho_n \rightarrow 0$, from the above equation we can get

$$g(\xi_0) = \lim_{n \rightarrow \infty} g_n(\xi_n) = \infty,$$

which contradicts $g'(\xi_0) = \frac{c_1}{c_2} - 1$. This shows that $g'(\xi) \neq \frac{c_1}{c_2} - 1$ on $\mathbb{C}$.

Since g is of order at most one, so is g' , it follows that

$$g'(\xi) = \frac{c_2}{c_1} - 1 + e^{b_0+b_1\xi}, \quad (3.19)$$

where b_0, b_1 are finite constants. We divide this case into two subcases.

Subcase 1. If $b_1 = 0$, from (3.19) we have

$$g(\xi) = \left(\frac{c_2}{c_1} - 1 + e^{b_0} \right) \xi + c_0, \quad (3.20)$$

where c_0 is a constant. Since $g = 0 \Rightarrow g' = 0$, from (3.20) we have $\frac{c_2}{c_1} + e^{b_0} = 1$. Then $g' \equiv 0$, hence g is a constant. This is a contradiction.

Subcase 2. If $b_1 \neq 0$, by (3.19) we obtain

$$g(\xi) = \left(\frac{c_2}{c_1} - 1 \right) \xi + \frac{1}{b_1} e^{b_0+b_1\xi} + B, \quad (3.21)$$

where B is a constant. Obviously, $g(\xi) = 0$ has infinitely many solutions. Suppose that there exists a point ξ_0 such that $g(\xi_0) = 0$. Then by (3.19), (3.21) and $g = 0 \Rightarrow g' = 0$, we get $\xi_0 = \frac{1-c_2-Bc_1b_1}{b_1(c_2-c_1)}$. This is a contradiction. Thus f is of finite order. $\square$

4. Proof of Theorem 1.2

If $R_1 \equiv 0$, then we can get $f = 0 \Rightarrow f' = 0$. It is easy to get $f \neq 0$. By Proposition 1.4 we know that f is of finite order.

If $R_1 \not\equiv 0$, then from Propositions 1.3 and 1.5, we know that f is of finite order.

Then by Lemma 2.5, we have

$$\frac{f' - R_1}{f - R_1} = \lambda, \quad (4.1)$$

where λ is a nonzero constant.

Rewrite (4.1) as

$$f' = \lambda f + (1 - \lambda)R_1. \quad (4.2)$$

Since f and R_1 have no common poles, from (4.2) we can get that f and R_1 are two entire functions. If $\lambda \neq 1$, integral (4.2), we have

$$f = e^{\lambda z} [A + H(z)]. \quad (4.3)$$

Here A is a constant and $H(z) = \int_0^z (1 - \lambda)R_1(s)e^{-\lambda s} ds$. Next, we shall distinguish two cases below:

Case 1. If $\deg(Q) \geq 1$, by a simple calculation, we can get $\sigma(H') = \sigma(R_1) = \deg(Q) \geq 1$, then by (4.3) we can have $\sigma(f) \leq \sigma(H) = \deg(Q)$. This is a contradiction.

Case 2. If $\deg(Q) = 0$, then we can get that R_1 is a polynomial and R_2 is a rational function. Thus we have

$$f = Ae^{\lambda z} + h(z), \quad (4.4)$$

where A is a nonzero constant and $h(z)$ is a polynomial. Thus, we have

$$f' = A\lambda e^{\lambda z} + h'(z). \quad (4.5)$$

Substituting (4.4) and (4.5) into (4.2), we deduce that

$$(\lambda - 1)R_1 - (\lambda h - h') \equiv 0. \quad (4.6)$$

Next, we will prove that $h(z) \equiv R_2(z)$. Suppose that $h(z) \not\equiv R_2(z)$, then

$$\bar{N}\left(r, \frac{1}{f - R_2}\right) = \bar{N}\left(r, \frac{1}{Ae^{\lambda z} + h(z) - R_2}\right) = T(r, e^{\lambda z}) + S(r, f) \neq S(r, f). \quad (4.7)$$

Let z_0 be a zero of $f - R_2$, by the assumption we have $f'(z_0) = R_2(z_0)$. Put z_0 into (4.4) and (4.5), we have

$$(\lambda - 1)R_2(z_0) = \lambda h(z_0) - h'(z_0).$$

If $(\lambda - 1)R_2 - (\lambda h - h') \not\equiv 0$, we have

$$\begin{aligned} \bar{N}\left(r, \frac{1}{f - R_2}\right) &\leq \bar{N}\left(r, \frac{1}{(\lambda - 1)R_2 - (\lambda h - h')}\right) \\ &= O(\log r) = S(r, f), \end{aligned}$$

which contradicts with (4.7). Hence,

$$(\lambda - 1)R_2 - (\lambda h - h') \equiv 0.$$

Comparing it with (4.6), we have $R_1 = R_2$, a contradiction. Thus, we prove that $h(z) = R_2(z)$. Then, from (4.4) and (4.6), we have

$$f = Ae^{\lambda z} + R_2(z)$$

and

$$(\lambda - 1)R_1 = \lambda R_2 - R_2'$$

which is (2).

If $\lambda = 1$, we obtain $f = f'$, which is (1). Thus Theorem 1.2 is completely proved.

5. Remarks

Remark 5.1. We add an example here to point out when the restriction on the order change to that: The order of R_1 is equal to the order of f and the rest of conditions are still the same, then case (2) in Theorem 1.2 cannot happen.

Example 2. Let $f = ze^{z^2} + e^{\frac{z}{2}}$, $R_1 = (4z^2 - z + 2)e^{z^2}$, and $R_2 = ze^{z^2}$. Then

$$\frac{f' - R_1}{f - R_1} = \frac{1}{2} \quad \text{and} \quad f \neq R_2.$$

Thus, it does not satisfy the assumption of Theorem 1.2.

Remark 5.2. At the end of this article, we pose the following question: Can we remove the condition: The order of R_1 is less than f ?

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